NOTES AND CORRESPONDENCE

Derivation of the Elliptic Condition for the Balance Equation in Spherical Coordinates

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The meteorological balance equation is a commonly used diagnostic relationship between the velocity and pressure fields for large-scale motions in the atmosphere. In its original form it relates the nondivergent horizontal velocity to the horizontal pressure field. References to the equation date back to, at least, the early 1950's [see Petterson (1953)]. Bolin (1955) and Charney (1955) discussed the equation and pointed out a certain condition that must be satisfied to assure that the equation is elliptic. Ellipticity is required to permit solution of the equation as a boundary value problem.

There has been some ambiguity in the literature concerning the exact form of the elliptic condition. One notes, for instance, the difference between Bolin and Charney with respect to the presence and absence, respectively, of the beta or Coriolis parameter variation term. Sometimes the absence of the beta term can be attributed to the assumption of a constant Coriolis parameter. This assumption may be made only at the time the ellipticity condition is discussed, i.e., see Hollmann (1966). In other cases it is not clear just why it is absent (Miyakoda, 1960).

If map-scale factors are included in the equation, questions may arise as to how they affect the elliptic condition. Shuman (1957) includes these factors in his elliptic requirement but relies on computational evidence for its correctness.

With the advent of global spherical models it became necessary, or at least of theoretical interest, to determine the elliptic condition for the balance equation in spherical coordinates. Because the derivation of the condition has been only briefly outlined in the meteorological literature and then just for nonspherical coordinate systems (e.g., Árnason, 1958; Kibel', 1963), it was decided to present the complete derivation of the elliptic condition for this case.

The balance equation in spherical coordinates is

$$f\nabla^2\psi-2J(v,u)-\beta u-\frac{1}{a^2}\left(1+\tan\varphi\frac{\partial}{\partial\varphi}\right)\times(u^2+v^2)=\frac{1}{\rho}\nabla^2 p, \quad (1)$$

where the horizontal velocity components u and v are the nondivergent components only. The other symbols are defined as follows:

- f Coriolis parameter
- β $\dfrac{1}{a}\dfrac{\partial f}{\partial\varphi}$
- a radius of earth
- φ latitude
- λ longitude
- ρ density
- p pressure
- ∇^2 Laplacian operator in spherical coordinates,

$$\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial}{\partial\varphi}$$

- J Jacobian operator in spherical coordinates,

$$J(A,B)=\frac{1}{a\cos\varphi}\frac{\partial A}{\partial\lambda}\left(\frac{1}{a}\frac{\partial B}{\partial\varphi}\right)-\frac{1}{a}\frac{\partial A}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial B}{\partial\lambda}\right)$$

- ψ stream function

In terms of the stream function, i.e., after writing u and v in terms of ψ, Eq. (1) is of the Monge-Ampère type. The Monge-Ampère equation can be written

$$Rr+2Ss+Tt+(rt-s^2)=V, \quad (2)$$

where r, s and t are second derivatives of the dependent variable with respect to the two dependent variables, and R, S, T and V are functions of the dependent variable, its first derivative, and the independent variables (Forsyth, 1959, p. 202).

The classification of Eq. (2) is determined by casting it in canonical form, i.e., determining the characteristics of the system. This procedure is discussed in detail by Forsyth in chapters 16 and 20. There he shows that the two characteristics are $-S\pm\sqrt{S^2-RT-V}$. In order to be elliptic, these characteristics must be complex, which requires

$$RT+V-S^2>0 \quad (3)$$

for the elliptic condition. Lewy (1937) writes the equation in the form

$$Ar+Bs+Ct+(rt-s^2)=E, \tag{4}$$

and refers to the condition

$$4(AC+E)-B^2>0, \tag{5}$$

which is equivalent to Eq. (3). Rellich (1932) and Kibel' (1963) also give the elliptic condition in a form comparable to (3).

To derive the elliptic condition, Eq. (1) is transformed into the form of Eq. (4) and then condition (5) is applied. The dependent variable is taken to be ψ. In order to obtain $s^2=(\psi_{xy})^2$ from the Jacobian term, ψ_{xy} must equal ψ_{yx}. (Subscripts refer to differentiation.) If $x=a(\cos\varphi)\lambda$ and $y=a\varphi$ are used as the independent variables, then the order of differentiation cannot be reversed without introducing an extra term. The method used here gives the same result; namely, to choose $x=\lambda$ and $y=\varphi$.

If ψ is substituted for u and v by the equations $u=\psi_\varphi/a$ and $v=\psi_\lambda/(a\cos\varphi)$ and $\psi_{\lambda\lambda}$, $\psi_{\lambda\varphi}$ and $\psi_{\varphi\varphi}$ replaced by r, s and t, respectively, Eq. (1) can be written after grouping terms as

$$\frac{f}{a^2\cos^2\varphi}r-\frac{4(\tan\varphi)\psi_\lambda}{a^4\cos^2\varphi}s+\left(\frac{f}{a^2}-\frac{2(\tan\varphi)\psi_\varphi}{a^4}\right)t$$
$$+\frac{2}{a^4\cos^2\varphi}(rt-s^2)=\frac{f\tan\varphi}{a^2}\psi_\varphi-\frac{\beta}{a}\psi_\varphi \tag{6}$$
$$+\frac{2\tan^2\varphi}{a^4\cos^2\varphi}(\psi_\lambda)^2+\frac{1}{\rho}\nabla^2p+K,$$

where

$$K=\frac{1}{a^2}\left[\left(\frac{\psi_\varphi}{a}\right)^2+\left(\frac{\psi_\lambda}{a\cos\varphi}\right)^2\right]. \tag{7}$$

The coefficient of $(rt-s^2)$ must be 1, so Eq. (6) is multiplied by $(a^4\cos^2\varphi)/2$. Then it is determined that

$$\left.\begin{aligned} A&=\tfrac{1}{2}a^2f\\ B&=-2(\tan\varphi)\psi_\lambda\\ C&=\frac{f(\cos^2\varphi)a^2}{2}-\frac{(\tan\varphi)\psi_\varphi}{\cos^2\varphi}\\ E&=\frac{a^4\cos^2\varphi}{2}\left[\frac{f(\tan\varphi)\psi_\varphi}{a^2}-\frac{\beta}{a}\psi_\varphi+\frac{2\tan^2\varphi}{a^4\cos^2\varphi}(\psi_\lambda)^2\right.\\ &\qquad\left.+\frac{1}{\rho}\nabla^2p+K\right]\end{aligned}\right\}, \tag{8}$$

if Eq. (1) is to correspond to Eq. (4). Substituting these into Eq. (5), simplifying and substituting for K gives the final condition,

$$\frac{1}{\rho}\nabla^2p+\frac{f^2}{2}-\frac{\beta}{a}\psi_\varphi+\frac{1}{a^2}\left[\left(\frac{\psi_\varphi}{a}\right)^2+\left(\frac{\psi_\lambda}{a\cos\varphi}\right)^2\right]>0. \tag{9}$$

This is essentially the equation given by Bolin (1955). The only addition is the term

$$\frac{1}{a^2}\left[\left(\frac{\psi_\varphi}{a}\right)^2+\left(\frac{\psi_\lambda}{a\cos\varphi}\right)^2\right].$$

Using standard scale analysis, i.e., Burger (1958), it can be shown that this last term is generally one to two orders of magnitude smaller than the sum of the other terms in Eq. (9). Thus, it may be concluded that the condition equivalent to Bolin's

$$\frac{1}{\rho}\nabla^2p+\frac{f^2}{2}-\frac{\beta}{a}\psi_\varphi>0 \tag{10}$$

is a good approximate form even for spherical coordinates. Scale analysis also shows that the beta term may not be neglected when considering global or low latitude models.

REFERENCES

Árnason, G., 1958: A convergent method for solving the balance equation. *J. Meteor.*, **15**, 220–225.

Bolin, B., 1955: Numerical forecasting with the barotropic model. *Tellus*, **7**, 27–49.

Burger, A. P., 1958: Scale consideration of planetary motions of the atmosphere. *Tellus*, **10**, 195–205.

Charney, J., 1955: The use of the primitive equations of motion in numerical prediction. *Tellus*, **7**, 22–26.

Forsyth, A. R., 1959: *Theory of Differential Equations*. Part 4, Vol. 6, New York, Dover Publications, Inc., 596 pp.

Hollmann, G., 1966: Zur Frage neuer diagnostischen Beziehungen zwischen Wind- und Druckfeld (Balancegleichungen) in einer barotropen Atmosphäre mit divergenter Strömung. *Breitr. Phys. Atmos.*, **39**, 99–111. (English translation by Meteorological Service of Canada.)

Kibel', I. A., 1963: *An Introduction to the Hydrodynamical Methods of Short Period Weather Forecasting*. New York, The Macmillan Company, 383 pp.

Lewy, H., 1937: *A priori* limitations for solutions of Monge-Ampère equations, II. *Trans. Amer. Math. Soc.*, **41**, 365–374.

Miyakoda, K., 1960: Numerical solution of the balance equation. Tech. Rept. Japan Meteorological Agency, No. 3, 15–34.

Petterson, S., 1953: On the relation between vorticity, deformation and divergence and the configuration of the pressure field. *Tellus*, **3**, 231–237.

Rellich, F., 1932: Zur ersten Randwertaufgabe bei Monge-Ampèreschen Differentialgleichungen vom elliptischen Typus: Differential geometrische Anwendungen. *Math. Ann.*, **107**, 505–513.

Shuman, F. G., 1957: Numerical methods in weather prediction: I. The balance equation. *Mon. Wea. Rev.*, **85**, 329–332.