

Approximate Equations of Motion for Gases and Liquids

JOHN A. DUTTON

Dept. of Meteorology, The Pennsylvania State University, University Park

AND GEORGE H. FICHTL

George C. Marshall Space Flight Center, NASA, Huntsville, Ala.

(Manuscript received 5 January 1968, in revised form 25 November 1968)

ABSTRACT

A set of conditions which justify the application of the Boussinesq approximation to compressible fluids is developed. Two cases are found and compared. In the first, in which the vertical scale of the motion can be of the same order of magnitude as the scale height of the medium, the perturbation momentum must be nondivergent and the effects of perturbations of pressure appear in several places. In the other case, where the vertical scale of the motion is much less than the scale height, the perturbation velocities are nondivergent and the perturbation pressure appears only in the pressure gradient force.

The approximate equations lead to linearized equations controlling the stability of wave motion which are formally equivalent to those for the same problem in the flow of a stratified medium which is incompressible in the sense that the flow is solenoidal. Thus, a variety of results about such motions are made applicable to the problems of convection and gravity wave motion in the atmosphere.

Various properties of the approximate equations are investigated; it is shown that acoustic modes are not permitted; quadratic forms which can serve as energies in various cases are developed; and integral methods of determining stability criteria are reviewed and applied.

In order to give the results wider applicability than to ideal gases, an ideal liquid is defined (c_p and the coefficients of expansion all being constant). The thermodynamic functions of this ideal liquid, including the entropy, internal energy and potential temperature, are determined explicitly.

1. Introduction

Approximate equations of motion allow students of fluid mechanics and atmospheric science to circumvent some of the difficulties associated with the full Navier-Stokes equations and still be able to study the mathematical properties of certain idealized flows. Such equations are useful in the analysis of two phenomena now attracting much interest in atmospheric science—convection and gravity wave motion.

Perhaps the most widely used approximate equations are known as those of Boussinesq (1903).¹ The crux of the Boussinesq approximation for a stratified, incompressible fluid is that variations of density are ignored except where they are multiplied by the acceleration of gravity in the vertical component of the equation of motion. An important assumption also made is that small causes produce small effects, an assumption discussed by Birkhoff (1960) and Van Dyke (1964). The extensive analysis by Chandrasekhar (1961) is an example of what can be accomplished with the Boussinesq equations.

In a recent study, Spiegel and Veronis (1960) extended the Boussinesq approximation to apply to certain flows of compressible fluids. Utilizing the basic

restriction that the vertical scale of the motion was small compared to the scale height of the reference state, they produced a set of equations which were formally equivalent to those of an incompressible fluid.

The severe restriction on the vertical scale was removed by Ogura and Phillips (1962), who emphasized that a rigorous derivation of the approximate equations for compressible fluids requires a limitation on the permitted frequencies of the motion. They derived an approximate set of equations in which acoustic modes cannot occur. This is important for numerical integration of the equations, because use of much longer time steps is possible.

In this article, a set of conditions is developed which can be used to justify a Boussinesq approximation for both deep and shallow convection simultaneously. The analysis shows that for deep atmospheric convection, in which the vertical scale of motion may be of the same order of magnitude as the scale height, the approximate continuity equation requires that the momentum field be solenoidal. For shallow convection, however, the restriction that the vertical scale of the motion be much less than the scale height permits a modification in this continuity equation so that the perturbation velocity becomes nondivergent. Another basic difference is that in shallow convection the effects of perturbations of pressure are retained only

¹ Eckart and Ferris (1956) point out that these equations are quite similar to a set of equations introduced by Oberbeck (1879).

in the pressure gradient force; in deep convection they appear in the first law of thermodynamics and in the equation of state.

We also remove the restriction present in the studies of both Spiegel and Veronis (1960) and Ogura and Phillips (1962) that the equations apply only to an ideal gas. To do so we consider the thermodynamics of a liquid, whose properties are somewhat less complex than those of actual liquids. An ideal liquid is defined here to be one in which the coefficients of isobaric and isothermal expansion and the coefficient c_p are all constant. This assumption should be permissible in the range of variation of state encountered, for example, in rotating convection experiments. We determine the entropy and internal energy of the ideal liquid, and also derive a potential temperature in analogy with that of an ideal gas.

The most significant consequence of our results is that the linearized approximate equations of motion for both deep and shallow convection and gravity waves in the atmosphere lead to wave motion equations which are shown to be *formally* equivalent to those of an incompressible fluid. Thus, the known results about the stability of flows in an incompressible fluid can be applied directly to these atmospheric problems. This is important because a major part of the interest in the role of gravity waves² and convection in atmospheric processes centers on how the stability properties of these motions are related to the generation and maintenance of turbulence in the atmosphere.

The recent and important theorems of Miles (1961) and Howard (1961) about the stability of wave motion in a stratified, incompressible medium can now be applied directly to atmospheric problems. In particular, the semicircle theorem is available, and the result that the critical Richardson number for gravity wave instability is 0.25 is justified for the atmosphere. We derive a necessary and sufficient condition for instability, which except for the case of the Richardson number being negative everywhere, depends on the solutions themselves. This illustrates the difficulty of obtaining sufficient conditions for instability which depend only on the structure of the reference state.

2. The equation of state

For gases or liquids, the existence of an equation of state of the form (see Appendix for table of symbols)

$$\alpha = \alpha(T, p) \quad (2.1)$$

is assumed. Expansion in a Taylor's series about some reference state to be represented by (α_0, p_0, T_0) pro-

duces

$$\alpha = \alpha_0 \left\{ 1 + \left[\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial T} \right)_p \right]_0 (T - T_0) + \left[\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial p} \right)_T \right]_0 (p - p_0) + \dots \right\}. \quad (2.2)$$

We shall denote the derivatives by

$$\epsilon = \left[\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial T} \right)_p \right]_0, \quad \eta = - \left[\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial p} \right)_T \right]_0. \quad (2.3)$$

We call ideal those gases for which the equation¹ of state

$$p\alpha = RT \quad (2.4)$$

holds. In this case the coefficients² are

$$\frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial T} \right)_p = T^{-1}, \quad \frac{1}{\alpha} \left(\frac{\partial \alpha}{\partial p} \right)_T = -p^{-1}, \quad (2.5)$$

so that

$$\epsilon = T_0^{-1}, \quad \eta = p_0^{-1}. \quad (2.6)$$

For an ideal gas, all of the higher order³ derivatives appearing in the series (2.2) involve higher order powers or products of the reciprocals of p or T . Therefore, the expression with the three terms displayed in (2.2) can be made as accurate as desired by limiting the excursions of p and T from p_0 and T_0 , respectively. Thus, with sufficient limitation on the size of $p' = p - p_0$ and $T' = T - T_0$, the approximate equation of³ state⁴ is

$$\alpha' / \alpha_0 = (T' / T_0) - (p' / p_0). \quad (2.7)$$

For non-ideal gases and for liquids, the values of the various derivatives in (2.2) must be determined by experiment. We shall assume that the higher order derivatives are such that sufficient restriction on p' and T' will yield a sufficiently accurate equation of state in the form

$$\alpha = \alpha_0 [1 + \epsilon(T - T_0) - \eta(p - p_0)]. \quad (2.8)$$

In analyzing idealized liquids, we shall assume that the variations permitted in the reference states are small enough that the two expansion coefficients may be taken as constant.

3. The equations of motion

The basic equations specifying the velocity fields, $\mathbf{v} = (u, v, w)$, are assumed to be those of Navier-Stokes for a nonrotating coordinate system with the acceleration of gravity g , in the vertical direction \mathbf{k} , i.e.,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha \nabla p - g \mathbf{k} + \nu [\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v})]. \quad (3.1)$$

²The recent interest in this subject was generated by an exposition of the properties of gravity waves by Hines (1960).

We have assumed that the ratio between the two viscosity coefficients in the non-isotropic part of the stress tensor is $-\frac{2}{3}$ and that the dynamic coefficient of viscosity, $\mu = \nu\rho$, is a constant. This coefficient is in general dependent on the temperature so that the latter assumption also requires a limitation on the permissible variations in temperature.

a. Reference state

The basic state upon which perturbations are superimposed presents formidable problems of its own. In situations which might arise in geophysics or laboratory experiments such a basic state would be evolving in time under the control of external influences, viscous effects and heat conduction, and interactions with any perturbations which might develop. The general approach in problems of the type being considered here is to rather arbitrarily reduce the complexity of the contributions of the basic state to the complete problem and emphasize the perturbation motion.

Perhaps the simplest way to accomplish this is to assume that the reference state may be considered to be in steady motion and that the effects of radiation, viscosity and heat conduction are not important in the basic state. Then we may choose a velocity $\mathbf{U} = \mathbf{i}_z U(z)$, and the equations of motion imply that

$$\nabla_2 p_0 = 0, \quad \alpha_0 \frac{\partial p_0}{\partial z} = -g, \tag{3.2}$$

in which the subscript 2 restricts the operator to the horizontal plane. This hydrostatic condition implies further that

$$\frac{\partial p_0}{\partial z} \nabla_2 \alpha_0 = -\alpha_0 \frac{\partial}{\partial z} \nabla_2 p_0 = 0, \tag{3.3}$$

and because of the relation

$$\nabla_2 \alpha_0 = \left(\frac{\partial \alpha_0}{\partial T_0} \right)_p \nabla_2 T_0 + \left(\frac{\partial \alpha_0}{\partial p_0} \right)_T \nabla_2 p_0, \tag{3.4}$$

it must be true that

$$\nabla_2 T_0 = 0. \tag{3.5}$$

Thus, the vertical variation of $U(z)$, $\rho_0(z)$, and $T_0(z)$ are arbitrary and can be chosen to model a case of interest by constructing a suitable vertical profile of the Richardson number, for example.

This approach to defining the basic state flow is not new. For example, the linear theory of the dynamic stability properties of parallel and nearly parallel flows in homogeneous media in the absence of gravitational body forces is based upon the Orr-Sommerfeld equation. The basic state velocity profile $U(z)$ appears in this equation and is considered to be an arbitrary function of z . However, physically speaking, the function $U(z)$ in this equation is the velocity distribution

for a basic state parallel flow and the exact solutions of the Navier-Stokes equations show that $U(z)$ can be only a constant, a linear function of z (plane Couette flow), or a quadratic function of z (plane Poiseuille flow). Nevertheless, many students of fluid mechanics use the Orr-Sommerfeld equation to examine linear dynamic stability properties of flows with velocity distributions $U(z)$ other than those just cited (see, for example, Lin, 1955). The justification of this philosophy must be examined for each class of flows one considers.

b. Perturbation equations

Upon substituting the perturbation forms of pressure and specific volume into the Navier-Stokes equations, the nonviscous vertical component terms on the right side of (3.1) become

$$-\alpha \frac{\partial p}{\partial z} - g = -\alpha_0 \left(1 + \frac{\alpha'}{\alpha_0} \right) \frac{\partial p'}{\partial z} + g \frac{\alpha'}{\alpha_0}. \tag{3.6}$$

In order to be able to ignore the product of perturbation quantities in the first term on the right side of the above equation, we must require

$$|\alpha'/\alpha_0| \ll 1. \tag{3.7}$$

Hence, we must ensure that the variations of p' , p_0 , T' and T_0 permitted in the fluid will guarantee (3.7) as well as justify the approximate equation of state (2.8).

With (3.7) applied to the coefficient of viscosity so that $\nu_0 = \mu\alpha_0$, the approximate equations of motion become

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha_0 \nabla p' + g \frac{\alpha'}{\alpha_0} \mathbf{k} + \nu_0 \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right]. \tag{3.8}$$

These equations, now linear in the thermodynamic variables, contain the essence of the Boussinesq approximation—variations in density or specific volume are ignored except when combined as a factor with the acceleration of gravity. For an incompressible fluid, the analysis is complete at this point. But for a compressible fluid, we must determine whether modifications in the other equations are necessary to ensure consistency, or whether simplifying assumptions may be possible for certain flows.

For the subsequent analysis, we shall need a consistent method to estimate orders of magnitude of various terms in the relevant equations. We shall employ various versions of the equations with all variables φ replaced by the Fourier representation

$$\varphi(x, y, z, t) = \hat{\varphi}(\omega, \kappa_1, \kappa_2, z) e^{i(\omega t + \kappa_1 x + \kappa_2 y)}. \tag{3.9}$$

The order of magnitude of $\hat{\varphi}$ will be denoted by $|\hat{\varphi}|_M$. If a vertical derivative of $\hat{\varphi}$ occurs, we shall express

the magnitude of the derivative by employing a scale L_ϕ defined by

$$L_\phi \left| \frac{\partial \hat{\phi}}{\partial z} \right|_M = |\hat{\phi}|_M. \tag{3.10}$$

The vertical component of the vector equation of motion is

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w = -\alpha_0 \frac{\partial p'}{\partial z} + g \frac{\alpha'}{\alpha_0} + \nu_0 [\nabla^2 w + \frac{1}{3} \mathbf{k} \cdot \nabla (\nabla \cdot \mathbf{v})]. \tag{3.11}$$

In order to estimate the magnitude of the local component of the acceleration, we employ the linearized equations with the effects of viscosity neglected. The representation (3.9) yields

$$i\omega \hat{w} = -\alpha_0 \frac{\partial \hat{p}'}{\partial z} + g \frac{\hat{\alpha}'}{\alpha_0}. \tag{3.12}$$

Various modifications of this equation are possible, depending on the type of flow being considered. For example, in studying quasi-hydrostatic motions it would be appropriate to ignore the acceleration term and to require that the other two terms be in phase but of opposite sign. In contrast, for the case of convection or gravity wave motion we assume that buoyancy is the driving force and that the specific volume term may be used to estimate the order of magnitude of the acceleration. However, if we wish to retain the cases in which the pressure gradient term is of the same magnitude as the buoyancy term, we must assume that the orders of magnitude are related by

$$\frac{\hat{p}_0}{\rho_0} \left| \frac{\hat{p}'}{\hat{p}_0} \right|_M \sim L_{p'} \left| \frac{\hat{\alpha}'}{\alpha_0} \right|_M. \tag{3.13}$$

But in order to utilize the assumption that the accelerations are governed in magnitude by the buoyancy force, we must also assume that the pressure and buoyancy terms tend to be out of phase. With these conditions, we have the estimate

$$|\partial w / \partial t|_M = |\omega \hat{w}|_M \sim g |\hat{\alpha}' / \alpha_0|_M. \tag{3.14}$$

This estimate, along with (3.13), is the foundation for our analysis of the other equations.

4. The equation of continuity

The equation of continuity

$$\frac{d\alpha}{dt} = \alpha \nabla \cdot \mathbf{v}, \tag{4.1}$$

may be expanded in the form

$$\begin{aligned} \frac{\partial \alpha'}{\partial t} + \mathbf{v} \cdot \nabla_2 \alpha' + w \frac{\partial \alpha'}{\partial z} + w \frac{\partial \alpha_0}{\partial z} \\ = \alpha_0 \left(1 + \frac{\alpha'}{\alpha_0} \right) \nabla \cdot \mathbf{v} = \alpha_0 \nabla_2 \cdot \mathbf{v} + \alpha_0 \frac{\partial w}{\partial z}. \end{aligned} \tag{4.2}$$

The condition (3.7) allows us to ignore the (α' / α_0) term compared to unity on the right side.

An analysis of the ratios of the orders of magnitudes of the various terms will indicate whether any of them may be ignored. We employ the Fourier decomposition (3.9), the definition of scales (3.10), and the result (3.14), along with the convention that $\kappa_i = 2\pi / L_i$. We shall denote typical horizontal values with a subscript H , and define the scale height H_α by

$$H_\alpha^{-1} = \frac{1}{\alpha_0} \frac{\partial \alpha_0}{\partial z}. \tag{4.3}$$

Division of the various terms of (4.2) by the last term on the left produces the set of ratios:

$$\begin{aligned} \left| \frac{\partial \alpha'}{\partial t} \right|_M : \left| w \frac{\partial \alpha_0}{\partial z} \right|_M &= \frac{|\omega|^2 |\hat{\alpha}'|_M}{|\omega| |\hat{w}|_M (\partial \alpha_0 / \partial z)} \\ &= \frac{|\omega|^2 H_\alpha}{g}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} |\mathbf{v} \cdot \nabla_2 \alpha'|_M : \left| w \frac{\partial \alpha_0}{\partial z} \right|_M &= \left| \frac{\hat{v}_{HKH}}{\hat{w}} \right|_M \left| \frac{\hat{\alpha}'}{\alpha_0} \right|_M H_\alpha \\ &= \left| \frac{\hat{v}_H}{\hat{w}} \right|_M \left(\frac{2\pi H_\alpha}{L_H} \right) \left| \frac{\alpha'}{\alpha_0} \right|_M, \end{aligned} \tag{4.5}$$

$$\left| \frac{\partial \alpha'}{\partial z} \right|_M : \left| w \frac{\partial \alpha_0}{\partial z} \right|_M = \frac{H_\alpha}{L_\alpha} \left| \frac{\alpha'}{\alpha_0} \right|_M, \tag{4.6}$$

$$|\alpha_0 \nabla_2 \cdot \mathbf{v}|_M : \left| w \frac{\partial \alpha_0}{\partial z} \right|_M = \left| \frac{\hat{v}_H}{\hat{w}} \right|_M \frac{2\pi H_\alpha}{L_H}, \tag{4.7}$$

$$|\alpha_0 \partial w / \partial z|_M : \left| w \frac{\partial \alpha_0}{\partial z} \right|_M = H_\alpha / L_w. \tag{4.8}$$

By virtue of (3.7), the conditions

$$|\omega|^2 \ll g / H_\alpha, \tag{4.9}$$

$$|\hat{v}_H / \hat{w}|_M \leq L_H / 2\pi H_\alpha, \tag{4.10}$$

$$L_w \sim H_\alpha, \tag{4.11}$$

$$L_w \sim L_\alpha, \tag{4.12}$$

are *sufficient* to validate the approximate equation of

continuity

$$\alpha_0 \nabla_2 \cdot \mathbf{v} + \alpha_0 \frac{\partial w}{\partial z} - w \frac{\partial \alpha_0}{\partial z} = 0, \tag{4.13}$$

which may be written as

$$\nabla \cdot (\alpha_0^{-1} \mathbf{v}) = \nabla \cdot \rho_0 \mathbf{v} = 0. \tag{4.14}$$

Because the scales L_w are permitted to be of the same order of magnitude as the scale height H_α , we shall refer to this case as that of *deep convection*. An example of a flow for which the deep convection equations are appropriate is gravity-wave motion which is harmonic in the vertical with wavelength L_3 . Then by (3.9) we have $L_w = L_3 / (2\pi) \sim H_\alpha$. Thus, vertical wavelengths of the order of tens of kilometers are permitted because H_α for the atmosphere is about 8 km.

The ratio of the last two terms on the left of (4.13), according to (4.8), is H_α / L_w so that if we multiply each of the ratios (4.4)–(4.8) by L_w / H_α , the new ratios are those which would be obtained by using $\alpha_0 \partial w / \partial z$ as a divisor. Thus, if

$$\left. \begin{aligned} |\omega|^2 &\ll g / L_w \\ |\hat{v}_H / \hat{w}|_M &\leq L_H / 2\pi L_w \\ L_w &\ll H_\alpha \\ L_w &\sim L_\alpha \end{aligned} \right\}, \tag{4.15}$$

we may use the approximate equation of continuity

$$\nabla \cdot \mathbf{v} = 0. \tag{4.16}$$

Because (4.15) restricts the permissible vertical scales to be much less than the scale height H_α , we shall refer to this case as that of *shallow convection*. An example would be atmospheric motion with vertical scales $L_3 = 2\pi L_w$ of the order of, say, $(0.01)2\pi H_\alpha$, or about half a kilometer. Another example would be rotating convection experiments utilizing water as the working fluid since the scale height in this case is extremely large.

An important property of the restrictions for shallow convection is that H_α has been replaced by L_w so that higher frequencies are permitted for the components of the flow with smaller vertical scales. Also, as the vertical scale $L_3 = 2\pi L_w$ approaches the horizontal scale, the magnitudes of the horizontal and vertical components of the velocity are allowed to be identical. Thus, these equations may be used in the study of atmospheric turbulence, and allow compressible problems to be studied approximately with the equations for incompressible motion.

The condition on the maximum frequency which is permitted is an essential part of the sufficient conditions for the validity of the approximate equations of continuity. The significance of this restriction on frequency was first pointed out by Ogura and Phillips (1962).

This condition for deep convection can be revised to agree with results to be presented later. For an ideal

gas in hydrostatic equilibrium we have for potential temperature

$$\frac{1}{\theta_0} \frac{\partial \theta_0}{\partial z} = \frac{1}{\alpha_0} \frac{\partial \alpha_0}{\partial z} + \frac{c_v}{c_p} \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} < \frac{1}{\alpha_0} \frac{\partial \alpha_0}{\partial z}, \tag{4.17}$$

and for an isothermal atmosphere, in particular,

$$\frac{1}{\theta_0} \frac{\partial \theta_0}{\partial z} = \frac{R}{c_p} \frac{1}{\alpha_0} \frac{\partial \alpha_0}{\partial z}. \tag{4.18}$$

We shall therefore provisionally replace condition (4.9) with the requirement that

$$|\omega|^2 \leq \omega_g^2 < g / H_\alpha, \tag{4.19}$$

where the Brunt-Väisälä frequency ω_g is defined (when potential temperature increases with height) by

$$\omega_g^2 = \frac{g}{\theta_0} \frac{\partial \theta_0}{\partial z}. \tag{4.20}$$

We shall verify that the linearized, inviscid approximate equations of motion for both ideal gases and liquids do not in fact permit frequencies larger than the Brunt-Väisälä frequency for nonamplifying wave motion superimposed on a shear-free basic velocity field.

The major distinction between the cases of deep and shallow convection which has appeared so far is the difference in the permitted magnitude of the vertical scale L_w , a difference which is reflected in the different equations of continuity appropriate to the two cases. For shallow convection, the Boussinesq approximation allows us to treat the fluid as incompressible; for deep convection, however, the approximate continuity equation requires that the momentum field $\rho_0 \mathbf{v}$ be solenoidal, and the expansion or contraction of parcels moving in the vertical is taken into account.

Another modification is also available now for shallow atmospheric convection. We define the scale height

$$H_i = p_0 / g \rho_0, \tag{4.21}$$

and from (4.17) we have the result

$$H_\alpha^{-1} = \left(\gamma_d + \frac{\partial T_0}{\partial z} \right) T_0^{-1} + (c_v / c_p) H_i^{-1}. \tag{4.22}$$

Because the first term on the right is of the order of $(50 \text{ km})^{-1}$, we may conclude that $H_\alpha \sim H_i$. Now in (3.13) we require that

$$L_{p'} \sim L_w, \tag{4.23}$$

so that we may write

$$\left(\frac{p'}{p_0} \right)_M \sim \left(\frac{\alpha'}{\alpha_0} \right)_M \frac{L_w}{H_\alpha}. \tag{4.24}$$

Thus, the pressure perturbation term may be omitted from the equation of state for shallow convection in the atmosphere.

5. The thermodynamic energy equation

The enthalpy form of the thermodynamic energy equation is³

$$dh - \alpha dp = T ds, \tag{5.1}$$

in which h is enthalpy and s is entropy. Taking $h = h(p, T)$ yields

$$\left(\frac{\partial h}{\partial T}\right)_p dT + \left[\left(\frac{\partial h}{\partial p}\right)_T - \alpha\right] dp = T ds. \tag{5.2}$$

Consideration of $h = h(s, p)$ gives

$$\left(\frac{\partial h}{\partial T}\right)_p = \left(\frac{\partial h}{\partial s}\right)_p \left(\frac{\partial s}{\partial T}\right)_p, \tag{5.3}$$

and by virtue of the fact that $T = (\partial h / \partial s)_p$ and the definition $c_p = T(\partial s / \partial T)_p$, we have

$$c_p dT + \left[\left(\frac{\partial h}{\partial p}\right)_T - \alpha\right] dp = T ds. \tag{5.4}$$

To obtain a more convenient form of the coefficient of dp , let us apply the equation to the reference state. Then (5.4) becomes

$$c_p \frac{\partial T_0}{\partial z} - \left[\left(\frac{\partial h}{\partial p}\right)_T - \alpha\right]_0 g \rho_0 = T_0 \left[\left(\frac{\partial s}{\partial T}\right)_p \frac{\partial T}{\partial z} + \left(\frac{\partial s}{\partial p}\right)_T \frac{\partial p}{\partial z} \right], \tag{5.5}$$

and upon use of one of Maxwell's reciprocity relations we have

$$\left[\left(\frac{\partial h}{\partial p}\right)_T - \alpha\right]_0 = \left[T \left(\frac{\partial s}{\partial p}\right)_T\right]_0 = - \left[T \left(\frac{\partial \alpha}{\partial T}\right)_p\right]_0 = - [T \epsilon \alpha]_0. \tag{5.6}$$

We define $A = [(\partial h / \partial p)_T - \alpha]$, and because A is a function only of thermodynamic variables, we assume that $A' / A_0 \ll 1$. For an ideal gas, it is obvious that $A = -\alpha$.

³ By starting in this manner, we avoid a problem encountered by Spiegel and Veronis (1960), who combined the exact forms of the energy and continuity equations, and thus faced the problem of having the divergence appear in the energy equation subsequent to their establishment of a continuity equation which required nondivergent motion.

It can be shown that the scaling assumptions already made allow us to conclude that the approximation

$$\frac{dp}{dt} = -g \rho_0 w \tag{5.7}$$

is valid. However, this result cannot be used in a temporal form of (5.4) without assurance that the terms involving p' are also small compared to other terms which are retained. The definitions of the form $T = T_0 + T'$ allow us to write

$$c_p \frac{dT'}{dt} + c_p w \frac{\partial T_0}{\partial z} + A_0 \left(1 + \frac{A'}{A_0}\right) \frac{d}{dt} \left[p_0 \left(1 + \frac{p'}{p_0}\right) \right] = T \frac{ds}{dt}, \tag{5.8}$$

which gives

$$c_p \frac{dT'}{dt} + c_p w \left[\frac{\partial T_0}{\partial z} - \frac{g}{c_p} \rho_0 A_0 \left(1 + \frac{A'}{A_0}\right) \left(1 + \frac{p'}{p_0}\right) \right] + A_0 \left(1 + \frac{A'}{A_0}\right) p_0 \frac{d}{dt} \left(\frac{p'}{p_0}\right) = T \frac{ds}{dt}. \tag{5.9}$$

The conditions (3.7) and (3.13) along with (4.23) justify neglect of p' / p_0 compared to unity. Hence, with the assumption about A' / A_0 , (5.9) may be put in the form

$$c_p \frac{dT'}{dt} + c_p w \left(\epsilon \frac{\partial T_0}{\partial z} - T' \frac{\partial \epsilon}{\partial z} + \frac{g \epsilon^2}{c_p} T_0 \right) - p_0 \alpha_0 \epsilon^2 T_0 \frac{d}{dt} \left(\frac{p'}{p_0}\right) = \epsilon T \frac{ds}{dt}. \tag{5.10}$$

For an ideal gas this yields

$$c_p \frac{dT'}{dt} \left(\frac{T'}{T_0}\right) + \frac{c_p w}{T_0} \left(\frac{\partial T_0}{\partial z} + \gamma_s\right) - R \frac{d}{dt} \left(\frac{p'}{p_0}\right) = \frac{ds}{dt}, \tag{5.11}$$

in which we have used (3.7), (3.13), and the equation of state to neglect T' / T_0 as compared with unity, and denoted the lapse rate of an isentropic atmosphere as $\gamma_s = -(\partial T / \partial z)_s = g / c_p$. Now, for shallow atmospheric convection we know from the results of Section 4 that the fractional pressure perturbations are small compared to those of temperature so that we have

$$c_p \frac{dT'}{dt} \left(\frac{T'}{T_0}\right) + \frac{c_p w}{T_0} \left(\frac{\partial T_0}{\partial z} + \gamma_s\right) = \frac{ds}{dt}. \tag{5.12}$$

This is the same approximate thermodynamic energy equation obtained by Spiegel and Veronis (1960) for shallow convection.

Substitution of the general equation of state (2.8) in (5.10) produces

$$\frac{d}{dt} \left(\frac{\alpha'}{\alpha_0} + \eta p' \right) - \frac{p_0 \alpha_0 \epsilon^2}{c_p} T_0 \frac{d}{dt} \left(\frac{p'}{p_0} \right) + \epsilon w \left(\frac{\partial T_0}{\partial z} + \frac{g \epsilon}{c_p} T_0 \right) = \frac{\epsilon T}{c_p} \frac{ds}{dt}, \quad (5.13)$$

where we have used either the assumption that ϵ is constant in a liquid or the fact that $T'/T_0 \ll 1$ in a gas. For an ideal gas this becomes

$$\frac{d}{dt} \left(\frac{\alpha'}{\alpha_0} + \frac{c_p}{c_p} \frac{p'}{p_0} \right) + \frac{w}{T_0} \left(\frac{\partial T_0}{\partial z} + \gamma_s \right) = \frac{1}{c_p} \frac{ds}{dt}. \quad (5.14)$$

According to the results of Section 4, we will have

$$|p'/p_0|_M : |\alpha'/\alpha_0|_M \sim L_w/H_\alpha \sim 1 \quad (5.15)$$

for deep convection, so the pressure term must be retained in that case.

An expansion of Poisson's equation for the potential temperature, and use of the hydrostatic relation for the reference state shows that (5.14) may be written as

$$\frac{d}{dt} \left(\frac{\theta'}{\theta_0} \right) + \frac{w}{\theta_0} \frac{\partial \theta_0}{\partial z} = \frac{1}{c_p} \frac{ds}{dt}. \quad (5.16)$$

Thus, the first law of thermodynamics (5.14) is a statement about entropy change, and for isentropic flow ($ds/dt=0$) reduces to $(ds'/dt) + w(\partial s_0/\partial z) = 0$.

The analysis of (5.14) for ideal liquids is deferred until Section 7.

All of the conditions necessary to validate the approximate equations of motion presented here have now been stated except for one more to be added in Section 8. It is worth emphasizing that after a solution of these equations is obtained in a special case, that solution must be tested to determine whether the conditions for the validity of the equations are in fact satisfied.

6. Additional comments on the equations of motion

The results of the previous sections now permit some additional analysis of the equations of motion.

A first point is that the restrictions permitted in the case of shallow convection may be used to simplify the momentum conservation equations. Let δ_s be 1 for shallow atmospheric convection and 0 for deep. Then the pressure and bouyancy terms in the vertical equation of motion may be written

$$-\alpha_0 \frac{\partial p'}{\partial z} + g \frac{\alpha'}{\alpha_0} = -\alpha_0 \frac{\partial p'}{\partial z} + g [\epsilon T' - (1 - \delta_s) \eta p']. \quad (6.1)$$

For the cases other than $\delta_s=1$, we must compare the two pressure terms which have a ratio

$$|g \eta p'|_M : |\alpha_0 \partial p'/\partial z|_M = g \eta L_w / \alpha_0 = p_0 \eta L_w / H_\alpha. \quad (6.2)$$

For deep atmospheric convection this term must be retained; we shall show that it may be neglected for water, as is obvious because $p_0 \eta \ll 1$.

Two main differences between deep and shallow convection have appeared. The first is that the continuity equation takes a different form, the second is that the effects of pressure perturbations must be retained for deep convection in the first law of thermodynamics, the equation of state, and the vertical component of the equation of motion, whereas they may be neglected in the study of shallow convection.

In the case of shallow convection, the approximate equation of continuity renders the motion field solenoidal so the irrotational part of the viscous dissipation may be omitted. But for deep convection, the approximate equation of continuity implies that

$$\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) = \nabla^2 \mathbf{v} - \frac{1}{3} \nabla (w/H_\alpha), \quad (6.3)$$

so that upon assuming that H_α is constant, we find the ratio

$$\frac{1}{3} \left| \frac{1}{H_\alpha} \frac{\partial w}{\partial z} \right|_M : \left| \frac{\partial^2 w}{\partial z^2} \right|_M = \frac{1}{3} \frac{L_w}{H_\alpha} \sim 1. \quad (6.4)$$

Since the vertical component of the gradient of the divergence is of the same order of magnitude as one of the terms of the Laplacian of the vertical component of velocity, the second term of (6.3) cannot be ignored.

7. The thermodynamics of an ideal liquid

The specification of the thermodynamic properties of a gas or a liquid is a necessary step in the derivation of approximate equations of motion due to the dependence of the analysis on thermodynamic coefficients. In this section we will develop the thermodynamics of what we will denote, for convenience of reference, as an ideal liquid. The restrictions are that the thermodynamic coefficients ϵ , η and c_p of an ideal liquid are all constant over the range of variation permitted in the fiducial states (α_0, p_0, T_0) in the equation of state (2.8) of the fluid; that is

$$\alpha = \alpha_0 [1 + \epsilon(T - T_0) - \eta(p - p_0)]. \quad (7.1)$$

Most studies of convection in liquids (e.g., Chandrasekhar, 1961) omit the usually small term in (7.1) involving the perturbation pressure since the isothermal compressibility is so small for common liquids. The retention of this term in our analysis does not cause any undue complication, however.

Our development will be based upon the Gibbs function G , whose differential is given by

$$dG = -s dT + \alpha dp. \quad (7.2)$$

Since $\bar{\alpha}$ this function must be an exact differential, we

know that

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial \alpha}{\partial T}\right)_p = -\alpha_0 \epsilon, \tag{7.3}$$

in which we have used (7.1).

Therefore, considering $s = s(p, T)$ and in view of the definition $c_p = T(\partial s / \partial T)_p$, we obtain

$$ds = \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp = c_p \frac{dT}{T} - \alpha_0 \epsilon dp. \tag{7.4}$$

We integrate this expression [along lines in a (p, T) diagram] from (p_0, T_0) to (p_0, T) holding p at p_0 , and then from (p_0, T) to (p, T) holding T constant. Thus, with $s_0 = s(p_0, T_0)$, the integrated form of (7.4) is

$$s = s_0 + \int_{T_0}^T c_p(p_0, T) \frac{dT}{T} - \alpha_0 \epsilon (p - p_0). \tag{7.5}$$

Denoting the derivative $T(\partial s / \partial T)_\alpha$ by c_v and observing from the equation of state (7.1) that

$$(\partial p / \partial T)_\alpha = \epsilon / \eta, \tag{7.6}$$

we have from (7.5) that

$$c_v = c_p(p_0, T_0) - \alpha_0 \epsilon^2 T / \eta. \tag{7.7}$$

Now invoking our assumption that c_p is constant for all allowable pressures and temperatures, we see that for an ideal fluid the coefficient c_v is dependent upon both the temperature and the specific volume of the reference state. It is therefore advantageous to avoid expressions in which c_v might appear. With constant c_p , (7.5) becomes

$$s = s_0 + c_p \ln(T / T_0) - \alpha_0 \epsilon (p - p_0). \tag{7.8}$$

If a potential temperature θ is defined by the differential transformation

$$ds = (c_p / \theta) d\theta, \tag{7.9}$$

we obtain from (7.8) the result

$$d \ln \theta = d \ln T - \frac{\alpha_0 \epsilon}{c_p} dp, \tag{7.10}$$

so that we may write

$$\theta = T \exp \left[- \int_{p_{00}}^p (\alpha_0 \epsilon / c_p) dp_1 \right], \tag{7.11}$$

in which p_{00} is the reference pressure at which $\theta = T$. The integral I in the exponent is to be interpreted as

$$I = \int_{r(p_{00})}^r \alpha_0 [p_0(r), T_0(r)] (\epsilon / c_p) \nabla p \cdot dr \approx \frac{\alpha_0 \epsilon}{c_p} (p - p_{00}). \tag{7.12}$$

Utilizing a hydrostatic isentropic state, we may calculate from (7.10) that

$$\gamma_s = - \left(\frac{\partial T}{\partial z} \right)_s = \alpha_0 \epsilon g \rho T / c_p \approx g \epsilon T / c_p, \tag{7.13}$$

in which we have again used (3.7) to obtain the approximate relation. For water at 300K, γ_s is of the order of $2.10^{-4} \text{K m}^{-1}$.

Returning to the Gibbs function, we substitute the result (7.8) and the equation of state (7.1) in (7.2), and integrate over the same path used to obtain the entropy, with the result that

$$G = c_p(T - T_0) - c_p T \ln(T / T_0) + s_0(T_0 - T) + \alpha(p - p_0) + \frac{\alpha_0 \eta}{2} (p - p_0)^2 + G_0. \tag{7.14}$$

This Gibbs function will allow us to determine any of the other thermodynamic potentials for an ideal fluid. In particular, because the internal energy u is related to G by

$$u = G + Ts - p\alpha, \tag{7.15}$$

we obtain

$$u = u_0 + c_p(T - T_0) + p_0(\alpha_0 - \alpha) + \frac{\alpha_0 \eta}{2} (p - p_0)^2 - \alpha_0 \epsilon (p - p_0) T. \tag{7.16}$$

Utilizing this result and (7.3), we may determine the derivatives

$$(\partial u / \partial T)_p = c_p - \alpha_0 \epsilon p, \tag{7.17}$$

$$(\partial u / \partial p)_T = \alpha_0 (\eta p - \epsilon T). \tag{7.18}$$

Since

$$d\alpha = \alpha_0 \epsilon dT - \alpha_0 \eta dp, \tag{7.19}$$

we determine from

$$du = \left(\frac{\partial u}{\partial T}\right)_p dT + \left(\frac{\partial u}{\partial p}\right)_T dp = T ds - p d\alpha, \tag{7.20}$$

that

$$c_p dT - \alpha_0 \epsilon T dp = T ds. \tag{7.21}$$

Comparison with (5.6) provides a verification of the assumption that $A' / A_0 \ll 1$.

Another form of the first law of thermodynamics can be obtained from

$$\left(\frac{\partial u}{\partial T}\right)_\alpha = c_p - \alpha_0 \epsilon T \left(\frac{\partial p}{\partial T}\right)_\alpha = c_v, \tag{7.22}$$

in which we used (7.6) and (7.7), and from

$$\left(\frac{\partial u}{\partial \alpha}\right)_T = \frac{\epsilon T}{\eta} - p, \tag{7.23}$$

so that according to (7.20)

$$du = c_v dT + [(\epsilon T / \eta) - p] d\alpha = T ds - p d\alpha, \tag{7.24}$$

and hence

$$c_v dT + (\epsilon T / \eta) d\alpha = T ds. \tag{7.25}$$

There are two points at which we used relations valid only for an ideal gas in determining relative size of quantities associated with the approximate equations of motion. We can now extend these results to ideal fluids.

First, we were able to replace the frequency limitation g/H_α with ω_σ^2 . We shall show that the natural limitation on the frequencies which actually occurs in the linear wave equations is

$$\omega_\sigma^2 = g \epsilon \left(\frac{\partial T_0}{\partial z} + \gamma_s \right), \tag{7.26}$$

which, for ideal gases, is identical with the definition (4.20). For the hydrostatic reference states, (7.12) and (7.13) imply that

$$\frac{1}{\theta_0} \frac{\partial \theta_0}{\partial z} = \frac{1}{T_0} \frac{\partial T_0}{\partial z} + \frac{\gamma_s}{T_0}, \tag{7.27}$$

so that for both fluids and ideal gases it is valid to write

$$\omega_\sigma^2 = \frac{g \epsilon T_0}{\theta_0} \frac{\partial \theta_0}{\partial z}. \tag{7.28}$$

The version (7.25) of the thermodynamic energy equation shows that for isothermal reference states it is true that

$$\frac{\epsilon \alpha_0}{c_p \eta} \left(\frac{1}{\alpha_0} \frac{\partial \alpha_0}{\partial z} \right) = \frac{1}{\theta_0} \frac{\partial \theta_0}{\partial z}. \tag{7.29}$$

To illustrate, for water we have the values $\epsilon \sim 2.5 \times 10^{-4} (\text{°K})^{-1}$, $\alpha_0 \sim 1 \text{ cm}^3 \text{ gm}^{-1}$, $c_p \sim 4.2 \times 10^7 \text{ erg gm}^{-1} (\text{°K})^{-1}$, $\eta \sim 5.10^{-11} \text{ cm}^2 \text{ dyn}^{-1}$, and so $\epsilon \alpha_0 / (c_p \eta) \sim 0.1$. Hence,

$$\frac{g}{\alpha_0} \frac{\partial \alpha_0}{\partial z} \sim 10 \frac{g}{\theta_0} \frac{\partial \theta_0}{\partial z} > 10 \frac{g \epsilon T_0}{\theta_0} \frac{\partial \theta_0}{\partial z} = 10 \omega_\sigma^2, \tag{7.30}$$

in which we have inequality because $\epsilon T_0 \sim 0.075$. Thus, for liquids we may also conclude that the condition

$$\omega^2 \ll (g/\alpha_0) \partial \alpha_0 / \partial z \tag{7.31}$$

is satisfied by

$$\omega^2 \leq \omega_\sigma^2. \tag{7.32}$$

The second condition considered was that the quantity $\eta p'$ in the equation of the vertical component of motion may be neglected if (using values for water again and cgs units)

$$g \eta L_w / \alpha_0 \sim 5.10^{-8} L_w \ll 1. \tag{7.33}$$

Thus, for $L_w \leq 20 \text{ km}$, it is safe to neglect the term $\eta p'$.

It is also possible to examine the pressure terms in the approximate thermodynamic energy equation (5.14).

The relevant ratios, with the aid of (3.13), (4.21) and (7.8) become

$$|\eta p'|_M : |\alpha'/\alpha_0|_M \sim L_p \eta p_0 / H_i \ll 1, \tag{7.34}$$

and

$$\frac{\epsilon^2 T_0 \alpha_0 p_0}{c_p} \left| \frac{p'}{p_0} \right|_M : \left| \frac{\alpha'}{\alpha_0} \right|_M \sim \frac{\epsilon^2 T_0 \alpha_0 p_0 L_w}{c_p H_i} = \left(1 - \frac{c_v}{c_p} \right) \eta p_0 \frac{L_w}{H_i} \ll 1. \tag{7.35}$$

The inequalities are subject to the condition that p_0 does not exceed about $2 \cdot 10^4 \text{ atm}$, in which case the shallow convection form of the first law of thermodynamics is appropriate for water.

The results (7.29), (7.27) and (7.13) combine to show that for an isothermal stratification of water we have $H_\alpha \sim 200 \text{ km}$. Thus, the existence of deep convection in water is doubtful under any conceivable circumstances.

8. Boundary conditions and energy forms

Energy equations accompanying the approximate equations of motion will be derived in this section after consideration of the boundary conditions to be applied in various cases.

The derivation of energy equations necessitates one change in the equations for deep convection. Specifically, based upon the result from (4.22) that for the atmosphere, $H_\alpha \sim (c_p/c_v) H_i$, we replace the pressure term in the first law of thermodynamics,

$$\frac{c_v p'}{c_p p_0} = \frac{c_v}{c_p g \rho_0 H_i} p', \tag{8.1}$$

with the term $p'/g\rho_0 H_\alpha$. This substitution, essential for energy conservation, is exact only when $\partial \theta_0 / \partial z$ vanishes, and thus is apparently the reason Ogura and Phillips (1962) chose an isentropic basic state.

The need for this substitution arises from the filtering of sound waves by elimination of α'/α_0 from the continuity equation; if the terms are retained an energy equation can be obtained without the need for the approximation as demonstrated by Eckart (1960) or by Eckart and Ferris (1956).

Thus, because the modification of the pressure term—a change which increases it about 15% in importance and is therefore consistent with the other approximations being made—arises from filtering, it would seem that the choice of the isentropic basic state in order to obtain energy equations is unnecessarily restrictive. It is, however, no longer true that the revised version of (5.14) becomes (5.16) for isentropic flow unless we also alter the definition of θ' with the same modification of the fractional pressure perturbation term.

The equations then, as derived for deep and shallow convection, may be stated as ($\delta_s = 1$ for shallow, $\delta_d = 1$

for deep)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha_0 \nabla p' + g \left(\delta_s \epsilon T' + \delta_a \frac{\alpha'}{\alpha_0} \right) \mathbf{k} + \nu_0 \left[\nabla^2 \mathbf{v} + \frac{\delta_a}{3} \nabla (\nabla \cdot \mathbf{v}) \right], \quad (8.2)$$

$$\delta_s \frac{d}{dt} (\epsilon T') + \delta_a \frac{d}{dt} \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{g \rho_0 H_\alpha} \right) + \frac{\omega_g^2}{g} w = \frac{T \epsilon}{c_p} \frac{ds}{dt}, \quad (8.3)$$

$$\nabla \cdot \mathbf{v} + \delta_a \frac{w}{\rho_0} \frac{\partial \rho_0}{\partial z} = 0, \quad (8.4)$$

$$\frac{\alpha'}{\alpha_0} = \epsilon T' - \delta_a \frac{p'}{p_0}, \quad (8.5)$$

in which the substitutions $\epsilon = T_0^{-1}$ and $\eta = p_0^{-1}$ are to be made for gases.

a. Boundary conditions

It is necessary to consider the conditions to be imposed upon the variables both in the medium and at the boundaries before proceeding to the determination of energy forms. As pointed out by Eckart (1960), it is convenient to confine the discussion to *regular solutions*, i.e., those for which the variables are differentiable and bounded everywhere in the medium and on the boundaries.

For the cases in which there is only a finite distance between the vertical boundaries, we shall require that w vanish on these vertical boundaries, be they rigid or free. In the latter case, these conditions filter external gravity waves from the solutions being considered. These conditions on w imply, according to (8.2) and (8.3), that at the boundaries

$$-\alpha_0 \frac{\partial p'}{\partial z} + g \left(\epsilon T' \delta_s + \frac{\alpha'}{\alpha_0} \delta_a \right) = 0, \quad (8.6)$$

$$\delta_s \frac{d}{dt} (\epsilon T') + \delta_a \frac{d}{dt} \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{g \rho_0 H_\alpha} \right) = 0, \quad (8.7)$$

in which we now adopt the isentropic assumption.

For the case of a fluid of infinite depth, as might be assumed for atmospheric problems, we shall require a rigid lower boundary at which the above conditions apply, and a diffuse upper region in which

$$\lim_{z \rightarrow \infty} p_0 = \lim_{z \rightarrow \infty} \rho_0 = \lim_{z \rightarrow \infty} \frac{\partial \rho_0}{\partial z} = 0. \quad (8.8)$$

We require that the perturbation pressure satisfy

$$\lim_{z \rightarrow \infty} p' = 0. \quad (8.9)$$

For shallow convection it is sometimes convenient to use the dynamic-kinematic upper boundary condition that

$$\lim_{s \rightarrow Z_T} dp/dt = 0, \quad (8.10)$$

where Z_T is the height of the upper boundary.

We shall use the condition of horizontal cyclic continuity in our analyses. If rigid walls are used as horizontal boundaries, appropriate conditions must be specified at these walls.

b. Energy forms

The form of energy which is to be used depends to some extent on the problem to which the equations are applied. For simplicity, we consider inviscid, isentropic motion in this section.

For deep convection we may take the scalar product of (8.2) and $\rho_0 v$ to obtain

$$\frac{\partial}{\partial t} \frac{\rho_0 v^2}{2} + \nabla \cdot \rho_0 \mathbf{v} \frac{v^2}{2} = -\nabla \cdot p' \mathbf{v} + \left(\frac{p'}{\rho_0 H_\alpha} + g \frac{\alpha'}{\alpha_0} \right) \rho_0 w, \quad (8.11)$$

and from (8.3) we find that

$$\frac{\partial}{\partial t} \left[\frac{\rho_0}{2 \omega_g^2} \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{\rho_0 H_\alpha} \right)^2 \right] + \nabla \cdot \left[\frac{\rho_0 \mathbf{v}}{2 \omega_g^2} \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{\rho_0 H_\alpha} \right)^2 \right] + \rho_0 w \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{\rho_0 H_\alpha} \right) = 0, \quad (8.12)$$

in which we have used a representative, but constant, value of ω_g^2 ; the linearized energy equations to be presented at the end of this section permit $\omega_g^2 = \omega_g^2(z)$. Upon integration over the entire fluid we have the energy conservation theorem

$$\frac{\partial}{\partial t} \int_V \frac{\rho_0}{2} \left[v^2 + \left(\frac{\alpha'}{\alpha_0} + \frac{p'}{\rho_0 H_\alpha} \right)^2 \omega_g^{-2} \right] dV = 0. \quad (8.13)$$

These equations make it clear that the change in the divisor of the perturbation pressure was necessary to secure a quadratic form. Linearization of the equation with the basic velocity $U = U(z)$ would yield a Reynolds stress term as an energy source.

The importance of the thermodynamic quantity which appears in the energy form was emphasized by Eckart and Ferris (1956) and has been given various names. As pointed out by Dutton and Johnson (1967), for ideal gases it is essentially a perturbation form of the available potential energy utilized in atmospheric studies. Thus, when frictional and diabatic effects are included we will not obtain the usual forms of energy losses and gains, but rather terms which show how these effects contribute to the rate of change of the sum of the kinetic and the available potential energy.

The rate of transformation from available to kinetic energy is given by

$$C(A,K) = \int_V \rho_0 w \left(g \frac{\alpha'}{\alpha_0} + \frac{p'}{\rho_0 H_\alpha} \right) dV. \quad (8.14)$$

We have assumed that w is bounded and therefore, because $|p'/g\rho_0 H_\alpha| = |p'/p_0| RT_0/(gH_\alpha) \ll RT_0/(gH_\alpha)$ and because $|\alpha'/\alpha_0| \ll 1$, if we assume that T_0 is bounded and that ρ_0 goes to zero fast enough so that the total mass of the basic state is finite, we can conclude that the rate of energy conversion is finite.

Two cases are possible in shallow convection. In the first, we neglect the term in the vertical equation of motion which expresses the effect of the density gradient on inertial forces and write

$$\alpha_0 \nabla p' = \nabla p' \alpha_0. \quad (8.15)$$

Then we have

$$\frac{\partial}{\partial t} \frac{v^2}{2} + \nabla \cdot \left(\frac{v^2}{2} \mathbf{v} \right) = -\nabla \cdot p' \alpha_0 \mathbf{v} + g \frac{T'}{T_0} w, \quad (8.16)$$

$$\frac{\partial}{\partial t} \frac{1}{2} \left(\frac{gT'}{\omega_g T_0} \right)^2 + \nabla \cdot \left(\frac{v}{2} \left(\frac{gT'}{\omega_g T_0} \right)^2 \right) + g \frac{v w}{T_0} = 0. \quad (8.17)$$

These equations, with the viscous term added, are thus appropriate for the study of turbulent motion forced by convection in a shallow layer because nonlinear energy conservation is assured once ω_g^2 is taken as a constant. It is worth noting that if the approximation (8.15) is not made, then the equations must be multiplied by $\rho_0 w$ to obtain a divergence from the pressure term, but that then a term $-v^2 \mathbf{v} \partial \rho_0 / \partial z$ appears in the inertial term. Thus, nonlinear energy equations of standard form do not seem to be possible without (8.15).

For both deep and shallow convection, we may linearize with v taken as a perturbation superimposed on a basic velocity profile $U(z)$ which is directed toward increasing x . Then the equation of motion yields

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \frac{\rho_0 v^2}{2} + \rho_0 w \frac{\partial U}{\partial z} = -\nabla p' \mathbf{v} + \left(\frac{p'}{\rho_0 H_\alpha} \delta_a + g \frac{\alpha'}{\alpha_0} \right) \rho_0 w, \quad (8.18)$$

and from the first law of thermodynamics we obtain

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left\{ \frac{\rho_0}{2\omega_g^2} \left[g \frac{\alpha'}{\alpha_0} + \delta_a \frac{p'}{\rho_0 H_\alpha} \right]^2 \right\} + \rho_0 w \left[g \frac{\alpha'}{\alpha_0} + \delta_a \frac{p'}{\rho_0 H_\alpha} \right] = 0. \quad (8.19)$$

These are the appropriate equations for the study of the linearized energetics of wave motion and provide

a harbinger of the development in the next two sections of a methodology for applying stability theorems for wave motion to shallow and deep convection problems simultaneously.

9. The linear wave equations

The assumption that the perturbation motion is inviscid, isentropic and harmonic in both time and the horizontal plane allows us to reduce the system (8.12)–(8.14) to a single equation with the vertical coordinate as the independent variable. Hence, we put, for each variable

$$\varphi(x, y, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\varphi}(\omega, \kappa_1, \kappa_2, z) e^{i(\omega t + \kappa_1 x + \kappa_2 y)} d\kappa_1 d\kappa_2, \quad (9.1)$$

where the eigenfrequency ω , which depends on κ_1, κ_2 and the structure of the basic state, must be determined subsequently. Since the solution is constructed from a complete function set, we may consider each Fourier mode separately, and may apply the boundary conditions to the Fourier coefficients individually.

It is convenient to define the canonical variables:

	Deep	Shallow
ξ	$\rho_0 v$	v
π	p'	$p' \alpha_0$
σ	α'/α_0^2	α'/α_0

which will aid in the simultaneous treatment of deep and shallow convection.

Substitution of the integrand of (9.1) into (8.2)–(8.4) and use of the canonical variables along with the notations $\Omega = (\omega + \kappa_1 U)$ and $D = d/dz$ gives the system of equations

$$i\Omega \hat{\xi}_j + \hat{\xi}_3 D U \delta_{j1} + i\kappa_j \hat{\pi} = 0, \quad j = 1, 2, \quad (9.2)$$

$$i\Omega \hat{\xi}_3 + (D - \delta_s H_\alpha^{-1}) \hat{\pi} - g \hat{\sigma} = 0, \quad (9.3)$$

$$i\Omega [\hat{\sigma} + \delta_a \hat{\pi} (g H_\alpha)^{-1}] + (\omega_g^2/g) \hat{\xi}_3 = 0, \quad (9.4)$$

$$i\kappa_1 \hat{\xi}_1 + i\kappa_2 \hat{\xi}_2 + D \hat{\xi}_3 = 0. \quad (9.5)$$

Combination of (9.5) and (9.2) and of (9.3) and (9.4) gives the two equations

$$\Omega D \hat{\xi}_3 - \kappa_1 \hat{\xi}_3 D U - i\kappa^2 \hat{\pi} = 0, \quad \kappa^2 = \kappa_1^2 + \kappa_2^2, \quad (9.6)$$

$$\Omega^2 \hat{\xi}_3 - i\Omega [D - (\delta_s - \delta_a) H_\alpha^{-1}] \hat{\pi} - \omega_g^2 \hat{\xi}_3 = 0. \quad (9.7)$$

Elimination of π between (9.6) and (9.7) yields

$$\Omega^2 D^2 \hat{\xi}_3 - \kappa_1 \Omega \hat{\xi}_3 D^2 U + \Omega H_\alpha^{-1} (\delta_a - \delta_s) [\Omega D \hat{\xi}_3 - \kappa_1 \hat{\xi}_3 D U] + \kappa^2 (\omega_g^2 - \Omega^2) \hat{\xi}_3 = 0. \quad (9.8)$$

The term multiplied by the δ 's arises from the inertia term (as can be seen by solving the wave equations with $\hat{\pi} = \hat{\sigma} = 0$) and thus represents the effect of stratification on the inertial forces.

Now the definition that

$$q_0 = \begin{cases} \rho_0, & \text{shallow convection} \\ \alpha_0, & \text{deep convection} \end{cases}, \quad (9.9)$$

allows (9.8) to be put in the form

$$\Omega^2 D(q_0 D \dot{\xi}_3) - \kappa_1 \Omega \dot{\xi}_3 D(q_0 DU) + \kappa^2 (\omega_\theta^2 - \Omega^2) q_0 \dot{\xi}_3 = 0. \quad (9.10)$$

This equation, which is mathematically equivalent to that considered in a variety of studies (e.g., Miles, 1961; Howard, 1961), is the basic differential equation for wave motion in either deep or shallow convection. It is worth noting that we have not made the approximation (8.15) in the case of shallow convection.

To determine the frequency limits associated with the linearized system of equations, let us put $U=0$, multiply (9.10) by the complex conjugate $\dot{\xi}_3$, and integrate in the vertical. Utilization of integration by parts and the boundary conditions at zero and at a top height Z_T (which may be infinite) gives

$$\int_0^{Z_T} \omega^2 |D \dot{\xi}_3|^2 q_0 dz = \kappa^2 \int_0^{Z_T} (\omega_\theta^2 - \omega^2) |\dot{\xi}_3|^2 q_0 dz. \quad (9.11)$$

Note that for both deep and shallow convection each of the velocity functions in the integrands is multiplied by either ρ_0 or $D\rho_0$ and so the integrals exist even in an infinite medium.

We observe that if the wavenumbers are real so that the motion is purely harmonic on horizontal planes, then the imaginary part of the frequency must vanish. With this restriction, every factor in (9.11) except for $(\omega_\theta^2 - \omega^2)$ is positive. If ω_θ is constant, the model requires that $|\omega| < \omega_\theta$; if ω_θ is not constant, it still must be true that $|\omega|$ is less than the maximum value of ω_θ present. If $DU=0$ but U = constant, then in (9.11) we would have Ω rather than ω and we would find a limiting frequency of $\omega = -\kappa_1 U \pm \omega_\theta$.

Therefore, the various approximations we have made have eliminated acoustic modes from the solutions of the linearized equations. This essentially occurs because of the neglect of the temporal change of specific volume in the equation of continuity. To use the phrase suggested by Ogura and Phillips (1962), the equations are soundproof if $DU=0$; some information on the frequencies permitted when arbitrary wind profiles are present will be developed in Section 10.

10. Stability theorems

Much of the interest in solutions of approximate perturbation equations such as we are considering here centers on whether the perturbations will amplify in time. The eigenfrequency ω is in general complex, and if the imaginary part of ω is negative, then according to (9.1), the magnitudes of the variables will grow exponentially. The question is whether conditions on the basic profile and the static stability can be deter-

mined which will guarantee stability or instability even though the actual solutions are not known.

A technique utilized by Howard (1961) provides an efficient method of generating stability theorems. We assume that the imaginary part ω_I of ω is not zero and put

$$\dot{\xi}_3 = \Omega^n F_n(z). \quad (10.1)$$

The quantity $\Omega = (\omega + \kappa_1 U)$ cannot vanish because ω_I does not.

Substitution of (10.1) into (9.10), multiplication by $\Omega^n \bar{F}_n$ (the overbar denotes complex conjugation), and integration produces

$$\begin{aligned} & \int_0^{Z_T} \Omega^{2n} \{ |DF_n|^2 + \kappa^2 |F_n|^2 \} q_0 dz \\ & - \int_0^{Z_T} \Omega^{2n-2} \{ n(n-1)\kappa_1^2 (DU)^2 + \kappa^2 \omega_\theta^2 \} |F_n|^2 q_0 dz \\ & + \int_0^{Z_T} \Omega^{2n-1} \{ (1-n)\kappa_1 D(q_0 DU) \} |F_n|^2 dz \\ & + \{ (n-1)\Omega^{2n-1} q_0 \kappa_1 DU |F_n|^2 \}_{z=Z_T} = 0. \quad (10.2) \end{aligned}$$

To obtain this result we have used several integrations by parts and the boundary condition that $\dot{\xi}_3 = \Omega^n F_n$ vanishes at the lower boundary in all cases. For finite distances between boundaries, the same condition eliminates all evaluations at the upper boundary. Therefore, the last term applies only in the case of an upper boundary at infinity, and was obtained with the aid of an evaluation of (9.6) at the diffuse upper boundary and with the deep convection requirement that $\lim_{z \rightarrow \infty} \hat{\pi} = 0$. Thus, (10.2) is valid for both deep and shallow convection in fluids with finite depths with the last term omitted, and for deep convection in infinite fluids with the last term retained. The last term contains a multiplication by $q_0 \rho_0^2 = \rho_0$ and thus vanishes if DU and U are bounded at infinity.

For $n = \frac{1}{2}$, the imaginary part of (10.2) yields Miles' (1961) theorem that the wave motion cannot be unstable if the Richardson number $Ri = \omega_\theta^2 / (DU)^2$ is such that

$$Ri > \kappa_1^2 / 4\kappa^2 \quad (10.3)$$

everywhere, and that if the motion is unstable, then the inequality must be reversed somewhere in the fluid. For $n=1$, (10.2) yields Howard's (1961) semicircle theorem and his equality on the growth rate is obtained with $n = \frac{1}{2}$. For finite depth, $n=0$ gives the Rayleigh theorem that the quantity

$$\kappa_1 D(q_0 DU) - 2(\omega_R + \kappa_1 U) q_0 \kappa^2 \omega_\theta^2 / |\Omega|^2 \quad (10.4)$$

must change sign somewhere if the motion is unstable (ω_R is the real part of ω). The theorem is also true when U and DU are bounded at infinity. Details of all these theorems are given in Howard's (1961) article.

An important consequence of our analysis of approximate equations is that these theorems can now be applied to deep convection in a finite or infinite atmosphere, and in particular, to the stability problems associated with atmospheric gravity waves.

It is possible to derive another result with (10.2), which so far as we know, has not appeared before. For $n=1$ and $\omega_\theta^2 \neq 0$, we obtain

$$\int_0^{Z_T} \Omega^2 |G_1|^2 q_0 dz - \int_0^{Z_T} \kappa^2 \omega_\theta^2 |F_1|^2 q_0 dz, \quad (10.5)$$

where

$$|G_1|^2 = |DF_1|^2 + \kappa^2 |F_1|^2, \quad (10.6)$$

and Z_T may be finite or infinite. Expansion of Ω^2 in (10.5) gives

$$\omega^2 I_2 + 2\omega I_1 + I_0 = 0, \quad (10.7)$$

where

$$I_0 = \int_0^{Z_T} \{(\kappa_1 U)^2 |G_1|^2 - \kappa^2 \omega_\theta^2 |F_1|^2\} q_0 dz, \quad (10.8)$$

$$I_1 = \int_0^{Z_T} \kappa_1 U |G_1|^2 q_0 dz, \quad (10.9)$$

$$I_2 = \int_0^{Z_T} |G_1|^2 q_0 dz. \quad (10.10)$$

The solution of (10.7) is

$$\omega = -\frac{I_1}{I_2} \pm \left(\frac{I_1^2}{I_2^2} - \frac{I_0}{I_2} \right)^{1/2}, \quad (10.11)$$

so that the necessary and sufficient condition for the assumed instability ($\omega_I \neq 0$) is

$$I_1^2 < I_0 I_2. \quad (10.12)$$

One consequence of this result⁵ is immediately apparent. The Schwarz inequality shows that

$$I_1^2 \leq I_2 \int_0^{Z_T} (\kappa_1 U)^2 |G_1|^2 q_0 dz = I_2 \left(I_0 + \int_0^{Z_T} \kappa^2 \omega_\theta^2 |F_1|^2 q_0 dz \right), \quad (10.13)$$

with equality only if U is constant. Thus, for all distributions of $U(z)$ which vary, however slightly, the motion is unstable if

$$\int_0^{Z_T} \omega_\theta^2 |F_1|^2 q_0 dz < 0. \quad (10.14)$$

Thus, if the atmosphere is statically unstable in the sense that $D\theta_0 < 0$ everywhere, then any perturbation motion controlled by these inviscid equations will be unstable. In any other case, the evaluation of the

condition (10.2) requires knowledge of the solutions of the equations. It would thus appear, except for the unlikely possibility that the integrals in (10.13) are invariant, that completely general sufficient conditions for instability which depend explicitly only on the external parameters will not be possible when the atmosphere is statically stable.

If the Richardson number is greater than $\kappa_1^2/(4\kappa^2)$ everywhere in the fluid we have stability; if it is negative everywhere we have instability provided that some shear exists. In the case of $\omega_\theta^2 \equiv 0$, one power of Ω is lost in (9.10) and the stability problem reduces to that of the inviscid Orr-Sommerfeld equation.

It therefore appears that when the Richardson number is less than $\kappa_1^2/(4\kappa^2)$ somewhere in the fluid but is not uniformly negative, each case must be treated separately until methods of obtaining general solutions of the eigenvalue problem for equations of the form of (9.10) are found.

Acknowledgments. We are indebted to Dr. Niels E. Busch, Danish Atomic Energy Commission, Risø, for several discussions of the material of this paper and for a thorough reading of the manuscript. His comments and suggestions led to improvement in the form and content of the article.

Profs. Alfred K. Blackadar and Hendrik Tennekes of The Pennsylvania State University both offered a series of constructive suggestions about the manuscript, and we are particularly grateful to an anonymous reviewer whose comments led to major improvements in this article.

The research reported here was partially supported by the National Science Foundation under Grant GA 1595X.

APPENDIX

Symbols Not Defined in Text

Physical Variables (All thermodynamic quantities in mechanical units)

p	pressure
T	absolute temperature
α	specific volume
ρ	density
θ	potential temperature: $\theta = T(1000/p)^{R/c_p}$ for ideal gases
c_v	specific heat at constant volume
c_p	specific heat at constant pressure
R	gas constant; $R = c_p - c_v$ for an ideal gas
\mathbf{v}	velocity vector
u	x component of \mathbf{v}
v	y component of \mathbf{v}
w	z component of \mathbf{v}
g	acceleration of gravity
h	specific enthalpy
s	specific entropy
G	Gibbs function

Coordinates

t time
 x, y, z Cartesian coordinates: z increases upward, x toward the east, y toward the north.

Mathematical operators

$\int_V dV$ integral over the entire fluid
 ∇ Hamiltonian operator (del); may be restricted by subscript
 d total derivative
 ∂ partial derivative

Subscripts

0 Refers to basic or reference state.
 T Implies evaluation on the top boundary.
 2 Restricts Hamiltonian operator to the horizontal plane.

REFERENCES

- Birkhoff, Garrett, 1960: *Hydrodynamics*. Princeton University Press, 184 pp.
- Boussinesq, J., 1903: *Theorie Analytique de Chaleur*, Vol. 2. Paris, Gauthier-Villars.
- Chandrasekhar, S., 1961: *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, 652 pp.
- Dutton, J. A., and D. R. Johnson, 1967: The theory of available potential energy and a variational approach to atmospheric energetics. *Advances in Geophysics*, Vol. 12, New York, Academic Press, 333–436.
- Eckart, C., 1960: *Hydrodynamics of Oceans and Atmospheres*. New York, Pergamon Press, 290 pp.
- , and H. G. Ferris, 1956: Equations of motion of the ocean and atmosphere. *Rev. Mod. Phys.*, **28**, 48–52.
- Hines, C. O., 1960: Internal atmospheric gravity waves at ionospheric heights. *Can. J. Phys.*, **38**, 1441–1481.
- Howard, L. N., 1961: Note on a paper of John W. Miles. *J. Fluid Mech.*, **10**, 509–512.
- Lin, C. C., 1955: *The Theory of Hydrodynamic Stability*. Cambridge University Press, 155 pp.
- Miles, John W., 1961: On the stability of heterogeneous shear flows. *J. Fluid Mech.*, **10**, 496–508.
- Oberbeck, A., 1879: Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperatur Differenzen. *Ann. Physik*, **7**, 271–292.
- Ogura, Y., and N. W. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, **19**, 173–179.
- Spiegel, E. A., and G. Veronis, 1960: On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, **131**, 442–447.
- Van Dyke, M., 1964: *Perturbation Methods in Fluid Mechanics*. New York, Academic Press, 229 pp.