Planetary Rossby Waves Propagating Vertically Through Weak Westerly Wind Wave Guides[1]

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ABSTRACT

The role of horizontal wind shears in the vertical propagation of planetary Rossby waves is investigated using an adiabatic linear model. We discuss wave guides formed by regions of weak westerly wind. If the wave guide is formed by trapping of waves between strong westerlies and/or the geometric poles, the ducting occurs as a wave propagation in discrete normal modes of the internal wave guide. On the other hand, for wave guides formed by one or more lines of zero wind, waves are absorbed rather than reflected at the zero wind line so that there are no normal modes of the wave guide. Disturbances excited in the lower stratosphere in the equatorial zero wind wave guide will terminate somewhere in the equatorial stratosphere, but eddy motions may be maintained in the tropics at higher levels by leakage from the Aleutian high planetary wave propagating vertically in a polar wave guide. The Aleutian high should be significantly attenuated by such leakage. The theory of zero wind line absorption suggests a planetary wave coupling with the biennial oscillation.

1. Introduction

During the winter and summer there are found great circumpolar westerly and easterly jets in the upper stratosphere. A transition from one regime to the other occurs in spring and fall. The object of this paper is to examine the role of horizontal wind shears in guiding the upward propagation of stationary planetary wave motions around the winter hemisphere jet. The prevailing zonal wind systems and the refraction of planetary waves inferred from theory are indicated schematically in Fig. 1.

Charney and Drazin (1961) considered the vertical propagation of stationary planetary waves under the further assumptions that the Coriolis parameter and the strength of mean zonal wind were independent of latitude, showing by their analysis that stationary, vertically propagating, eddy wave motions occur in the presence of a mean zonal wind only when the zonal wind is westerly, and that there is some maximum speed of the westerly wind above which there can be no propagation. Presently available data on winds in the upper stratosphere during the summer establish the essential correctness of Charney and Drazin's theoretical result concerning the absence of penetration of planetary waves into a region of easterly flow.

Observational evidence accumulated in the last several years shows that planetary wave motions do indeed occur in the upper stratosphere (Finger *et al.*, 1966) apparently reaching as high as the lower thermosphere (Newell and Dickinson, 1967) during most of the winter. Such eddy wave motions propagating vertically out of the troposphere dominate the general circulation of the wintertime stratosphere as evidenced by such recent observational studies as those of Oort (1964), Julian and Labitzke (1965), Muench (1965), Boville (1967) and Perry (1967). Numerical models exhibiting the importance of eddy wave energy propagating from the troposphere have been discussed by Peng (1965a,b), Nitta (1967) and Byron-Scott (1967). Lindzen (1967) has described an equatorial β-plane perturbation model indicating that planetary waves will propagate above the equator through westerly winds of any strength. Wallace (1966) has suggested that such waves are possibly an important intermediate mechanism for driving the biennial oscillation. Eddy motions in the lower stratosphere determine the latitudinal and downward transports of ozone and radioactive trace substances (Newell, 1963), and in the mesosphere and lower thermosphere may determine variations in the concentration of such important trace substances as atomic oxygen and nitric oxide.

It is important to note that actual zonal winds vary greatly with latitude. Consequently, it is to be expected that any successful quantitative prediction of the upward propagation of planetary Rossby waves cannot be based only on the middle latitude zonal winds. Indeed, for the idealized model of an atmosphere on a spherical earth in constant angular rotation, half a dozen normal modes can propagate vertically for zonal winds >38 m sec^{-1}, the lowest normal mode propagating upward as a stationary disturbance through an eastward rotating atmosphere with angular velocity more than twice as

[1] An early version of this theory was presented at the November 1966 Meeting, American Meteorological Society, El Paso, Tex.

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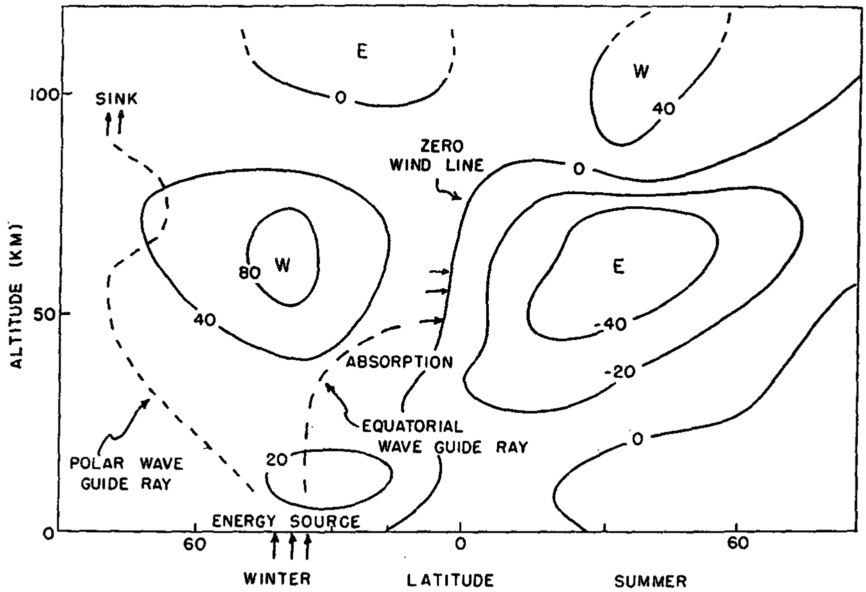

FIG. 1. Schematic sketch of the winter and summer zonal wind systems as functions of height. Also indicated schematically are planetary wave ray paths for the weak westerly wind wave guides in the winter hemisphere.

great as that of the earth (Dickinson, 1968). Models that neglect horizontal wind variation and dissipation apparently will predict an unreasonably large leakage of disturbances into the upper atmosphere.

Due to the complexity of atmospheric hydrodynamic processes, analytical studies such as this one cannot be expected to yield numerically exact solutions for atmospheric motions. Their primary purpose is to give a qualitative understanding of the important processes and parameters and to provide crude numerical results. In this spirit we concern ourselves primarily with solutions for a special class of zonal winds that permit separation of the vertical and latitudinal dependence of solutions, making liberal use of various asymptotic approximations, not necessarily because the approximations are mathematically justifiable for the value of the physical parameters appropriate to the problem at hand, but rather because the assumed approximations give increased insight into the nature of planetary waves. More general analytical treatments that have been examined by the author do not appear to give any further physical insights and are not especially convenient for numerical calculations.

2. Formulation

The spatial variables used in this study are λ, longitude; $y=\tanh^{-1}(\sin\varphi)$, where φ is latitude; and $z=\log(p_0/p)$, where p is pressure and p_0 some reference pressure. Our horizontal variables are Mercator coordinates, λ increasing eastward and y increasing northward. This paper assumes that zonally averaged winds with speeds small compared to that of the earth's rotation are specified as functions of latitude, pressure and time, and is concerned with the approximate description of planetary scale, asymmetric eddy motions. These motions are assumed to vary slowly on the time scale of a day, and hence to satisfy approximately the following equations [cf., for example, Dickinson (1968), Eqs. (19b) and (11)]:

$$\frac{D}{Dt}\Delta\psi+\{\sigma-[\sigma^{-1}(\sigma\bar{u})_\varphi]_\varphi\}\frac{\partial\psi}{\sigma\partial\lambda}-\mu\left(\frac{\partial w}{\partial z}-w\right)=0, \quad (2.1)$$

$$\frac{D}{Dt}\frac{\partial\psi}{\partial z}-\bar{u}_z\frac{\partial\psi}{\sigma\partial\lambda}+w\frac{S}{\mu}=0, \quad (2.2)$$

which are linearized vorticity and thermodynamic equations, respectively. Here ψ is a nondimensional eddy stream function, the term $\mu=\sin\varphi=\tanh y$ is the Coriolis parameter, $\sigma=\cos\varphi=\operatorname{sech} y$, t is time measured in units of $(2\Omega)^{-1}$, $w=dz/dt$, $\bar{u}(y,z,t)$ is the mean zonal wind measured in units of $(2\Omega a)$, where a is the radius of the earth, $\Delta=\sigma^{-2}(\partial^2/\partial y^2+\partial^2/\partial z^2)$ is the Laplacian operator on a unit sphere, $S=R(2\Omega a)^{-2}(\partial\bar{T}/\partial z+R\bar{T}/C_p)$ is the static stability, where $\bar{T}$ is the mean temperature assumed to depend only on z, C_p specific heat at constant pressure, R the gas constant, $D/Dt=\partial/\partial t+(\bar{u}/\sigma)\partial/\partial\lambda$ the linearized substantial derivative, and Ω the earth's angular rotation. A bar over a quantity indicates that it has been zonally averaged. A prime denotes the value of the deviation of a quantity from its zonal average. When convenient, subscripts are used to indicate differentiation of a function with respect to the subscript. Dimensional eddy winds u', v', geopotential height h', and temperature T' are related to

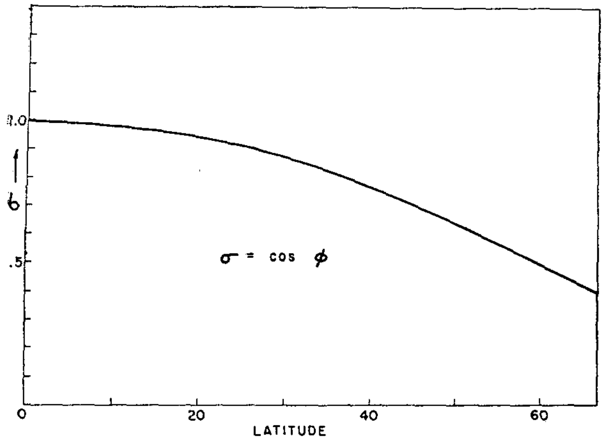

FIG. 2. Nondimensional gradient of planetary vorticity, cos φ, as a function of latitude.

ψ by

$$\left.\begin{aligned} u' &= -2\Omega a\sigma^{-1}\psi_y \\ v' &= 2\Omega a\sigma^{-1}\psi_\lambda \\ gh' &= (2\Omega a)^2\mu\psi \\ T' &= (2\Omega a)^2\mu R^{-1}\psi_z \end{aligned}\right\}. \tag{2.3}$$

Vertical motion is obtained from ψ by means of (2.2).

The error resulting from use of (2.1) and (2.2), rather than the exact linearized system, for study of planetary waves propagating through westerly winds will be negligible except in the tropics (Dickinson, 1968). Significant error may result from neglect of nonlinear terms or diabatic heating, but these error sources will not be discussed here. The variable Coriolis parameter and spherical geometry in the system (2.1), (2.2) allows a uniformly valid treatment of disturbances in middle and high latitudes and may be used to continue solutions from middle latitudes into tropical regions [as discussed, for example, by Charney (1963)].

The rate of zonal wind change is assumed to be sufficiently slow that it may be neglected for the computation of eddy motions. Solutions can then be Fourier analyzed into disturbances of the form

$$\psi(\lambda,y,z,t)=\Psi(y,z)e^{im(\lambda-ct)}. \tag{2.4}$$

Substitution of (2.4) into (2.1) and (2.2) and elimination of w yields a partial differential equation in y and z,

$$L\Psi+\frac{\sigma^2 r(y,z)}{\bar{u}-\sigma c}\Psi=0, \tag{2.5}$$

where the ellipic operator L is defined by

$$L=\left[\frac{\partial^2}{\partial y^2}-m^2+(\sigma^2\mu^2e^z)\frac{\partial}{\partial z}\frac{e^{-z}}{S}\frac{\partial}{\partial z}\right], \tag{2.6}$$

and $r(y,z)$ by

$$r(y,z)=\sigma-\{[\sigma^{-1}(\sigma\bar{u})_\varphi]_\varphi+\mu^2e^z(e^{-z}\bar{u}_z/S)_z\}. \tag{2.7}$$

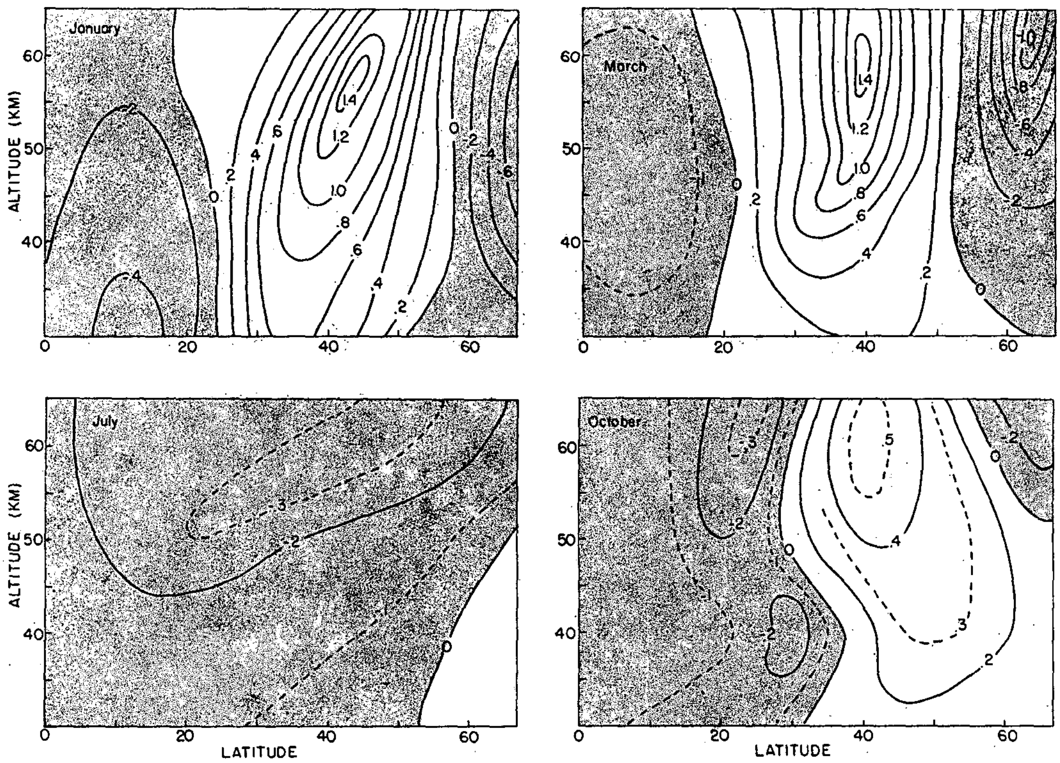

FIG. 3. Nondimensional gradient of mean zonal wind vorticity in the upper stratosphere, $-[(\cos\varphi)^{-1}(\bar{u}\cos\varphi)_\varphi]_\varphi$, for January, March, July and October.

The parameter $r(y,z)$ consists of three terms: 1) $\sigma=\cos\varphi$, shown in Fig. 2, is the nondimensional Rossby β parameter; 2) $-[\sigma^{-1}(\sigma\bar{u})_\varphi]_\varphi$, shown in Fig. 3, is the nondimensional gradient of vorticity of the mean zonal winds; and 3) the vertical wind shear term $-\mu^2e^z(e^{-z}\bar{u}_z/S)_z$, shown in Fig. 4. As indicated by the derivation of Charney and Stern (1962), $r(y,z)$ may be interpreted as proportional to the gradient of mean potential vorticity. We have computed the second and third terms of $r(y,z)$ for altitudes of 30–65 km and for the months of January, March, July and October by using the mean zonal winds and temperatures of the CIRA 1965 Standard Atmosphere (COSPAR, 1965). Because of observational uncertainties and year-to-year fluctuations in the wind data, aggravated by errors from the finite difference computation of wind shears, these parameters cannot be very accurately defined. The range of values we may expect for $r(y,z)$, as indicated by Figs. 3 and 4, is sufficient information for our discussion. Also, we show in Fig. 5 the values of S for altitudes of 30–80 km and for the same time periods.

The boundary conditions at the bottom and sides are the specification of Ψ on some lower boundary $z=z_0$, and the condition that as $|y|\rightarrow\infty$, $\Psi\sim e^{-m|y|}$. The top boundary condition is the requirement as $z\rightarrow\infty$, that solutions are selected which either decay exponentially or give maximum outward eddy flux of energy $e^{-z}\overline{gw'h'}$.

In order to simplify analytical discussion of solutions to (2.5), only disturbances with zero phase speed are considered and it is assumed that $\bar{u}(y,z)$ belongs to a class of zonal winds of the form

$$\bar{u}(y,z)=\frac{r(y,z)}{\beta}\frac{U(y)V(z)}{\mu^2U(y)+V(z)}, \tag{2.8}$$

where β is a nondimensional constant. Further discussion assumes $r(y,z)$ to be always greater than zero so that sign changes of $\bar{u}(y,z)$ are always associated with a sign change of $U(y)$ or $V(z)$. It is helpful in the discussion of solutions using (2.8) to choose a $U(y)$ which at least roughly models the actual latitudinal variation of zonal winds. If $V(z)$ is large compared to $U(y)$, then according to (2.8), $\bar{u}(y,z)\simeq[r(y,z)/\beta]U(y)$. Hence, it is plausible, at least for regions of strong horizontal shear, that if β is chosen to be equal to some mean value of $r(y,z)$, then we can assume that $U(y)$ models $\bar{u}(y,z)$. The summing of Figs. 2, 3 and 4 indicates that β, taken as the mean value of $r(y,z)$ for some region, will generally be between zero and one. We ignore the possibility indicated by these figures that $r(y,z)$ changes sign in polar latitudes during some months and at some levels. It then follows that (2.8) describes a class of winds with zeros of $\bar{u}$ either along lines of constant y or along lines of constant z.

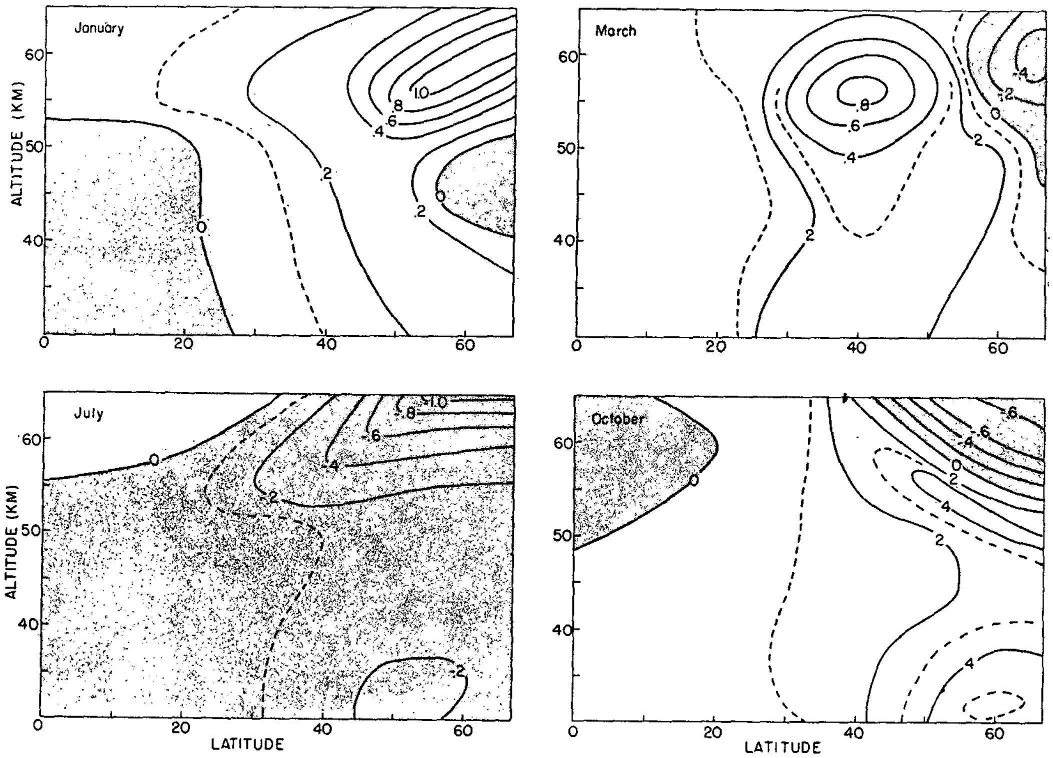

FIG. 4. Nondimensional vertical wind shear parameter, $-\sin^2\varphi e^z(e^{-z}\bar{u}_z/S)_z$ in the upper stratosphere for January, March, July and October.

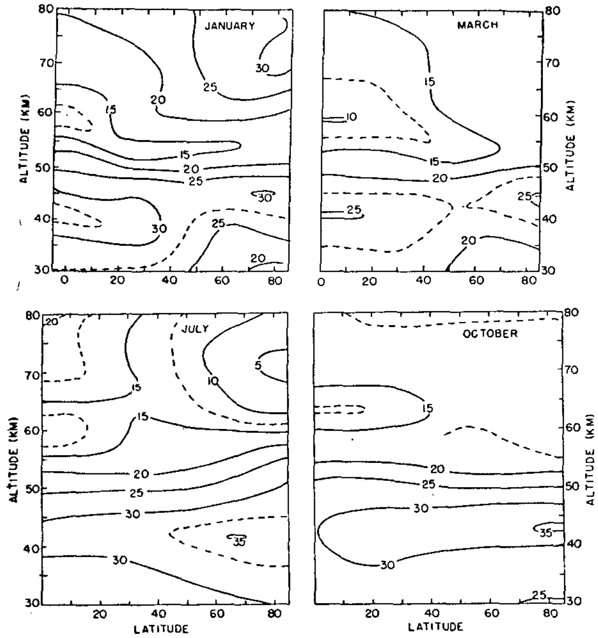

FIG. 5. Nondimensional static stability,
$$S=(2\Omega a)^{-2}R[\bar{T}_z+(R/C_p)\bar{T}]$$
when divided by 10^3, for the upper stratosphere and mesosphere for January, March, July and October.

Assuming (2.8) and taking $c=0$, we separate variables by anticipating solutions of the form

$$\Psi(y,z)=(Se^z)^{\frac{1}{2}}Y(y)Z(z). \tag{2.9}$$

After substitution of (2.9) into (2.5) and separation of variables, we find Y and Z are determined by

$$\left\{\frac{\partial^2}{\partial y^2}-m^2+\sigma^2\left[\frac{\beta}{U(y)}-\mu^2\gamma^2\right]\right\}Y(y,\gamma)=0, \tag{2.10}$$

$$\left\{\frac{\partial^2}{\partial z^2}-\tfrac{1}{4}(1+\epsilon)+S\left[\frac{\beta}{V(z)}+\gamma^2\right]\right\}Z(z,\gamma)=0. \tag{2.11}$$

We call (2.10) the *latitudinal structure equation* and (2.11) the *vertical structure equation.*

In the above, γ^2 is the separation of variables constant, and $\epsilon(z)=4S^{\frac{1}{2}}[(S^{-\frac{1}{2}})_{zz}-(S^{-\frac{1}{2}})_z]$ is a parameter which vanishes for an isothermal atmosphere. Fig. 6, sketched from the CIRA data, indicates the mean altitude variation of $\epsilon(z)$ from 30–80 km, showing $\epsilon(z)$ to be small except for dipole peaks centered at the stratopause, where there occurs a sharp decrease in stability. For (2.11) to have wavelike solutions, it is necessary that $\{[\beta/V(z)]+\gamma^2\}>(1+\epsilon)/(4S)$. The following discussion assumes that $\epsilon\ll 1$ and that $U(y)$ and $V(z)$ are positive. Eq. (2.11) will then always have wavelike solutions for $\gamma^2>1/(4S)$. In the upper stratosphere the stability $S\sim 0.025$, so for γ^2 roughly 10 or greater, we may expect (2.11) to have wavelike solutions at all levels. The amplitude of propagating waves will increase in the vertical as $(Se^z)^{\frac{1}{2}}$, according to (2.9). For $0<\gamma^2<1/(4S)$, the solutions to (2.11) may be evanescent, but $Z(z)$ should decay less rapidly than $e^{-z/2}$, so again from (2.9) the geopotential will increase with height, provided S decreases sufficiently slowly. In the following discussion we shall refer only to the modes with $\gamma^2>1/(4S)\sim 10$ as *propagating modes.*

Physical discussion of solutions to (2.5) may be made in terms of the eddy fluxes of heat, $\overline{v'T'}$, of momentum, $\overline{u'v'}$, and the vertical and horizontal fluxes of energy proportional to $\overline{gw'h'}$ and $\overline{gv'h'}$. Let $s=\sigma^{-2}mSe^z(2\Omega a)^3$. Then assuming (2.3), (2.4), (2.8) and (2.9), these fluxes may be written in terms of $Y(y)$ and $Z(z)$ as

$$\left.\begin{aligned}\overline{v'T'}&=-[(\sigma\mu s/R)|Y|^2]\,\mathrm{Im}(ZZ_z^*)\\ \overline{u'v'}&=[s/(2\Omega a)|Z|^2]\,\mathrm{Im}(YY_y^*)\\ \overline{gw'h'}&=-[\sigma\mu^2(s/S)|Y|^2]\bar{u}\,\mathrm{Im}(ZZ_z^*)\\ \overline{gv'h'}&=-[s|Z|^2]\bar{u}\,\mathrm{Im}(YY_y^*)\end{aligned}\right\}. \tag{2.12}$$

In these expressions Im denotes the "imaginary part of" and an asterisk denotes complex conjugate. The first two of these expressions are obtained directly from the definitions (2.3). The expression for w' is obtained from Eq. (2.2). The height h' is evaluated to a higher order

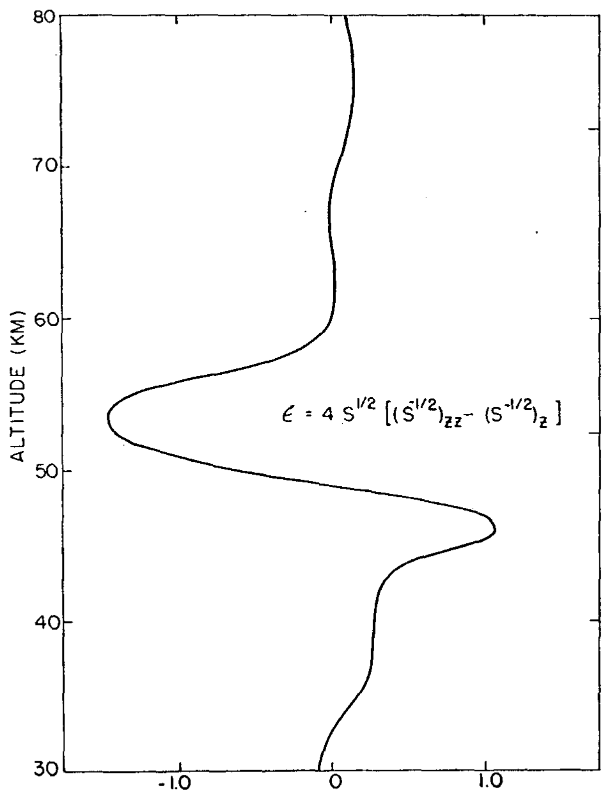

FIG. 6. Mean profile of the parameter $\epsilon=4S^{\frac{1}{2}}(S_{zz}^{-\frac{1}{2}}-S_z^{-\frac{1}{2}})$.

approximation based on the eddy equation of motion in the zonal direction in order to obtain the contribution to $\overline{v'h'}$, the h' given by (2.3) being uncorrelated with v'. The term $\overline{v'h'}$, defined by (2.12), gives the horizontal energy flux obtained from the energy integral for the system (2.1)–(2.2). Eliassen and Palm (1961) have derived more general expressions for all the flux terms in (2.12), which do not require the assumption of quasi-geostrophic motions or of the separation of variables (2.8)–(2.11).

It is a well known property of the differential equation, $\phi_{xx}+f(x)\phi=0$, that the Wronskian W of two solutions, ϕ_1 and ϕ_2, is a constant except possibly at singularities of $f(x)$. That is, $(d/dx)W(\phi_1,\phi_2)=0$, using the definition $W(\phi_1,\phi_2)=(\phi_1\phi_{2,x}-\phi_2\phi_{1,x})$. In particular, if $f(x)$ is real, then if ϕ is a solution, its complex conjugate ϕ^* is also a solution, and $W(\phi,\phi^*)=2\text{Im}(\phi\phi_x^*)$ $=$constant. Hence, since $Y(y)$ and $Z(z)$ satisfy (2.10) and (2.11), respectively, if $U\neq0$, $V\neq0$, it follows that

$$\left.\begin{aligned} &\frac{d}{dz}\text{Im}(ZZ_z^*)=0 \\ &\frac{d}{dy}\text{Im}(YY_y^*)=0 \end{aligned}\right\}. \tag{2.13}$$

Consequently, evaluating eddy fluxes from solutions to (2.10) and (2.11), $U(z)>0$, $V(z)>0$, we conclude:

1) The eddy momentum flux $\overline{u'v'}$ is independent of y and the eddy heat flux $\overline{v'T'}$ is independent of z except for a slowly varying factor S.

2) The energy flux $g\overline{w'h'}$ is upward for positive (northward) heat flux, while the energy flow $g\overline{v'h'}$ is oppositely directed to the eddy momentum flux $\overline{u'v'}$.

3) The energy flux goes to zero along the lines in y, z space where U or V, and hence $\bar{u}$, approach zero. Similar conclusions were stated by Eliassen and Palm (1961).

Since (2.10) and (2.11) can have wavelike solutions for $\bar{u}>0$, but not for $\bar{u}<0$, we can expect jumps in $\text{Im}(YY_y^*)$ and $\text{Im}(ZZ_z^*)$ where $\bar{u}=0$. This will be established in Section 4. We shall define the term *wave absorption* to denote the fact that the energy flux is toward a singular line in the $\bar{u}>0$ region while vanishing for $\bar{u}<0$, and we shall use the term *wave emission* to denote the fact that the energy flux is away from the singular line for $\bar{u}>0$, while again vanishing for $\bar{u}<0$. These definitions are consistent with the fluxes obtained by solving initial value problems with sources in the $\bar{u}>0$ and $\bar{u}<0$ regions, respectively. Note, however, since the energy fluxes vanish when $\bar{u}=0$, that energy is not absorbed or emitted at the singular line, and also that "absorption" implies divergence of $\overline{u'v'}$ in an infinitesimal layer at the the singular line.

In the absence of singular lines or dissipation there is no real coupling between stationary planetary waves and the mean zonal flow. Considering the kinetic and available potential energies, zonal and eddy, given by the wave and the zonal flow, respectively, we find there is a cycling from one form of energy to another, with no net losses or gains. These conclusions follow from a straightforward generalization of the nonlinear perturbation analysis of Charney and Drazin to the present model. Changes of the zonal flow by singular line coupling will be discussed elsewhere.

3. Nonsingular vertical wave guides

The purpose of this section is to examine the latitudinal structure of planetary waves for various kinds of vertical wave guides determined by equivalent zonal winds $U(y)$, $V(z)$, and hence $\bar{u}(y,z)$, which remain everywhere positive. In general, planetary waves can be reflected by regions of strong westerlies or by the geometrical poles of our spherical coordinate system. The latter reflection is directly a consequence of the assumed disturbance form (2.4) with constant angular wavenumber m. The linear wavenumber $k=ma\sigma$ becomes very large as the poles are approached, and propagation is no longer possible. If zonal winds are everywhere westerly, standing waves of planetary latitudinal extent will be guided into vertical propagation by the confining sphericity of the earth. We shall designate this phenomenon *the planetary wave guide.*

If the zonal flow is broken up into pronounced regions of strong westerlies, i.e., zonal winds of order 100 m sec^{-1}, and regions of weak westerlies, i.e., zonal winds of order 10 m sec^{-1}, then at least some of the normal mode disturbances in the planetary wave guide defined above can be expected to be largely confined to a more local wave guide formed by a region of weak westerlies. Except for the coefficients changing with latitude, multiplying the z derivatives of L [cf. (2.6)], (2.5) is analogous to the usual two dimensional *reduced wave equation* with a variable "index of refraction," $[\sigma^2 r(y,z)/\bar{u}]-m^2$. Hence, there occur reflections when the "index of refraction" becomes sufficiently small, near the poles, where $\sigma^2\ll1$, or in regions of westerlies which are sufficiently strong that $r(y,z)/\bar{u}\ll1$. Standing waves may thus be ducted either by multiple reflections between the north or south pole and a region of strong westerlies, or, alternatively, by multiple reflections between two regions of strong westerlies. We denote these two possibilities the *polar cap* and *middle latitude* wave guides, respectively. The analysis can be restricted to a local wave guide by making simplifying assumptions about the geometry and about the equivalent zonal wind $U(y)$. If a polar cap or a mid-latitude wave guide exists, the atmospheric zonal winds outside the trapping region will have but a small influence on the trapped modes. In particular, easterly zonal winds on the far side of a strong westerly trapping region should have little effect on the structure of trapped modes.

The solutions of (2.10), for the examples to be discussed, are normal mode solutions that satisfy two-point boundary conditions. These normal modes can exist only for certain discrete values of the separation parameter $\gamma=\gamma_j$. The vertical structure of these normal modes is determined by the solution of (2.11) with $\gamma=\gamma_j$. The inhomogeneous bottom boundary conditions used for solving (2.11) in this case are obtained from projection of the lower boundary conditions, given as a function of latitude, onto the latitudinal normal mode.

In general, the number of propagating modes that can occur will depend on the wave guide width and inversely on the minimum wind strength in the wave guide. For numerical discussion of locally guided waves, we shall use atmospheric parameters appropriate to the stratospheric Aleutian high. We infer from the examples to be discussed below, for the observed range of stratospheric winds, that it is unlikely that there can occur significant vertical propagation of modes with longitudinal wavenumber >2. It is not obvious whether the polar cap or mid-latitude wave guide example best represents the actual polar wave guide formed north of the stratospheric jet, because reflections can occur not only when $\sigma=\cos\varphi$ becomes small at high latitudes, but also because $r(y,z)$, as indicated by Figs. 3 and 4, can become negative north of 60N, at least at some altitudes during the winter. Our model can simulate this reflection condition with large $U(y)$.

We now give a more quantitative discussion using the horizontal structure equation (2.10).

a. Planetary wave guide

The assumption of this model is that $U(y)$ remains finite and greater than zero for all y. Solutions of (2.10) which are bounded for $|y|\rightarrow\infty$ are necessarily proportional to $e^{-m|y|}$ for sufficiently large $|y|$, since (2.10) for $|y|\gg1$ becomes

$$\left[\frac{\partial^2}{\partial y^2}-m^2+O(e^{-2|y|})\right]Y=0.$$

The $-m^2$ angular wavenumber factor requires the reflection of planetary waves near the geometrical poles. There is a countable infinity of eigenvalues γ_j and associated orthonormal eigenfunctions $\phi_j(y)$ giving solutions to (2.10) that are bounded for both large negative and large positive y. Since these eigenfunctions form a complete set, an arbitrary function of latitude—in particular, the value of Ψ at a lower boundary, $\Psi=\Psi(y,z_0)$—can be expanded as a sum of these eigenfunctions.

If $U(y)$ is a constant, the eigenfunctions are spheroidal wave functions and the expansion is essentially that described by Dickinson (1968). If $U(y)$ varies sufficiently slowly, the eigenfunctions and eigenvalues of (2.10) should resemble those discussed in the author's earlier work. Lindzen (1967) has described β-plane models for such modes.

To summarize, for a planetary wave guide with zonal winds given by (2.8), we can solve (2.5) by first finding the eigensolutions and eigenvalues of the horizontal structure equation (2.10), then by synthesizing these solutions with solutions $Z(z,\gamma)$ to the vertical structure equation (2.11). The solution to (2.5) will hence be of the form

$$\Psi(y,z)=\sum_j\left[\frac{S(z)e^z}{S(z_0)e^{z_0}}\right]^{\frac{1}{2}}\phi_j(y)\frac{Z(z,\gamma_j)}{Z(z_0,\gamma_j)}\times\int_{-\infty}^{\infty}\phi_j(y')\Psi(y',z_0)dy'. \quad (3.1)$$

b. Polar cap wave guide

Assume now that disturbances are confined to a polar cap region such that $y>1$. We may then approximate $\sigma=\text{sech}\,y$ by $2e^{-y}$, and $\mu=\tanh y$ by 1. Hence, (2.10) becomes

$$\left\{\frac{\partial^2}{\partial y^2}-m^2+4e^{-2y}\left[\frac{\beta}{U(y)}-\gamma^2\right]\right\}Y(y,\gamma)=0. \quad (3.2)$$

We shall assume that $U(y)$ monotonically increases with decreasing y. There will be only a finite number of real eigenvalues γ, for which solutions to (3.2) are bounded as $y\rightarrow\infty$, and for which solutions decay exponentially for $y\rightarrow-\infty$. However, the approximate model (3.2) also has non-trapped solutions for all $\gamma^2<0$ which, as $y\rightarrow-\infty$, and $U(y)\rightarrow\infty$, are proportional to $e^{y/2}\exp[\pm2i|\gamma|e^{-y}]$, and hence are bounded. The geopotential of these modes will decay exponentially in the vertical for small enough $\beta/V(z)$. Since, furthermore, the assumptions necessary to derive our approximate model are only valid for the trapped modes, we shall not further discuss the nontrapped continuous spectrum modes.

It is here convenient to introduce as a new independent variable $\theta=2e^{-y}$ (which for $y\gg1$ is equivalent to colatitude), so that the Mercator coordinate point $y=\infty$ (i.e., the north pole) maps onto $\theta=0$. Eq. (3.2) then becomes

$$\left[\frac{\partial^2}{\partial\theta^2}+\frac{1}{\theta}\frac{\partial}{\partial\theta}-\gamma^2+\frac{\beta}{U(\theta)}-\frac{m^2}{\theta^2}\right]Y=0. \quad (3.3)$$

If we assume now that $U(\theta)$ may be approximated by $U_0'\theta$, where U_0' is a constant shear, then (3.3) has a solution bounded as $\theta\rightarrow\infty$, proportional to a Whittaker confluent hypergeometric function

$$Y\sim\theta^{-\frac{1}{2}}W_{\kappa,m}(2\gamma\theta), \quad (3.4)$$

where $\kappa=\beta/(2\gamma U_0')$. The solution (3.4) also is to be

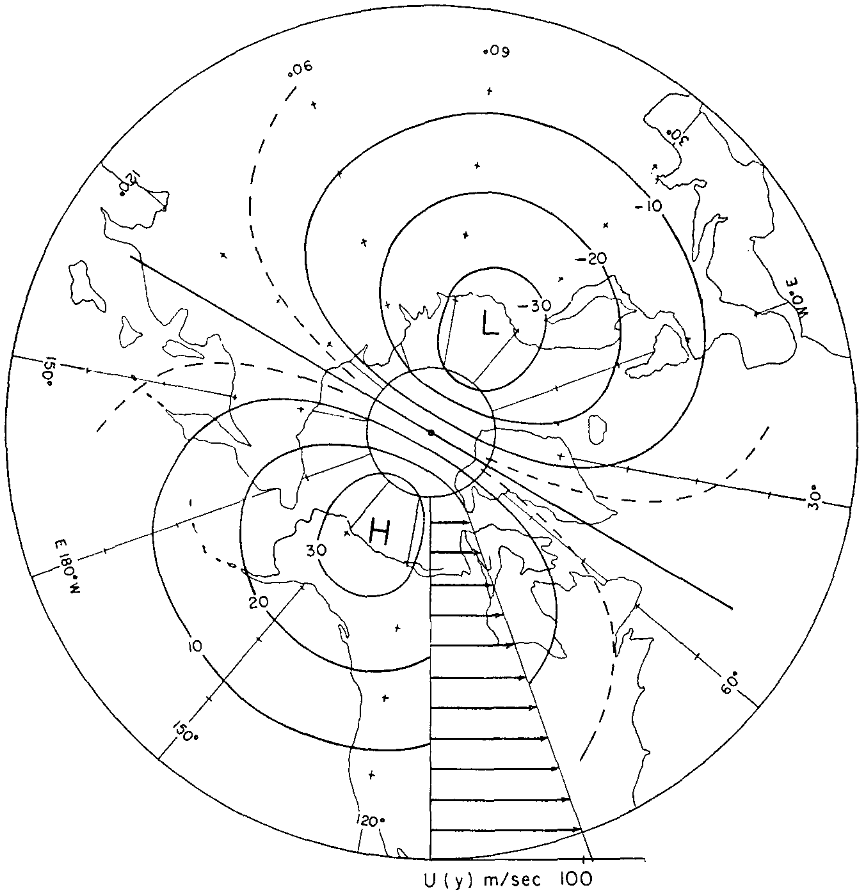

FIG. 7. The lowest wavenumber one mode geopotential for the *polar wave guide* example as described in the text, the amplitude being given in decameters. The arrows show the assumed latitudinal variation of the effective mean zonal wind.

bounded for $\theta\rightarrow 0$, so we must take $\kappa=(j+m)+\frac{1}{2}$, with $j\geq 0$ an integer. The eigenvalue γ then is

$$\gamma=\gamma_j=\frac{\beta}{2U_0'}(j+m+\tfrac{1}{2})^{-1}. \quad (3.5)$$

When γ is given by (3.5), the eigensolutions of (3.4) thus obtained may be written

$$Y=(2\gamma_j\theta)^m e^{-\gamma_j\theta}F(-j,\,2m+1;\,2\gamma_j\theta)$$
$$=\frac{j!(2m)!}{(j+2m)!}(2\gamma_j\theta)^m e^{-\gamma_j\theta}L_j^{(2m)}(2\gamma_j\theta), \quad (3.6)$$

where the associated Laguerre polynomial $L_j^{(\alpha)}(x)$ is defined by

$$L_j^{(\alpha)}(x)=e^x x^{-\alpha}\frac{1}{j!}\frac{d}{dx}(x^{\alpha+j}e^{-x}), \quad (3.7)$$

the special function notation used being that of Abramowitz and Stegun (1964). Solutions to (3.3) decay exponentially for $\theta>\theta_T$, where the turning point θ_T is defined by $[\beta/U(\theta_T)-\gamma^2-(m^2-\frac{1}{4})/(\theta_T^2)]=0$. With the eigenvalues (3.5), θ_T is given approximately by

$$\theta_T\simeq(2U_0'/\beta)\left[1+\left(1-\frac{m^2-\frac{1}{4}}{(j+m+\frac{1}{2})^2}\right)^{\frac{1}{2}}\right](j+m+\tfrac{1}{2})^2.$$

Examination of horizontal wind shears obtained from the CIRA 1965 atmosphere indicates that the winter zonal winds in the stratosphere and mesosphere monotonically increase equatorward between the pole and approximately 40N. If we take U_0' to correspond to the mean shear in this region and assume $\theta_T\sim 1$, solutions to (3.3) can be expected to approximate solutions to (2.10) and roughly model actual high latitude planetary waves. The winds of the winter upper stratosphere typically increase from zero at the pole to between 40 and 80 m sec^{-1} in middle latitudes, giving a magnitude of U_0' between 0.05 and 0.1. Somewhat smaller values will be appropriate in late fall after the changeover, or at lower and higher levels. We take β to be unity for our discussion. For $\beta/U_0'\simeq 10$, the $m=1$, $j=0$ mode has $\theta_T<1$ and hence is "trapped," while for $\beta/U_0'\simeq 20$, the $m=1, j=0$ and $j=1$, and the $m=2, j=0$

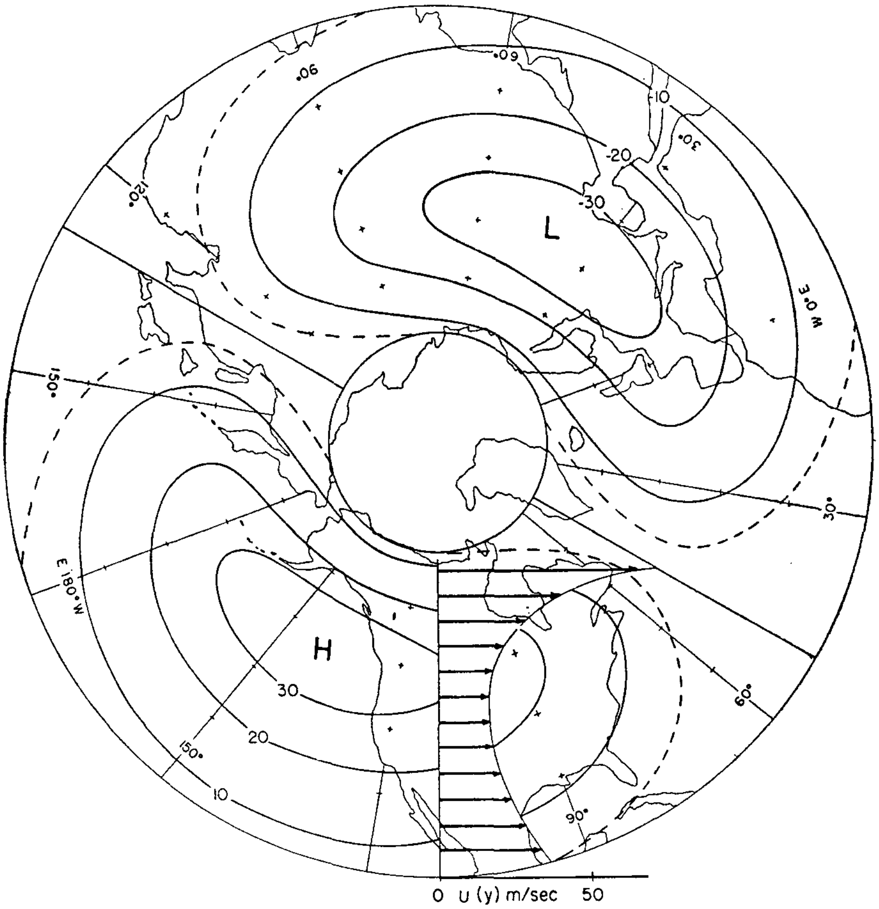

FIG. 8. Same as Fig. 7 except for the *mid-latitude wave guide* example described in the text.

modes have $\theta_T \lesssim 1$ and so are "trapped." Wavenumber one and two modes "fit" our model wave guide for relatively weak westerly winds, while only the lowest $m=1$ wave will fit for high values of westerly winds. In Fig. 7, we depict in dimensional units the geopotential height corresponding to the $m=1, j=0$ mode. For this figure we take $\beta/U_0'=10$, so $\gamma=10/3$. The phase and amplitude, here arbitrary, are normalized to give rough agreement with 50-mb planetary waves observed for January 1958 (Teweles, 1963).

c. Mid-latitude wave guide

Our third example assumes a region of weak $U(y)$ centered at $y=y_0$, outside of which the strength of the $U(y)$ wind rapidly increases. We look for trapped modes which decay exponentially for large enough $|y-y_0|$. Eq. (2.10) is assumed to be approximated by

$$\left[\frac{\partial^2}{\partial y^2}+\frac{\sigma_0^2\beta}{U(y)}-(m^2+\sigma_0^2\mu_0^2\gamma^2)\right]Y=0, \qquad (3.8)$$

where σ_0 and μ_0 are constants. (This model could alternatively be obtained by the usual middle latitude β-plane approximations.)

In order to discuss an explicit solution, we assume the wind to be modeled by the profile

$$U(y)=U_0\cosh^2\left(\frac{y-y_0}{w}\right), \qquad (3.9)$$

where U_0 is the minimum wind strength, the parameter w measuring the width of the wave guide. Assuming (3.9), (3.8) may be written

$$\left[\frac{\partial^2}{\partial \tilde{y}^2}+N(N+1)\,\mathrm{sech}^2\tilde{y}-M^2\right]Y(\tilde{y})=0, \qquad (3.10)$$

where

$$\left.\begin{aligned}\tilde{y}&=(y-y_0)/w\\ N&=-\tfrac{1}{2}+\tfrac{1}{2}\left(\frac{4\sigma_0^2w^2\beta}{U_0}+1\right)^{\frac{1}{2}}\\ M&=w(m^2+\sigma_0^2\mu_0^2\gamma^2)^{\frac{1}{2}}\end{aligned}\right\}. \qquad (3.11)$$

Eq. (3.10) is solved by the hypergeometric function

$$Y(\tilde{y})=(\mathrm{sech}^2\tilde{y})^{M/2}\times F\left(M-N,\, M+N+1,\, 1+M;\, \frac{e^{-\tilde{y}}}{e^{\tilde{y}}+e^{-\tilde{y}}}\right), \qquad (3.12)$$

which decays exponentially for $(y-y_0)>w$, but grows at a faster than exponential rate for $(y-y_0)<-w$, unless $N-M=0$ or a positive integer j. When $N-M=j$, the series defined by (3.12) terminates and Y decays exponentially for $(y-y_0)<-w$. This occurs provided $\gamma=\gamma_j$ satisfies

$$\gamma_j=(\sigma_0\mu_0)^{-1}\left[\frac{(N-j)^2}{w^2}-m^2\right]^{\frac{1}{2}}. \qquad (3.13)$$

We take, as possible typical values, $\beta\simeq1$, $U_0\simeq1/48$ (corresponding to a minimum wind of $\sim$20 m $\sec^{-1}$), $\sigma_0{}^2\simeq\mu_0{}^2\simeq\frac{1}{2}$; we also take the characteristic width $w=\frac{1}{2}$, corresponding to westerly winds doubling in strength at 25° latitude on each side of y_0 and rapidly becoming large at a greater distance from y_0. We then have $N=2$, $\gamma_j=4[(2-j)^2-m^2/4]^{\frac{1}{2}}$. Depending on the value of m, one or more of the γ_j will be imaginary.

Recalling that discrete modes exist when $(N-M)\geq0$, we see that the maximum possible number of discrete modes is $N+1$, i.e., 3 for the numerical values assumed above. All discrete modes decay exponentially when $|y-y_0|>w$. In Fig. 8 we have plotted, for the numerical values assumed above, the horizontal structure of geopotential height fields obtained from the stream function defined by (3.12) for the $j=0$, $m=1$ mode. Again, the arbitrary amplitudes and phases have been chosen to be suggestive of those observed in the stratosphere. We find for the above example that the lowest two modes (i.e., $j=0$, 1) of wavenumber 1 and the lowest mode of wavenumber 2 have $\gamma^2>10$, and hence are vertically propagating modes.

4. Continuation of Rossby wave solutions across singular lines

In Section 3 we studied planetary wave models constructed under the assumption that $\bar{u}(y,z)>0$ for all latitudes. We now consider what happens when $\bar{u}$ vanishes somewhere. Let us recall that the advection of potential vorticity by the mean zonal wind $\bar{u}$ is proportional to the product of $\bar{u}$ and $L\Psi$, where the perturbation potential vorticity $L\Psi$, defined by (2.6), the most highly differentiated term in (2.5). A partial differential equation is called *singular* where the coefficient of the most highly differentiated term vanishes. Thus, lines on which $\bar{u}(y,z)=0$ are called *singular lines.* One can find exponentially decaying or growing solutions to the equation for Ψ in the region $\bar{u}<0$ and wavelike solutions near the singular line in the region $\bar{u}>0$. However, we find that only the propagating wave solutions for the region $\bar{u}>0$ match across the singularity to an exponentially decaying solution in the region $\bar{u}<0$.

Singular lines are a kind of boundary, since solutions to the reduced wave equation for Ψ are not uniquely obtained without specifying some condition across the singular line. If there are sources of energy in the $\bar{u}>0$ region only, then $|\Psi|$ should not grow exponentially going into the $\bar{u}<0$ region away from the singular line. We then find it necessary to use a wave solution for Ψ in the $\bar{u}>0$ region that is absorbed rather than reflected at the singular line.

More generally, the requisite matching condition across the $\bar{u}=0$ line can be uniquely determined for sources in the $\bar{u}>0$ or $\bar{u}<0$ region by solution of an initial value problem. Again the solutions in the $\bar{u}>0$ region are propagating waves, absorbed or emitted along the $\bar{u}=0$ line. Propagating wave solutions *absorbed* at the $\bar{u}=0$ line imply a *divergence* of momentum through a thin layer where $\bar{u}=0$, while wave solutions which are *emitted* imply a momentum *convergence.*

The conclusions stated above are now established for vertically aligned singular lines given by wind profiles which satisfy (2.8). The separated equations (2.10) and (2.11) are again obtained from (2.9). Now $U(y)=0$ for some value of y, $y=y_s$. We examine the propagation of Rossby waves in the region of weak westerlies adjacent to the vertical line on which $\bar{u}$ vanishes. (Horizontally aligned zero wind lines where $V(z)=0$ could similarly be studied.) In the neighborhood of $y=y_s$, (2.10) may be approximated by

$$\left[\frac{\partial^2}{\partial y^2}+\frac{\alpha}{y-y_s}\right]Y(y)=0, \qquad (4.1)$$

where α, assumed greater than zero, is given by $\alpha=\beta\sigma^2(y_s)/U_y(y_s)$, $\alpha>0$ implying westerlies to the north of easterlies. An equation of the form (4.1) can be obtained for the actual orientation of a singular line provided α is defined properly and y is taken to be a coordinate normal to the zero wind line.

Eq. (4.1) is well known in differential equation theory, where it is used, as it is here, to obtain approximate solutions to certain differential equations with singular coefficients, when the singularity is a simple pole. [For example, Miles (1964) has used a solution of (4.1) to obtain an asymptotic expression for the "Charney baroclinic instability problem."]

Sources are assumed to occur in the region of positive y north of the singular line, and (4.1) is to be solved. For such sources $Y(y)$ should decay exponentially with decreasing y for $y\ll y_s$. The solution for $y<y_s$ is then the Bessel function

$$Y(y)=(y_s-y)^{\frac{1}{2}}2\pi^{-\frac{1}{2}}K_1[2\alpha^{\frac{1}{2}}(y_s-y)^{\frac{1}{2}}]. \qquad (4.2)$$

If we now take y to be a complex variable, then (4.2) is an analytic function for finite y except for a branch point at $y=y_s$. We make this function single-valued by assuming a branch line in the complex y plane on the ray, arg $(y-y_s)=\pi/2$. Then for real $y>y_s$ we can write (4.2) as

$$Y(y)=-\pi^{\frac{1}{2}}(y-y_s)^{\frac{1}{2}}H_1{}^{(2)}[2\alpha^{\frac{1}{2}}(y-y_s)^{\frac{1}{2}}]. \qquad (4.3)$$

On the other hand, if we had chosen the branch line on the ray, arg $(y-y_s)=-\pi/2$, we would use the

Hankel function $H_1^{(1)}$ rather than $H_1^{(2)}$ in (4.3). This Hankel function is the appropriate choice for the $y>y_s$ solution to (4.1) when sources are to the south of the point $y=y_s$, i.e., $y<y_s$. The branch lines were located by an initial value analysis; that is, we assumed a time independent source to be switched on in the $y\gg y_s$ region at $t=0$ and determined from contour integration the inversion of the Laplace transform of the solution. The branch line to the left of the imaginary axis in the Laplace transform variable plane maps onto the positive imaginary axis of $y-y_s$, so that as $t\rightarrow\infty$ the assumed location of the branch line in the complex y plane is verified.

The asymptotic approximations to (4.2) and (4.3) show the evanescence of (4.2) for large negative $y-y_s$ and the wave propagation of (4.3) for large positive $y-y_s$. For $(y_s-y)\gg\alpha^{-1}$, (4.2) is approximated by

$$Y(y)=\alpha^{-\frac{1}{4}}(y_s-y)^{\frac{1}{4}}\{1+O[\alpha^{-\frac{1}{2}}(y_s-y)^{-\frac{1}{2}}]\}\times\exp[-2\alpha^{\frac{1}{2}}(y_s-y)^{\frac{1}{2}}],\quad(4.4)$$

while for $(y-y_s)\gg\alpha^{-1}$, (4.3) is approximated by

$$Y(y)=\alpha^{-\frac{1}{4}}(y-y_s)^{\frac{1}{4}}\{1+O[\alpha^{-\frac{1}{2}}(y-y_s)^{-\frac{1}{2}}]\}\times\exp[-2i\alpha^{\frac{1}{2}}(y-y_s)^{\frac{1}{2}}-i\pi/4].\quad(4.5)$$

This expression indicates a positive, northeast-southwest tilt of constant phase lines on a constant pressure surface, as seen by reference to (2.4). In a local coordinate system moving with the zonal flow, the phases propagate outward from $y=y_s$.

The following small $y-y_s$ approximations show the nature of the $y=y_s$ singularity:

$$Y(y)=(\alpha\pi)^{-\frac{1}{2}}[1+\alpha(y_s-y)\log(y_s-y)+O(y_s-y)],\quad(4.6)$$

for $0<(y_s-y)\ll\alpha^{-1}$, while

$$Y(y)=(\alpha\pi)^{-\frac{1}{2}}\{1-\alpha(y-y_s)[\log(y-y_s)+i\pi]+O(y-y_s)\},\quad(4.7)$$

for $0<(y-y_s)\ll\alpha^{-1}$.

The relations (4.3)-(4.7) may be derived from (4.2) from the integral representation,

$$K_1(x)=x\int_1^\infty e^{-xt}(t^2-1)^{\frac{1}{2}}dt.\quad(4.8)$$

We easily see from (4.6) and (4.7) that $\mathrm{Im}(YY_y^*)$ jumps from 0 to 1 as y crosses y_s going from smaller to larger y. Hence, we may write

$$\frac{d}{dy}\mathrm{Im}(YY_y^*)=\delta(y-y_s),\quad(4.9)$$

which together with (2.12) shows the divergence of $\overline{u'v'}$ at $y=y_s$.

We shall now discuss briefly some aspects of the continuation of our solutions to large values of $|y-y_s|$. In the following discussion we shall denote the local latitudinal wavenumber squared by $Q(y)$, i.e.,

$$Q(y)=\sigma^2\left[\frac{\beta}{U(y)}-\gamma^2\mu^2\right]-m^2.\quad(4.10)$$

The poles and zeros of $Q(y)$ are called *turning points*, since they separate regions where $Q(y)>0$, and hence where $Y(y)$ oscillates as a wave, from regions where $Q(y)<0$, and hence where $Y(y)$ grows or decays exponentially. Let us now consider the solution to (2.10) when Q has no zeros and a single pole at $y=y_s$, so that $Q(y)<0$ for $y<y_s$, $Q(y)>0$ for $y>y_s$. For $y\ll y_s$, solutions to (2.10) may be approximately evaluated by the phase integral (i.e., WKB) method. For $y\ll y_s$,

$$Y(y)\simeq|Q|^{-\frac{1}{4}}\exp\left[-\int_y^{y_s}|Q|^{\frac{1}{2}}dy\right].\quad(4.11)$$

As $y\rightarrow y_s$, (4.11) reduces to (4.4) and hence matches to the solution (4.2). Then taking $y>y_s$, (4.3) gives the continuation of the solution across the singular line. Again if the energy source is in the $y>y_s$ region, the wave phases will tilt to the northeast. We also see that for $y\gg y_s$, (4.3) matches to the phase integral solution

$$Y(y)=Q^{-\frac{1}{4}}\exp\left[-i\int_{y_s}^{y}Q^{\frac{1}{2}}dy-i\pi/4\right].\quad(4.12)$$

In summary, the approximate solution we obtain to (2.10) for eddy energy flux toward (and phase propagation and momentum flux away from) the singular line is (4.12) for $y\gg y_s$, (4.2) and (4.3) for $y\sim y_s$, and (4.11) for $y\ll y_s$ (In practice, $|y-y_s|\gg0$ refers to values of y for which $|y-y_s|>\alpha^{-1}$.)

When there is more than one singular line, solutions valid near each singular line must be matched in order to obtain uniformly valid approximate solutions to (2.10). For instance, assume a belt of easterlies with westerlies on each side. Then $Q(y)$ has two poles, denoted by $y=y_r$ and $y=y_s$, $y_s>y_r$. Also assume $Q(y)>0$ for $y>y_s$ or $y<y_r$, while $Q(y)<0$ between y_r and y_s. Then we find an absorbed wave approximate solution of the form

$$Y\simeq Q^{-\frac{1}{4}}\exp\left[-i\int_{y_s}^{y}Q^{\frac{1}{2}}dy-i\pi/4\right],\quad(4.13)$$

for $(y-y_s)\gg0$, matches to a solution of the form

$$Y\simeq TQ^{-\frac{1}{4}}\exp\left[-i\int_{y_r}^{y}Q^{\frac{1}{2}}dy+i\pi/4\right],\quad(4.14)$$

for $(y_r-y)\gg0$, where T is the transmission coefficient,

$$T=\exp\left[-\int_{y_r}^{y_s}|Q|^{\frac{1}{2}}dy\right].\quad(4.15)$$

We have assumed

$$\int_{y_T}^{y_s} |Q|^{\frac{1}{2}} dy \gg 1$$

in order to justify applying the asymptotic approximation (4.11) for the solutions near each singular point. For a belt of easterlies wide enough for (4.15) to apply, the attenuation to this approximation depends only on the wave evanescence, not on the presence of a singular line. This evanescence reduces the energy flux or momentum transport of a wave incident on one side of the easterly region approximately by T^2 as the wave emerges from the other side, while no fluxes occur in the easterly region. In the next section we examine in detail the solution when $Q(y)$ has a zero as well as a pole.

5. Singular line wave guides

Planetary waves can propagate in the region of weak westerlies bounded by the mid-latitude stratospheric westerly jet and by the equatorial zero wind line that separates the summer stratospheric easterlies from the winter westerlies. Fig. 9, taken from Reed (1966), indicates the variation with height of the zero wind line in the upper stratosphere and lower mesosphere. The zero wind line of the lower stratosphere in winter fluctuates widely in latitude from one year to the next as a consequence of the biennial periodicity of zonal winds.

In this section we obtain approximate solutions for the latitudinal structure of perturbations occurring between a model zero wind line and a barrier formed by a westerly jet. As will be shown, there are no normal mode solutions for such a wave guide that satisfy two point-boundary conditions. The separable model (2.8) is again assumed and solutions are given as the product of solutions of (2.10) and (2.11). Furthermore, let us again assume for simplicity that $U(y)$ has only one sign change, occurring at $y=y_s$, but now that $Q(y)$, as defined by (4.10), has a single zero located at $y=y_T$. Then (2.10) has exponential-like solutions in the region for $y>y_T$ or $y<y_s$, and wavelike solutions for $y_s<y<y_T$. We denote the independent solutions that satisfy the conditions that they either be bounded for $y\to+\infty$ or for $y\to-\infty$ as Y_+ and Y_-, respectively. The asymptotic behavior of these two solutions for $|y|\to\infty$, as obtained from (2.10), may be given as

$$Y_\pm \simeq |Q|^{-\frac{1}{4}} \exp\left[\mp\int^y |Q|^{\frac{1}{2}} dy\right], \qquad (5.1)$$

where $Q(y)$, defined by (4.10), is now negative for $|y|\to\infty$. It is known from the theory of ordinary differential equations that any solution to (2.10) can be written as a linear combination of a *fundamental set* of two independent solutions, which here are taken to be $Y_\pm$ with asymptotic behavior (5.1).

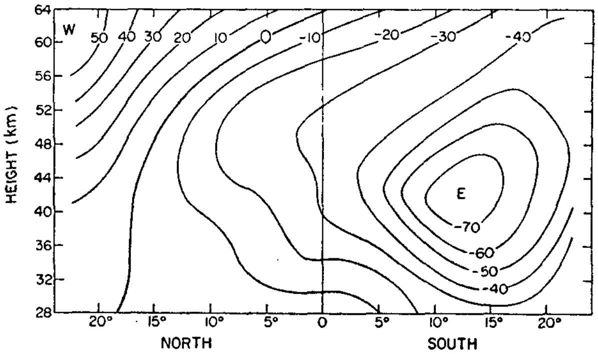

FIG. 9. January–July mean zonal winds for the equatorial stratosphere and lower mesosphere, according to Reed (1966).

The following simple argument establishes the absence of normal mode solutions in a singular line wave guide. First, by definition, a normal mode solution to (2.10) is a solution which for some value of γ satisfies both the boundary conditions for $y\to-\infty$ and for $y\to+\infty$, or, in other words, satisfies the condition that Y_+ be proportional to Y_-. It was shown in the previous section that as we continue the Y_- solution across the singular line $y=y_s$, Y_- becomes proportional to a traveling wave solution [cf. (4.3)] and the quantity $\mathrm{Im}(Y_-Y_{-,y}^*)$ jumps from 0 to 1. We may easily determine from (5.1) that as $y\to\infty$, $\mathrm{Im}(Y_+, Y_{+,y}^*)=0$. But from (2.13), the quantity $\mathrm{Im}(Y,Y_y^*)$ is independent of y for any solution to (2.10), except for possible jumps at singular lines. Thus, for all $y>y_s$,

$$\mathrm{Im}(Y_+, Y_{+,y}^*)=0,$$
$$\mathrm{Im}(Y_-, Y_{-,y}^*)=1.$$

This proves that Y_- can never be proportional to Y_+. Hence, *there are no normal mode solutions*. As a corollary, there is no normal mode expansion of solutions to the singular line wave guide equation.

In order to obtain an approximate description of the latitudinal structure of planetary waves in the singular line wave guide, $Q(y)$ is expanded in a Laurent series about $y=y_s$, i.e., (4.10) is written

$$Q(y)=\frac{\alpha}{y-y_s}-b+c(y-y_s)+\cdots, \qquad (5.2)$$

where α was defined following (4.1), and

$$b(\gamma)=\beta\sigma^2(y_s)\left[\frac{U_{yy}(y_s)}{2U_y^2(y_s)}+\frac{2\mu(y_s)}{U_y(y_s)}\right]+m^2+\gamma^2\sigma^2(y_s)\mu^2(y_s). \qquad (5.3)$$

We assume α and b to be greater than zero.

We assume that α/b is a small parameter so that the turning point $y=y_T$ is determined approximately by

$$y_T\simeq\frac{\alpha}{b(\gamma)}+y_s+\mathrm{O}(\alpha/b)^2.$$

It then follows for $y_s \lesssim y \lesssim y_T$ that (2.10) is approximated by the tractable equation

$$\left[\frac{\partial^2}{\partial y^2}+\frac{\alpha}{y-y_s}-b\right]Y(y,\gamma)=0. \tag{5.4}$$

Let us now determine solutions to (5.4) such that the asymptotic forms of these solutions, obtained for large $|y|$, match to the large $|y|$, asymptotic solutions (5.1) throughout the region where the expansion (5.2) is valid. The two independent solutions to (5.4) which satisfy the above requirement are

$$\left.\begin{aligned} Y_+ &= c_+W_{\kappa,\frac{1}{2}}[2b^{\frac{1}{2}}(y-y_s)] \\ Y_- &= c_-W_{-\kappa,\frac{1}{2}}[-2b^{\frac{1}{2}}(y-y_s)] \end{aligned}\right\}. \tag{5.5}$$

We have used the definitions

$$\left.\begin{aligned} c_+=2(\pi/\kappa)^{\frac{1}{2}}/\Gamma(\kappa), \quad c_-=(\kappa\pi)^{-\frac{1}{2}}\Gamma(\kappa+1) \\ \kappa=\alpha/(2b^{\frac{1}{2}}) \end{aligned}\right\}, \tag{5.6}$$

and $W_{\kappa,\frac{1}{2}}(x)$ is the Whittaker function (Abramowitz and Stegun, 1964).

For $|x|\gg 1$, we may use the asymptotic approximations

$$\left.\begin{aligned} W_{\kappa,\frac{1}{2}}(x) &\simeq x^{\kappa}e^{-x/2} \\ W_{-\kappa,\frac{1}{2}}(-x) &\simeq (-x)^{-\kappa}e^{x/2} \end{aligned}\right\}. \tag{5.7}$$

The relations (5.7) establish that (5.5) matches to (5.1) when $|y-y_s|$ is large, but still sufficiently small that $Q(y)\simeq[\alpha/(y-y_s)-b]$. As in the previous section, we use a branch line, $\arg(y-y_s)=\pi/2$, to make the functions single-valued.

Now let us obtain approximate expressions for (5.5) valid in the wave guide region. These may be obtained by a variety of standard techniques. For y between y_s and y_T, we use the approximate expressions

$$\left.\begin{aligned} Y_+ &= 2Q^{-\frac{1}{4}}\cos\left\{\frac{\alpha}{b^{\frac{1}{2}}}\phi\left[\frac{b}{\alpha}(y-y_s)\right]+\left(\frac{1}{4}-\frac{\alpha}{2b^{\frac{1}{2}}}\right)\pi\right\} \\ Y_- &= Q^{-\frac{1}{4}}\exp\left\{-i\left(\frac{\alpha}{b^{\frac{1}{2}}}\phi\left[\frac{b}{\alpha}(y-y_s)\right]\right)-i\pi/4\right\} \end{aligned}\right\}, \tag{5.8}$$

where the phase function $\phi(x)$ is defined by

$$\phi(x)=\int_0^x(1/x-1)^{\frac{1}{2}}dx=\sin^{-1}x^{\frac{1}{2}}+x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}. \tag{5.9}$$

For values of x near zero or one, we may use for $\phi(x)$ the approximations

$$\left.\begin{aligned} \lim_{x\to 0}\phi(x) &= 2x^{\frac{1}{2}} \\ \lim_{x\to 1}\phi(x) &= \pi/2-\tfrac{2}{3}(1-x)^{\frac{3}{2}} \end{aligned}\right\}. \tag{5.10}$$

Strictly speaking, the phase integral approximations (5.8) are asymptotically exact as $\alpha\to\infty$. The function (5.5) may also be approximated in the neighborhood of $y=y_s$ by the Bessel functions used in the previous section, and near $y=y_T$ by Airy functions. We see by comparing (5.7) and (5.8) with (4.4) and (4.5) that the solution Y_- matches to (4.2) of the previous section.

To proceed further with the description of vertical propagation, it is necessary to solve (2.11) and combine with solutions to (2.10). Let us briefly indicate the relevant computation. We write (2.11) as

$$\left\{\frac{\partial^2}{\partial z^2}+[\gamma^2-g(z)]\right\}Z(z,\gamma)=0, \tag{5.11}$$

where we use the definition

$$g(z)=\tfrac{1}{4}(1+\epsilon)-\beta S/V(z). \tag{5.12}$$

Let us assume $\gamma^2>g(z)$, so that (5.11) has wavelike solutions. Then by using a phase integral approximation, we obtain the wave solutions

$$Z(z,\gamma)=[\gamma^2-g(z)]^{-\frac{1}{4}} \times\exp\left[\pm i\int^z[\gamma^2-g(z')]^{\frac{1}{2}}dz'\right]. \tag{5.13}$$

The product of (5.13) and (5.5) [cf. (2.9)] gives approximate solutions to (2.5) when the separability condition (2.8) is assumed. Such a solution is wavelike in y between the zero wind line, $y=y_s$, and the turning points, $y=y_T$, and decays exponentially in y for $y<y_s$ or $y>y_T$. Thus, we see that the region $y_s<y<y_T$ forms a wave guide within which there can be significant vertical propagation of Rossby waves. Reference to (5.3) gives the dependence of y_T on the parameter γ.

6. Vertical propagation in a singular line wave guide

In the previous section we obtained plane wave solutions of the horizontal structure equation for a singular line wave guide. The Y_- solution, the solution which decays going into the region $y<y_s$, is absorbed rather than reflected at the zero wind line [cf. (5.8)]. In our previous analysis, we assumed a fixed value for the separation constant γ. The latitudinal disturbance

wavelength at a given latitude is a function of this parameter γ. Since there are no discrete normal modes, a disturbance excited by an arbitrary source at the bottom boundary will be described by a "Fourier integral" over elementary disturbances with single latitudinal wavelengths.

We find that all *wave packets* leaving the bottom boundary are *absorbed* by the *singular line* within a finite distance from the source point. Wave packets with an initial direction of propagation inclined toward the singular line are refracted into a path more orthogonal to the singular line, while wave packets with an initial direction of propagation inclined away from the singular line are refracted into a more vertical path and eventually reflected at the other side of the wave guide into a path toward the singular line.

In order to establish quantitatively the above statements, we formally sketch the solution for vertically propagating planetary waves excited by an arbitrary source at the bottom boundary of a singular line wave guide. An integral transform technique is used to obtain the exact solution as an integral over the separation-of-variables parameter. Various approximations are made to reduce the integrand to a product of exponential waves, so that the resulting integral may, in principle, be evaluated by the method of stationary phase. It follows from such an asymptotic integration that, for each value of the separation constant, the integrand gives a nonzero contribution to the integral only on certain curves in (y,z) space, the paths of wave packets being called *rays*. This integration is equivalent to the summing of infinitesimal contributions from individual wave packets propagating in all directions.

A quantitative description of the height attenuation of planetary waves is not easily obtained, since such attentuation will, in general, depend on the details of wave sources. However, our approximate model gives an expression for the maximum vertical distance a ray can propagate before it is absorbed at a singular line. This computation suggests that rays can propagate vertically at best a few tens of kilometers in the equatorial singular line wave guide. Thus, planetary waves excited in equatorial regions in the lower stratosphere will probably be absorbed by the zero wind line even before they reach the stratopause.

It is sufficient for our purposes to examine the solutions obtained for a vertically unbounded atmosphere and for point source excitation of (2.5). General solutions can be obtained by integrating over the influence function so obtained. Thus, we consider the equation

$$L\Psi+\frac{\sigma^2 r(y,z)}{\bar{u}}\Psi=-\delta(y-y')\delta(z), \qquad (6.1)$$

where L is defined by (2.6), r by (2.7), and $\bar{u}$ is of the form (2.8).

We shall further simplify the analysis by assuming the stability S to be constant, $g(z)$, defined by (5.12), to be zero, and the atmosphere to be unbounded in z. We then seek solutions to (6.1) which are represented by the Fourier Integral

$$\Psi(y,z)=\frac{(Se^z)^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty}d\gamma\,\exp[i(S^{\frac{1}{2}}\gamma)z]Y(y,\gamma). \qquad (6.2)$$

Using the fact that

$$\delta(z)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{ikz}dk,$$

we then reduce (6.1) to the horizontal structure equation,

$$\left\{\frac{\partial^2}{\partial y^2}-m^2+\sigma^2\left[\frac{\beta}{U(y)}-\mu^2\gamma^2\right]\right\}Y(y,\gamma)=-\delta(y-y'). \qquad (6.3)$$

For $y\neq y'$, (6.3) has a solution Y_+ which satisfies the condition that solutions be bounded as $y\rightarrow\infty$, and a solution Y_- which satisfies the condition that solutions be bounded as $y\rightarrow-\infty$ (cf. Section 5). Using these two independent solutions we may solve (6.3) by the standard Green's function procedure; that is, the solution to (6.3) is given by

$$Y(y,\gamma)=\frac{Y_+(y_>)Y_-(y_<)}{W[Y_+(y'),Y_-(y')]}, \qquad (6.4)$$

where $y_>$ denotes the greater of y or y', $y_<$ the lesser of y or y', and $W(Y_+, Y_-)$ is the Wronskian of Y_+ and Y_-.

Substitution of (6.4) into (6.2) and integration over y then gives us a formal mathematical solution to (6.1). The restrictions that our geometry be unbounded in z and that $g(z)$ be zero can in principle be relaxed. That is, we can construct an eigenfunction expansion from solutions of (2.11) that satisfy given boundary conditions (cf. Titchmarsh, 1962), which may be written as

$$\delta(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z(z,\gamma)d\gamma,$$

again giving the separated equation (6.3).

It is sufficient for our purposes to use the approximate solutions given by (5.5) to represent $Y_\pm$ in (6.4). The Wronskian of these solutions as written, and hence the denominator of (6.4), is $2\exp[-i(\alpha/2b^{\frac{1}{2}})\pi]$. Assuming both source point and point of observation are in the wave guide region $y_s<y<y_T$, we furthermore use (5.8) to approximate (5.5). Substituting (5.8) into (6.4), the

integral (6.2) becomes

$$\Psi(y,z)=\frac{e^{z/2}S^{\frac{1}{2}}}{\pi}\int_{-\infty}^{\infty}d\gamma\frac{\exp[i(\gamma S^{\frac{1}{2}})z]}{[Q(y)Q(y')]^{\frac{1}{2}}}\{\exp[i\Phi(y_<)+i\Phi(y_>)]+\exp[i\Phi(y_<)-i\Phi(y_>)]\},\quad(6.5)$$

where we define the latitudinal phase Φ as

$$\Phi(y)=-\left\{\frac{\alpha}{b^{\frac{1}{2}}}\phi\left[\frac{b}{\alpha}(y-y_s)\right]+\left(\frac{1}{4}-\frac{\alpha}{2b^{\frac{1}{2}}}\right)\pi\right\}.$$

The function $\phi(x)$ is defined by (5.9) and $b(\gamma)$ by (5.3).

The integral (6.5) is evaluated approximately by the method of stationary phase. The phase of (6.5) is stationary when

$$\frac{d}{d\gamma}[\gamma S^{\frac{1}{2}}z\pm\Phi(y_>;\gamma)+\Phi(y_<;\gamma)]=0.\quad(6.6)$$

The "$\pm$" refers to the first and second terms, respectively, of (6.5). We argue that asymptotically for large z, the integrand of the integral (6.5) contributes to the integral only when the phase is stationary. Geometrically, (6.6) is the criterion for an envelope of a family of surfaces in (y,z) space, $\gamma S^{\frac{1}{2}}z\pm\Phi(y_>)+\Phi(y_<)=0$, depending on the parameter γ. Physically, at any point in space, the phases of the partial waves associated with various values of γ can be expected to cancel by destructive interference unless partial waves over a continuous range of γ have the same phase. This occurs only on the envelope of the family of phase surfaces.

Inverting (6.6) to obtain $\gamma(y,z)$ and substituting this expression for γ in $\Phi(y)$, we can, in principle, obtain an expression for wave phases which is independent of γ. The stationary phase evaluation of the integral then is straightforward. However, it is not possible in the present problem to obtain from (6.6) a simple closed form for $\gamma(y,z)$. Rather, let us consider the rays defined by (6.6) as functions of the parameter γ. Disturbances may be considered to propagate along such rays [cf., for instance, Whitham (1961)]. After substituting the definitions of $\phi(\gamma)$ and $b(\gamma)$, we find the derivative of Φ to be

$$\frac{d\Phi(y;\gamma)}{d\gamma}=\frac{\alpha b_\gamma}{b^{\frac{3}{2}}}\left[\tilde{\phi}\left(\frac{b(y-y_s)}{\alpha}\right)-\pi/4\right],\quad(6.7)$$

where we use the definition

$$\tilde{\phi}(x)=\sin^{-1}x^{\frac{1}{2}}-x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}.\quad(6.8)$$

Let $Z(\gamma)$ denote the factor $\pi S^{-\frac{1}{2}}\alpha b^{-\frac{3}{2}}b_\gamma$.

Then from (6.6) and (6.7), we find the ray paths are given by

$$z=\begin{cases}\pi^{-1}Z(\gamma)\{\tilde{\phi}[b(y_>-y_s)/\alpha]-\tilde{\phi}[b(y_<-y_s)/\alpha]\}\\ \pi^{-1}Z(\gamma)\{\pi/2-\tilde{\phi}[b(y_>-y_s)/\alpha]-\tilde{\phi}[b(y_<-y_s)/\alpha]\}\end{cases}.\quad(6.9)$$

We can distinguish between direct rays and once reflected rays, the once reflected rays having undergone a deflection at the turning point $y-y_s=\alpha/b$. The maximum vertical travel distance for direct rays is $\frac{1}{2}Z(\gamma)$, which occurs for a source at the turning point, while the maximum vertical travel distance for a once reflected ray is $Z(\gamma)$, which occurs for a source at $y-y_s=0$. Note that $\tilde{\phi}(y-y_s)=\pi/4$ when $(y-y_s)\simeq0.84\alpha/b$. We have sketched in Fig. 10 the ray trajectories for two special cases, i.e., the source is at $y'-y_s=0$ or $y'-y_s=1.0$ in Fig. 10a, and the source is at $y'-y_s=0.84\alpha/b$ in Fig. 10b. For a source lying in the region (b/α) $(y-y_s)<0.84$, the direct ray will travel between 0 and $Z(\gamma)/4$, while the once reflected ray will travel between $Z(\gamma)/4$ and $Z(\gamma)/2$. For a source with (b/α) $(y'-y_s)$ between 0.84 and 1.0, the direct and once reflected rays will travel to heights in the range $\frac{1}{4}Z$–$\frac{1}{2}Z$ and $\frac{3}{4}Z$–Z, respectively.

We now shall choose values of the parameters in $Z(\gamma)$ which are appropriate to the observed zero wind line in the winter tropical upper stratosphere. Using these values, we shall obtain an estimate of the maximum $Z(\gamma)$ that may occur. First let us simplify the definition of $b(\gamma)$ by assuming the wind curvature at y_s is small, and use the fact that y_s is in the tropics. That is, we take

$$(U_{yy}/U_y^2)\ll1,$$
$$\mu(y_s)\ll1,$$
$$\sigma(y_s)\simeq1,$$

so that $b(\gamma)$ [cf. (5.3)] may be written

$$b(\gamma)\simeq m^2+\gamma^2\mu^2(y_s).\quad(6.10)$$

The parameter α may now be written [cf. (4.1)] as

$$\alpha=\beta/U_y(y_s).$$

We may then take $Z(\gamma)$ to be

$$Z(\gamma)=\frac{2\pi\gamma\mu^2(y_s)\beta/U_y(y_s)}{S^{\frac{1}{2}}[m^2+\gamma^2\mu^2(y_s)]^{\frac{3}{2}}}.\quad(6.11)$$

This expression has a maximum, $Z_\gamma(\gamma)=0$, when

$$\gamma^2=m^2/2\mu^2(y_s),$$

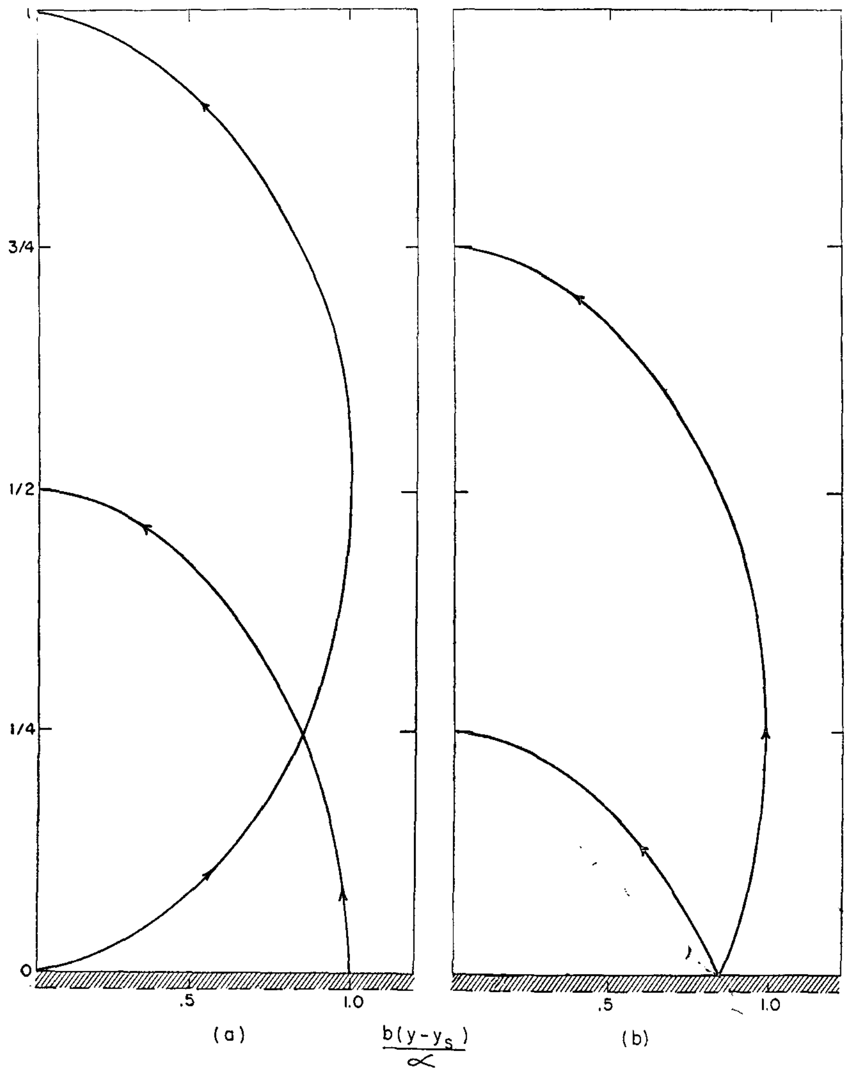

FIG. 10. The ray paths of a planetary wave in a singular line wave guide as given by (6.9), for a), the source point at either $(y-y_s)=0$ or $(y-y_s)=\alpha/b$, and b) the source point at $(y-y_s)=0.84\alpha/b$.

this maximum value being

$$Z_{\max}=\frac{Z_0}{m^2}\sin\varphi(y_s), \qquad (6.12)$$

where

$$Z_0=(4\sqrt{3}/9)[S^{\frac{1}{2}}U_y(y_s)]^{-1}\pi\beta.$$

For our numerical discussion we take $\beta\simeq1$, $S^{\frac{1}{2}}\simeq0.2$ and $U_y(y_s)$ to be approximately $\frac{1}{4}$, which corresponds to zonal winds increasing from zero to roughly 40 m sec^{-1} at 10° latitude away from the singular line, as indicated by Fig. 9. This gives $Z_0\simeq50$. We now take $\mu(y_s)\simeq0.2$, corresponding to a singular line ~12° from the equator which is roughly the location of the zero wind line in the upper stratosphere. (We see from Fig. 9 that the zero wind line crosses the equator at an altitude of ~65 km.) We then have

$$Z_{\max}\simeq10/m^2. \qquad (6.13)$$

Hence, referring to the discussion of vertical travel distances following (6.9), we put an upper limit on the distance a planetary wave ray may travel upward in the stratosphere before it is attenuated on a zero wind line.

To translate the ray trajectories given by (6.9) into vertical distances (km), we note that $dz=Hdh$, where H is scale height and h is geometrical distance. Hence, taking H of the stratosphere to be roughly 7 km, we multiply the z distances by 7 km. Assuming then (6.13), we thus estimate that direct rays have a maximum vertical travel distance (km) of $18/m^2$ to $35/m^2$, and once reflected rays a maximum travel distance of $35/m^2$ to $70/m^2$.

The above computation suggests that stationary planetary waves in the equatorial zero wind wave guide should be largely attenuated at the zero wind line before propagating vertically more than $\sim$25 km. The possibility of a small leakage of the $m=1$ partial waves components into the lower thermosphere through the equatorial wave guide is not ruled out by the above analysis, but if we take into account, qualitatively, the equatorward tilt of the actual zero wind line, such leakage seems highly unlikely. One may formulate the Hamilton-Jacobi ray equations for Eq. (4.5) with arbitrary zonal winds, whence it is found that, in general, the slope of rays is proportional to the square of the Coriolis parameter. It follows that ray paths are bent towards the equator and become completely horizontal as the equator is reached. The reader should be cautioned that the validity of the model becomes increasingly doubtful as the equator is approached.

7. Concluding remarks

Stationary planetary waves are excited at the earth's surface by topography and by the temperature difference between oceans and continents. This forcing is generally considered to be of largest magnitude in middle latitudes. There is then the possibility that these waves may propagate to great heights in the atmosphere, i.e., the lower thermosphere. However, the theory of planetary wave propagation shows that planetary waves will be evanescent in regions of easterly wind or in regions of sufficiently strong westerlies. Charney and Drazin (1961) established these conclusions for vertical propagation through wind systems independent of latitude and concluded from their theory that the summer easterly and winter westerly jets would keep most planetary wave energy trapped within the troposphere. Such trapping is much more complete during the summer.

This paper considers a linearized model for planetary waves which includes latitudinal variation of zonal winds and is valid on a spherical earth except near the equator. There exist wave ducts in regions of weak westerly wind. The analysis in Section 3 establishes the latitudinal normal mode structure of stationary planetary waves occurring in a duct formed by trapping between geometric poles and/or regions of strong westerly wind. The westerly jet of the winter upper stratosphere will act as a wave barrier in middle latitudes, creating such a wave guide in high latitudes in winter. Planetary wave disturbances generated in middle latitudes at the earth's surface and propagating into the stratosphere will be diffracted into this *polar wave guide*, as may be inferred from the "wave equation" analogy of (2.5).

Planetary wave disturbances propagating out of the troposphere can also be diffracted into an *equatorial wave guide* formed between the westerly jet of the winter hemisphere and the zero wind line transition to the stratospheric summer easterlies. Section 4 gives an approximate description of solutions to (2.5) in the neighborhood of a vertically inclined singular wind line. This analysis shows planetary wave disturbances to be absorbed rather than reflected along zero wind lines, such as that of the equatorial wave guide. Planetary waves incident on a vertically inclined singular line excite a divergence of eddy horizontal momentum flux away from the singular line, while those incident from below on a horizontally oriented singular line induce a jump in the northward heat transport at the singular line from a positive to a zero value. In particular, stationary planetary waves propagating out of the summer troposphere and incident on the stratospheric easterlies have such a jump. If sources are present in the region of easterly winds, the singular line will *emit* wave motions. Waves incident on an easterly barrier between two weak westerly propagation regions, which are exponentially attenuated in passing through as an evanescent wave, are emitted at the far side of the barrier by the far singular line.

Section 5 extends the description of disturbances in a vertically inclined singular line wave guide to include a second internal vertical boundary formed by a region of strong westerlies. Waves are then contained by one absorbing and one reflecting boundary. *No normal mode solutions exist in such a wave guide.* Consequently, planetary waves propagating vertically are represented not by a sum of vertically propagating normal mode solutions but rather by an integral over elementary waves propagating at all different angles to the vertical. Section 6 gives an approximate description of ray path vertical propagation in the singular line wave guide. Only those rays which never reach the zero wind line can propagate energy to very high levels. It appears that only a very small fraction of the total energy which enters the equatorial wave guide in the lower stratosphere will propagate higher than the stratopause.

The separable models used in our analysis are not directly applicable to transient planetary waves because of the restriction that singular lines are located on constant y or z coordinate surfaces. The results can be applied qualitatively to transient disturbances with a single phase speed c by making a Galilean transformation into a coordinate system translating with speed c relative to the earth. The analysis can also be generalized somewhat by inclusion of locally orthogonal coordinates defined to be along or normal to surfaces of constant $r(y,z)/u$ [cf. (2.7)]. The zero wind lines would then be correctly located. More realistic quantitative results could be obtained with the specification of $r(y,z)$ and $\bar{u}(y,z)$ from observations and the numerical solution of (2.5). Critical features of (2.5) include the unboundedness of the integration region for large y and positive z, singular lines where $\bar{u}=\sigma c$, and those regions where $\sigma^2 r(y,z)/(\bar{u}-\sigma c)$ is positive and sufficiently large that solutions have wavelike oscillation of phases.

The observed wintertime planetary waves are of sufficiently large amplitude to invalidate any precise numerical results obtained with a linear model. The validity of linearization is especially doubtful along singular lines, where for stationary waves the eddy wind amplitudes necessarily become large compared to the amplitude of the zonal wind. The above criticism applies to all linear theories of shear flow "critical points," so ubiquitous in the geophysical fluid dynamics literature. Included in this category are hydrodynamic stability theory (Lin, 1966), Miles' theory of water wave generation (cf. Lighthill, 1962), and the internal gravity wave theory of Booker and Bretherton (1967). As indicated by the discussion in these references, various arguments may be advanced to show that matching across the critical point (i.e., singular line) gives a valid description of disturbances on each side of the critical point, even though stationary wave solutions very near that point are of doubtful validity. Slow changes of the zonal winds and source strength will, in practice, limit the amplitude of disturbances at the singular line. Numerical solutions of the nonlinear planetary wave initial-value problem could profitably be used to determine the limitations of linear theory near the singular lines. The primary dissipation in the upper stratosphere is believed to be the diabatic damping of temperature in the form of a Newtonian cooling mechanism, this damping being especially important in regions of weak zonal winds.

Turning now to stratospheric motion phenomena that can be interpreted in terms of the theory, the *Aleutian high* may be considered a vertically propagating planetary wave in the polar wave guide. Significant vertical attenuation should occur by leakage of wave energy through the westerly jet barrier into the equarial wave guide, where waves can be absorbed along the zero wind line.

The biennial oscillation of the lower stratosphere should greatly modulate stationary planetary waves propagating upward from the troposphere in the subtropics. For instance, the latitude of the January mean zero wind line at 10 mb varies from north of 30N to south of the equator as the biennial zonal wind oscillation progresses downward (Wallace, 1967). Also, conversely, the absorption of planetary waves by the fluctuating zero wind line and the consequent periodicity in the location and strength of $\overline{u'v'}$ convergences may influence the biennial oscillation. Wallace (1966) has given some statistical evidence to support the hypothesis that vertically propagating planetary waves are modulated by the biennial zonal winds, and has suggested that this modulation might in turn produce the oscillating Reynolds' stresses he observed. The theory of planetary waves propagating near singular lines should prove useful for further evaluation of these hypotheses.

While it has not been our intent here to discuss stationary planetary waves in the troposphere, the analysis of this paper should provide guidance for the improvement of theoretical models of these waves. The current models have invariably assumed zonal winds independent of latitude and a normal mode latitudinal structure determined by the confining sphericity of the earth, and have frequently assumed confining rigid walls for β-plane studies. This study suggests that trapping associated with latitudinal variation of zonal winds may sometimes simulate the effect of such rigid walls, but considering the observed tropospheric winds, there should usually be at least one singular line turning point with absorption rather than reflection. The use of latitudinal normal mode expansions and associated concepts such as planetary wave "resonances" is then without even qualitative justification.

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