# The Influence of Changes in Surface Roughness on the Development of the Turbulent Boundary Layer in the Lower Layers of the Atmosphere

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ABSTRACT

This paper describes the development of a turbulent boundary layer in a neutral atmosphere downwind of an abrupt change of surface roughness. Both a single change and two subsequent changes are treated.

For the treatment of a single abrupt change, a theory of Townsend forms the starting point. It is proved, according to this theory, that no surface layer adapted to the underlying surface roughness can exist behind the change. It is then shown how Townsend's theory can be modified in such a way that this discrepancy is removed.

The modified theory gives nearly the same velocity profile as does the original. However, it leads to greatly different values of the surface shear stress behind the change, a matter of importance in calculating turbulent transport.

It is shown that a correct evaluation of the height of the adapted layer is of importance for the determination of surface shear stress and roughness height from measured velocity profiles.

The modified form of Townsend's theory is then extended to the case of two subsequent abrupt changes of surface roughness. From a numerical example it is seen that the growth of the thickness of the adapted layer with distance downwind of the second abrupt change is of the same order as that behind the first change. The analysis can be extended in a similar way to the case of three or more subsequent abrupt changes of surface roughness.

## 1. Introduction

It is customary in micrometeorology to describe the air flow in the layers adjacent to the earth's surface as turbulent flow along a flat plate of uniform surface roughness. Under neutral or near-neutral conditions the wind profile is then given by

$$U_0=(\tau_0)^{\frac{1}{2}}k^{-1}\ln(z/z_0), \tag{1}$$

where $U_0$ denotes the mean flow velocity, assumed to be in the $x$ direction, parallel to the surface; $\tau_0$ the kinematic surface shear stress; $z_0$ the roughness height; $k$ von Kármán's constant; and $z$ the vertical coordinate. We shall denote this wind profile as that of a boundary layer adapted to the roughness of the underlying surface. Such an adapted turbulent boundary layer is only present if the wind has been blowing steadily over a region of homogeneous roughness for some time. In general, areas with more or less constant, though mutually differing, roughnesses alternate. It is reasonable to assume in passing such areas that the flow does not immediately adapt itself at all levels to the local surface roughness but does so only in a layer adjacent to the surface. The height of the layer, in which the influence of the new roughness is felt, the so-called internal boundary layer, increases with distance downwind from the point of change in roughness.

In this paper we shall closely examine the development of the internal boundary layer and the accompanying velocity profile in an originally adapted turbulent boundary layer behind a change of surface roughness. This problem has been previously studied by Elliott (1958), Panofsky and Townsend (1964) and Townsend (1965a, b). From a numerical example it will be shown that while the velocity profiles calculated according to the theories of these authors agree reasonably well, the calculated values of the surface shear stress differ widely. One would consequently expect the same to hold for other forms of turbulent transport, such as heat and water vapor.

We have attempted to indicate the height $h'$ of the layer adjacent to the surface which can be regarded as adapted to the new roughness. The new roughness height and the surface shear stress can then be deduced from the velocity profile

$$U=k^{-1}[\tau_1(x)]^{\frac{1}{2}}\ln(z/z_1),\quad \text{for } z<h'\leqslant h, \tag{1a}$$

where $h$ is the height of the internal boundary layer. It is proved for the profile derived by Townsend that such a layer does not exist. It will be shown that this is a

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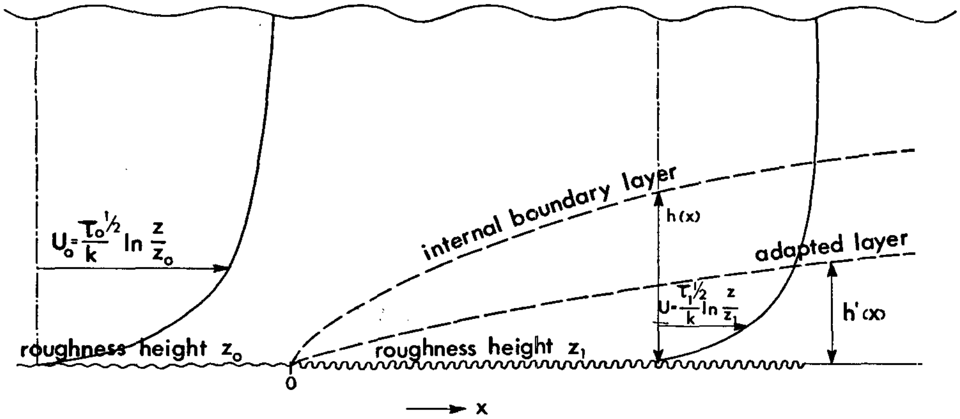


FIG. 1. Schematic representation of the development of an internal boundary layer behind a stepwise change of surface roughness.

consequence of a simplification introduced in the course of the analysis.

In this paper we shall show how this discrepancy can be removed. A revised profile will be derived that does satisfy the condition (1a).

An important result of the solution, clearly demonstrated by the given numerical example, was the fact that a complete adaptation of the boundary layer to the new surface roughness did not occur until the distance downstream of the abrupt change was of the order of several kilometers. This means in the atmosphere that we rarely meet a fully adapted turbulent boundary layer, because terrain is seldom found with a uniform roughness height over distances of that magnitude.

In view of this fact we have extended Townsend's theory to the more general case of two subsequent abrupt changes of surface roughness. The distance between the subsequent abrupt changes is assumed to be small in comparison with the distance required for complete adaptation to the surface roughness behind the first abrupt change. The development of the turbulent boundary layer behind the second abrupt change in surface roughness now represents a situation which occurs more frequently in the atmosphere than the development after a single abrupt change. The given solution is illustrated by a numerical example. It can be extended without much difficulty to the case of more than two subsequent abrupt changes in surface roughness.

## 2. A single abrupt change in surface roughness; Townsend's theory

Assume for $x<0$ a horizontal surface with a uniform roughness height $z_0$ and an adapted turbulent boundary layer in a neutral atmosphere. The mean velocity is in the positive $x$ direction. Then upstream of $x=0$, the vertical distribution of the mean velocity is given by Eq. (1), in which $\tau_0$ is independent of $x$. At $x=0$ an abrupt change of surface roughness occurs; for $x>0$, the roughness height is $z_1$ (see Fig. 1). We now want to calculate the influence of the change in roughness at $x=0$ on the wind profile and the surface shear stress.

Townsend introduced the assumption of a self-preserving development of the change of the velocity profile for $x>0$. It appears that the desired quantities can then be found by applying similarity arguments without the need arising for introducing further assumptions. There are two contributions to the change in velocity profile for $x>0$, one due to the acceleration $V(z)$ and the other due to the vertical displacement $\delta(z)$ of the streamlines. For $x>0$ the profile may then be written as

$$U=U_0+V(z)-u_0^*\delta(z)/(kz), \tag{2}$$

where $u_0^*=\tau_0^{\frac{1}{2}}$ is the friction velocity for $x<0$, and $U_0$ represents the original velocity profile given by (1). It is clear that for small values of $z$ the profile is rapidly adapted to the new friction velocity; thus,

$$U=[\tau_1(x)]^{\frac{1}{2}}k^{-1}\ln(z/z_1) \tag{3}$$

serves as an inner boundary condition. Here $u_1^*(x)=[\tau_1(x)]^{\frac{1}{2}}$ is the friction velocity for $x>0$.

Using the continuity equation it is possible to derive a relation between $\delta(z)$ and $V(z)$. To find a solution for $V(z)$, Townsend assumes the self-preserving form

$$V=u_1k^{-1}f(\eta), \tag{4}$$

in which $u_1$ is a velocity scale and $\eta=z/l_1$ with a length scale $l_1$. Both scales $u_1$ and $l_1$ are functions of $x$, whereas $f(\eta)$ is a universal function, independent of $x$ owing to the assumed self-preservation.

Townsend also assumes a self-preserving distribution of the shear stress of the form

$$\tau=(u_0^*)^2+\tau_sF(\eta), \tag{5}$$

in which $\tau_s=\tau_1-(u_0^*)^2$, so that $F(\eta)=1$ for $\eta=0$; $F(\eta)$ is also a universal function. Substitution of (4) and (5) in the equation of motion gives

$$-\eta(df/d\eta)=dF/d\eta, \qquad (6)$$

although an explicit form for $f(\eta)$ and $F(\eta)$ can be obtained only by making an assumption about the interaction between the velocity field and the turbulent motion. To do this Townsend uses the "mixing-length" transfer relation

$$dU/dz=\tau^{\frac{1}{2}}/(kz), \qquad (7)$$

which leads to

$$df/d\eta=\eta^{-1}F. \qquad (8)$$

The combination of (6) and (8) then gives

$$F(\eta)=e^{-\eta},\ f(\eta)=\mathrm{Ei}(-\eta), \qquad (9)$$

where Ei is the exponential integral.

Eq. (9) gives the solution for the velocity profile if the scales $u_1$ and $l_1$ are known. These scales are determined by using the inner boundary condition (3), together with the assumed universal character of $f(\eta)$ and $F(\eta)$.

Townsend finally arrives at

$$U=U_0+u_1k^{-1}[\mathrm{Ei}(-\eta)(1+P_1)-P_1(1-e^{-\eta})\eta^{-1}], \qquad (10)$$

where

$$P_1=[\ln(l_1/z_0)-C_0]^{-1}, \qquad (11)$$

$$l_1[\ln(l_1/z_1)-M-1]=2k^2x, \qquad (12)$$

$$M=\ln(z_0/z_1), \qquad (13)$$

$$u_1=-Mu_0^*\{[\ln(l_1/z_1)-C][1+P_1]\}^{-1}, \qquad (14)$$

and $C=\gamma$, $C_0=1+\gamma$.

For the surface shear stress we thus have

$$\tau_1^{\frac{1}{2}}=u_0^*+u_1(1+P_1). \qquad (15)$$

## 3. Discrepancy between Townsend's profile and his inner boundary condition

For small values of $\eta$ the function $\mathrm{Ei}(-\eta)$ can be approximated by $\ln\eta+\gamma$ and $\exp(-\eta)$ by $1-\eta$. Substitution of these approximations in (10) yields

$$U=U_0+u_1k^{-1}[(\ln\eta+\gamma)(1+P_1)-P_1]. \qquad (10a)$$

After some transformation this can be written as

$$U=k^{-1}\ln(z/z_1)[u_0^*+u_1(1+P_1)]-Mu_0^*k^{-1}-P_1u_1k^{-1}-u_1k^{-1}[\ln(l_1/z_1)-\gamma](1+P_1). \qquad (10b)$$

According to (15), the coefficient of $\ln(z/z_1)/k$ equals $\tau_1^{\frac{1}{2}}$ while (14) can be written as

$$Mu_0^*+u_1[\ln(l_1/z_1)-\gamma](1+P_1)=0, \qquad (15a)$$

which for (10b), leads to

$$U=\tau_1^{\frac{1}{2}}k^{-1}\ln(z/z_1)-u_1k^{-1}P_1. \qquad (10c)$$

For small values of $z$ the term $u_1k^{-1}P_1$ is not negligible with respect to $\tau_1^{\frac{1}{2}}k^{-1}\ln(z/z_1)$. This indicates that it is not possible, with the formulae given by Townsend, to indicate a layer satisfying the inner boundary condition (3). We shall show now how this discrepancy can be removed.

## 4. The revised Townsend profile

In the derivation of (14) for small $\eta$ Townsend used an approximation for the relation between $V(z)$ and $\delta(z)$, in which the integral $\int_0^\eta Vd\eta$ appeared. As a first approximation of this integral for small values of $\eta$ he used

$$\int_0^\eta Vd\eta=\eta V.$$

However, having found the relations (9), we are now able to give an exact expression for this integral, i.e.,

$$\int_0^\eta Vd\eta=u_1k^{-1}\int_0^\eta \mathrm{Ei}(-\eta)d\eta$$
$$=u_1k^{-1}[\eta\mathrm{Ei}(-\eta)-1+e^{-\eta}]. \qquad (16)$$

For small $\eta$, Eq. (16) gives

$$\int_0^\eta Vd\eta=\eta V-\eta u_1k^{-1}, \qquad (16a)$$

leading to

$$u_1=-Mu_0^*(1+P_1)^{-1}[\ln(l_1/z_1)-C+(1+P_1^{-1})^{-1}]^{-1} \qquad (17)$$

as a revised expression for the velocity scale, without altering the expressions found for $f(\eta)$, $F(\eta)$ and $l_1$.

It is now easy to prove that a domain satisfying the inner boundary condition does exist, although differing from that given by Townsend. While the greatest discrepancies occur close to the surface, it is clear from (15) that the surface shear stress also undergoes a marked change. This change is very important for the calculation of transport phenomena.

## 5. Two subsequent abrupt changes in surface roughness

### *a. The profile velocity* $V_2(z)$

We again assume for $x<0$ a horizontal surface with a uniform roughness height $z_0$ and an adapted turbulent boundary layer in a neutral atmosphere. The mean velocity is in the positive $x$ direction. At $x=0$ an abrupt change of surface roughness occurs such that for $0<x<L$ the roughness height is $z_1$. At $x=L$ there is another abrupt change of surface roughness such that for $x>L$ the roughness height is $z_2$ (see Fig. 2).

We have assumed that the distance $L$ is small with respect to the distance needed for a complete adaptation of the turbulent boundary layer to the roughness height

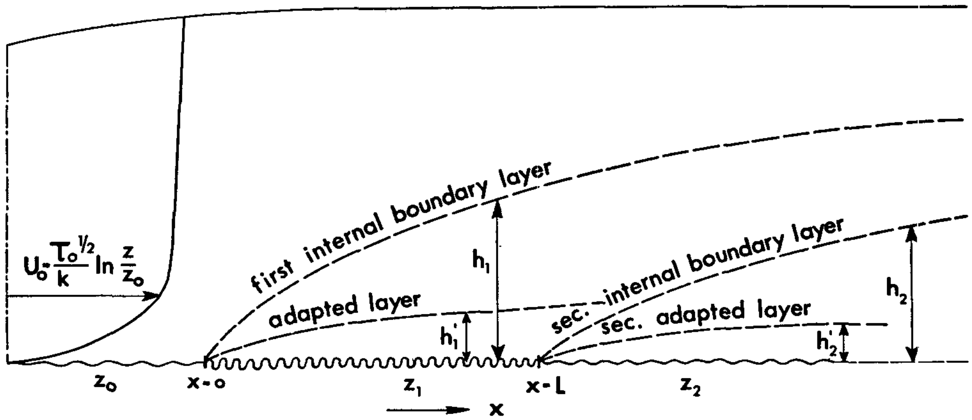


FIG. 2. Schematic representation of the development of a second internal boundary layer behind the second abrupt change of surface roughness.

$z_1$. Upstream of $x=0$, the vertical distribution of the mean velocity is $U_0(z)$, given by Eq. (1). The velocity profile $U(z)$ and the distribution of the surface shear stress $\tau_1(x)$ for $0<x<L$ are given by Eqs. (10) and (15), respectively.

We now want to calculate the influence of the second abrupt change in roughness at $x=L$ on the wind profile and on the distribution of the surface shear stress for $x>L$. Our treatment of this problem is to a great extent analogous to the solution given by Townsend for the case of one single abrupt change in surface roughness.

We shall indicate the velocity profile for $x>L$ by $U_2=U_2(z)$. The production of turbulent energy per unit mass at $x\approx L$ is of the order of

$$\tau_1(L)\partial U/\partial z \simeq [\tau_1(L)]^{\frac{3}{2}}/kz, \tag{18}$$

where $\tau_1(L)$ is the kinematic surface shear stress at $x=L$. Since the turbulent kinetic energy per unit mass is of the order of $3\tau_1(L)$, within a time $t\ll 3kz[\tau_1(L)]^{-\frac{1}{2}}$, only small changes can occur in this kinetic energy. During this same time $t$ the fluid has traveled a distance $x_1 \ll x-L$, where, in comparison, we have

$$(x-L)=3kzU[\tau_1(L)]^{-\frac{1}{2}}. \tag{19}$$

Eq. (19) defines a surface $z(x)$, which we shall name the second critical surface.

From the above considerations it follows that at great heights above the second critical surface the stress gradient has the same value as upstream; thus, flow acceleration is negligible and the only modification in the velocity profile is caused by a vertical displacement of the streamlines.

On the other hand, as soon as a parcel of air has traveled a distance large compared with $3kzU\tau_1(L)^{-\frac{1}{2}}$, a complete adaptation to the local situation must have taken place. This means, that in a layer adjacent to the surface, with height small compared with the height of the second critical surface, the turbulent boundary layer has adapted itself to the roughness of the underlying surface. Hence, in that layer the velocity profile is given by

$$U_2=\tau_2^{\frac{1}{2}}k^{-1}\ln(z/z_2), \tag{20}$$

with $\tau_2=\tau_2(x)$ being the kinematic surface shear stress for $x>L$; Eq. (20) will be called the inner boundary condition. It is clear that the height of the second critical surface can be interpreted as a measure of the height of the second internal boundary layer. We shall describe the velocity profile $U_2$ as the change in the velocity profile $U(z)$ given by Eq. (10) which at the same distance, $x>L$, would exist if there were no abrupt change in roughness at $x=L$. The change in the velocity profile $U(z)$ occurs as a result of the streamline displacement $\delta(z)$ and accelerations of the flow; this contribution to the change in $U(z)$ is called $V_2(z)$.

For the velocity profile $U_2(z)$ we now can write

$$U_2(z)=U(z)-\delta(z)\partial U/\partial z+V_2(z). \tag{21}$$

The problem is to find solutions for $V_2(z)$ and $\delta(z)$. We first use the equation of continuity to derive a relation between $V_2(z)$ and $\delta(z)$; thus,

$$u_0{}^*\delta k^{-1}=-[\ln(l_2/z_1)-C_{0,2}]^{-1}\int_0^z V_2(\zeta)d\zeta, \tag{22}$$

where as a first approximation for small values of $z$,

$$u_0{}^*\delta k^{-1}=-V_2z[\ln(l_2/z_1)-C_{0,2}]^{-1}. \tag{23}$$

The derivation of (22) is given in Appendix A. The velocity profile is now determined if we can solve for $V_2(z)$.

*b. Determination of $V_2(z)$ using the self-preserving character of the flow*

If the distribution $V_2(z)$ is assumed to be self-preserving, we can write

$$V_2(z)=u_2k^{-1}f_2(z/l_2), \tag{24}$$

where $u_2$ is the velocity scale for $x>L$, $l_2$ the length scale for $x>L$, and $f_2(z/l_2)$ a universal function, independent of $x$. Using (23) and (10), we find for small values of $z$ [owing to the inner boundary condition, Eq. (20)] that

$$f_2(z/l_2)=u_2^{-1}(P_2+1)^{-1}\{[\tau_2^{\frac{1}{2}}-u_0^*-u_1(1+P_1)]\times\ln(z/z_2)+u_0^*M_2-u_1(1+P_1)\ln(z_2/l_1)+u_1\gamma(1+P_1)+u_1P_1\},\quad (25)$$

where

$$P_2=u_0^{*-1}[u_0^*+u_1(1+P_1)][\ln(l_2/z_1)-C_{0,2}]^{-1},\quad (26)$$

and

$$M_2=\ln(z_0/z_2).\quad (27)$$

We now choose the velocity scale $u_2$ so that

$$\tau_2^{\frac{1}{2}}=u_0^*+u_1(1+P_1)+u_2(1+P_2).\quad (28)$$

The expression

$$\ln(l_2/z_2)+u_2^{-1}(P_2+1)^{-1}[u_0^*M_2-u_1(1+P_1)\times\ln(z_2/l_1)-u_1\gamma(1+P_1)+u_1P_1]$$

must then be a constant, say $C_2$, independent of $x$. Thus,

$$f_2(\eta_2)=\ln\eta_2+C_2,\ \eta_2=z/l_2,\quad (29)$$

while the first approximation to the velocity scale is

$$u_2=-[\ln(l_2/z_2)-C_2)]^{-1}(P_2+1)^{-1}[u_0^*M_2-u_1(1+P_1)\times\ln(z_2/l_1)-u_1\gamma(1+P_1)+u_1P_1].\quad (30)$$

Comparison with (14) shows that $u_2$ and $u_1$ are of the same order of magnitude. Based on the supposed self-preserving character of the flow, we assume

$$\tau=\tau_1+\tau_{s2}F_2(\eta_2),\quad (31)$$

in which $\tau_{s2}=\tau_2-\tau_1$, so that $F_2(\eta_2)=1$ for $\eta_2=0$. Thus, $F_2(\eta_2)$ is another universal function, independent of $x$, which implies that the change in the shear stress is also assumed to have a self-preserving form.

Making use of (28), we find for large values of $l_1/z_1$ that

$$\tau_{s2}\simeq 2u_2u_0^*.\quad (32)$$

Substituting (21), (24), (31) and (32) in the equation of motion, assuming $(U_2-U)\ll U$, and ignoring the change in velocity due to the displacement of the streamlines in regard to $V_2(z)$, we find with the help of the continuity equation, that

$$U[f_2du_2/dx-u_2\eta_2l_2^{-1}(dl_2/dx)df_2/d\eta_2]+(dU/dz)\{u_2\eta_2f_2dl_2/dx-[d(u_2l_2)/dx]\int f_2d\eta_2\}=2ku_2u_0^*l_2^{-1}dF_2/d\eta_2.\quad (33)$$

Since $f_2$ and $F_2$ are universal functions independent of $x$, (33) must be valid independent of $x$, which means that the coefficients of $f_2$, $\eta_2f_2$, $\eta_2df_2/d\eta$, $\int f_2\eta_2$ and $dF_2/d\eta_2$ must either be constant or negligibly small. For large values of $\ln(l_2/z_2)$ we may ignore all coefficients on the left-hand side of Eq. (33) with regard to the coefficient of $\eta_2df_2/d\eta_2$. For the coefficient of $\eta_2df_2/d\eta_2$ we have as a good approximation

$$Uu_2l_2^{-1}dl_2/dx\simeq u_0^*{}_2(kl_2)^{-1}\ln(l_2/z_0)dl_2/dx.$$

Since this must equal the coefficient of $dF_2/d\eta_2$, we have

$$(dl_2/dx)\ln(l_2/z_0)=2k^2.\quad (34)$$

After integration, using the boundary condition $l_2=0$ for $x=L$, we find that

$$l_2[\ln(l_2/z_1)-M-1]=2k^2(x-L),\quad (35)$$

giving for (33) the final result

$$-\eta_2df_2/d\eta_2=dF_2/d\eta_2.\quad (36)$$

Making use of the "mixing-length" transfer relation (7), we obtain

$$\partial U/\partial z+u_2(kl_2)^{-1}df_2/d\eta_2=(kz)^{-1}(\tau_1+2u_2u_0^*F_2)^{\frac{1}{2}}$$

as another relation between $f_2(\eta_2)$ and $F_2(\eta_2)$; to a close approximation we then have

$$\eta_2df_2/d\eta_2=F_2.\quad (37)$$

The combination of (36) and (37) leads to

$$F_2(\eta_2)=e^{-\eta_2},\quad (38)$$

and

$$f_2(\eta_2)=\mathrm{Ei}(-\eta_2)=-\int_{\eta_2}^{\infty}e^{-\zeta}\zeta^{-1}d\zeta.\quad (39)$$

These solutions appear to be completely analogous to (9) of Section 3. For small values of $\eta_2$ the function $\mathrm{Ei}(-\eta_2)$ can be approximated by $\ln\eta_2+\gamma$, which in combination with (29) gives

$$C_2=\gamma.\quad (40)$$

Finally, the constant $C_{0,2}$ of Eq. (22) is found to be (see also Appendix A)

$$C_{0,2}=1+\gamma+(u_1/u_0^*)(1+P_1)+M-u_1(1+P_1)u_0^{*-1}\times\ln(l_2/l_1)+P_1u_1/u_0^*.\quad (40a)$$

Substitution of the above given formulae in (24), (22) and (21) gives for the velocity profile:

$$U_2=U+u_2k^{-1}[(P_2+1)\mathrm{Ei}(-\eta_2)-(1-e^{-\eta_2})\eta_2^{-1}(P_2)].\quad (41)$$

*c. A more accurate approximation for the velocity scale* $u_2$

In a manner analogous to Section 3, it is easy to prove, for small values of $z$, that the velocity profile $U_2(z)$ given in (41) together with the expressions for $u_2$ and $\tau_2$ does not satisfy the inner boundary condition, Eq. (20). Again, this discrepancy is caused by the poor approximation of $V_2z$ for $\int_0^zV_2dz$ for small values of $z$. Since $V_2(z)$ is known, we now use (16a) as a better

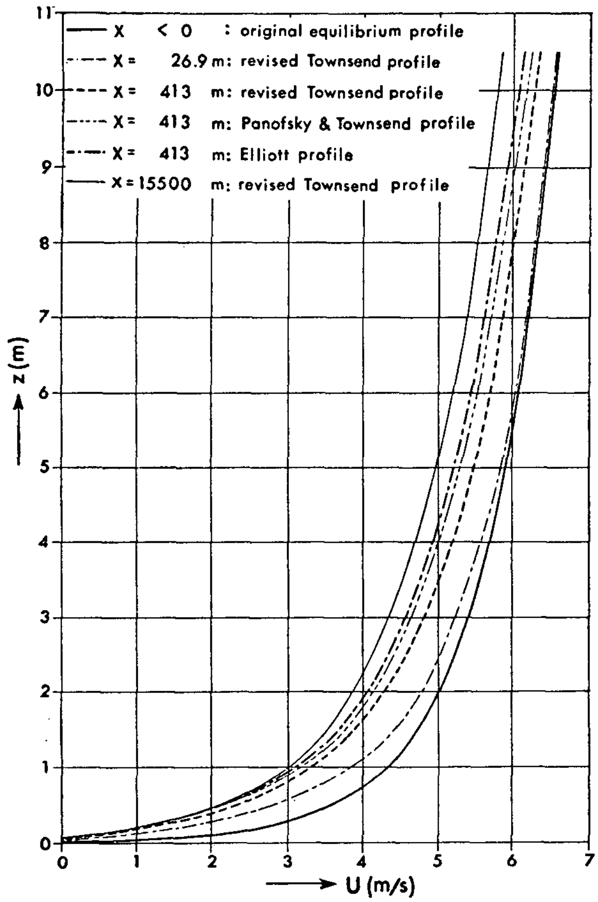


FIG. 3. Some calculated velocity profiles behind a single abrupt change of surface roughness. See text for modal parameters.

approximation for this integral to obtain a modified form of the velocity scale $u_2$; thus,

$$u_2 = -(P_2+1)^{-1}[\ln(l_2/z_2) - C_2 + 1 - (P_2+1)^{-1}]^{-1} \times [u_0{}^* M_2 - u_1(1+P_1)\ln(z_2/l_1) - u_1\gamma(1+P_1) + u_1 P_1]. \quad (30a)$$

The expressions found for $f_2(\eta_2)$ and $l_2$ remain unchanged. It is easy to prove now that a domain, satisfying the inner boundary condition (20), does exist.

TABLE 1. Calculated length and velocity scales, and depths of the internal boundary layer, at various distances downwind of a single abrupt change in surface roughness. See text for assumed initial conditions.

| $x$ (m) | $l_1$ [Eq. (14)] (m) | $u_1$ [Eq. (14)] (m sec$^{-1}$) | $u_1$ [Eq. (17)] (m sec$^{-1}$) | $h$ [Elliott] (m) | $h$ [Panofsky and Townsend] (m) |
|---|---|---|---|---|---|
| 11.3 | 1 | 0.2798 | 0.2493 | 2.85 | 6.60 |
| 18.8 | 1.5 | 0.2403 | 0.2200 | 4.28 | 9.00 |
| 26.9 | 2 | 0.2187 | 0.2029 | 5.70 | 11.5 |
| 81.5 | 5 | 0.1707 | 0.1628 | 13.9 | 22.0 |
| 185 | 10 | 0.1468 | 0.1416 | 26.7 | 43.5 |
| 413 | 20 | 0.1289 | 0.1253 | 50.3 | 80.0 |
| 1175 | 50 | 0.1111 | 0.1038 | 117.1 | 177 |
| 2600 | 100 | 0.0993 | 0.0976 | 221 | 345 |
| 15,500 | 500 | 0.0817 | 0.0808 | (922) | — |

## 6. Numerical examples and conclusions

### *a. A single abrupt change in surface roughness*

Features of the previously treated theories and the difference between the Townsend and revised profiles can best be illustrated by numerical examples, using appropriate numerical values for various quantities occurring in the formulae.

For $x<0$ we assume $z_0=10$ mm and take $U_0$ as 5 m sec$^{-1}$ at $z=2$ m, and the thickness of the layer having a logarithmic velocity distribution as 50 m (Rijkoort, 1968). For $x>0$ we assume $M=-2$ so that $z_1=73.9$ mm. For $x<0$ and $z_0<z\leqslant 50$, we use the velocity distribution according to (1). This gives $u_0{}^*=0.3775$ m sec$^{-1}$. The resulting characteristic length and velocity scales from Townsend [Eqs. (12) and (14)] and from the revised approximation, Eq. (17), as well as the $h$ values from Elliott and from Panofsky and Townsend are given in Table 1 for various distances downwind of a single abrupt change in surface roughness, $h$ denoting the thickness of the internal boundary layer. Some calculated wind profiles are given in Table 2 and Fig. 3, while the surface shear stresses based on the various

TABLE 2. Calculated velocity profiles (m sec$^{-1}$) at various distances downwind, for a single abrupt change of surface roughness, obtained from Elliott (E), Panofsky and Townsend (P&T) and Townsend's revised theory (T). See text for assumed initial conditions.

| $z$ (m) | $U(z)$ ($x<0$) | $U(z)$ ($x=26.9$ m) | | | $U(z)$ ($x=185$ m) | | | $U(z)$ ($x=413$ m) | | | $U(z)$ ($x=2600$ m) | | | $U(z)$ ($x=15{,}500$ m) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | E | P&T | T | E | P&T | T | E | P&T | T | E | P&T | T | T |
| 0.1 | 2.17 | 0.42 | 0.42 | 0.45 | 0.38 | 0.39 | 0.40 | 0.37 | 0.38 | 0.39 | 0.36 | 0.36 | 0.37 | 0.34 |
| 0.2 | 2.83 | 1.37 | 1.40 | 1.53 | 1.26 | 1.29 | 1.35 | 1.23 | 1.25 | 1.30 | 1.17 | 1.19 | 1.25 | 1.14 |
| 0.3 | 3.21 | 1.93 | 1.96 | 2.14 | 1.77 | 1.81 | 1.90 | 1.73 | 1.76 | 1.84 | 1.65 | 1.68 | 1.74 | 1.60 |
| 0.5 | 3.69 | 2.63 | 2.68 | 2.90 | 2.42 | 2.47 | 2.59 | 2.36 | 2.41 | 2.49 | 2.26 | 2.29 | 2.35 | 2.18 |
| 1 | 4.35 | 3.59 | 3.63 | 3.89 | 3.29 | 3.36 | 3.52 | 3.21 | 3.28 | 3.40 | 3.07 | 3.12 | 3.21 | 2.96 |
| 1.5 | 4.73 | 4.15 | 4.18 | 4.41 | 3.81 | 3.89 | 4.06 | 3.71 | 3.78 | 3.91 | 3.55 | 3.60 | 3.70 | 3.42 |
| 2 | 5.00 | 4.55 | 4.57 | 4.78 | 4.17 | 4.25 | 4.42 | 4.07 | 4.14 | 4.28 | 3.89 | 3.95 | 4.05 | 3.75 |
| 3 | 5.38 | 5.10 | 5.10 | 5.24 | 4.68 | 4.77 | 4.94 | 4.57 | 4.65 | 4.80 | 4.37 | 4.43 | 4.53 | 4.21 |
| 5 | 5.87 | 5.81 | 5.74 | 5.80 | 5.34 | 5.42 | 5.58 | 5.20 | 5.29 | 5.45 | 4.97 | 5.04 | 5.16 | 4.79 |
| 10 | 6.52 | 6.52 | 6.51 | 6.49 | 6.20 | 6.27 | 6.38 | 6.05 | 6.14 | 6.27 | 5.79 | 5.87 | 5.99 | 5.58 |
| 20 | 7.18 | 7.18 | 7.18 | 7.16 | 7.08 | 7.09 | 7.13 | 6.91 | 6.97 | 7.06 | 6.61 | 6.69 | 6.81 | 6.38 |
| 50 | 8.04 | 8.04 | 8.04 | 8.03 | 8.04 | 8.03 | 8.03 | 8.04 | 8.01 | 8.01 | 7.69 | 7.77 | 7.86 | 7.41 |

TABLE 3. Calculated surface shear stresses $\tau_w$(N m$^{-2}$) at various distances downwind for a single change of surface roughness, the initial $x<0$ equillibrium value being 0.1710 N m$^{-2}$. See text for assumed initial conditions.

| $x$ (m) | $\tau_w$ (Elliott) | $\tau_w$ (Panofsky and Townsend) | $\tau_w$ (Townsend) | $\tau_w$ (Townsend, revised) |
|---|---|---|---|---|
| 11.3 | 0.3135 | 0.4230 | 0.6745 | 0.6043 |
| 18.8 | 0.3025 | 0.3983 | 0.5677 | 0.5251 |
| 26.9 | 0.2958 | 0.3817 | 0.5148 | 0.4837 |
| 81.5 | 0.2785 | 0.3477 | 0.4107 | 0.3973 |
| 185 | 0.2687 | 0.3219 | 0.3654 | 0.3573 |
| 413 | 0.2607 | 0.3043 | 0.3343 | 0.3290 |
| 1175 | 0.2518 | 0.2867 | 0.3055 | 0.3024 |
| 2600 | 0.2462 | 0.2752 | 0.2876 | 0.2854 |
| 15,500 | — | — | 0.2627 | 0.2617 |

theories are given in Table 3 and Fig. 4. While the calculations in most cases, as explained below, have been carried out for a distance downstream equal to 15,500 m, it is difficult to say how far downstream the developed formulae hold.

The length parameter $l$ is of the same order of magnitude as the height of the critical surface [see (19) and (35)]. The total thickness of the boundary layer in the atmosphere, however, is difficult to assess. From the numerical example given here, it follows that the layer adapted to the roughness $z_1$ has a thickness $h'$ of the order of magnitude of $0.1\,l_1$. This order of magnitude is in reasonable agreement with the results of Brooks (1961). The variation of $h'$ with distance according to the various theories is given in Fig. 5. From this it follows that a reasonable estimation of the new equilibrium profile is found at a distance for which $h'=50$ m; this gives $x=15{,}500$ m ($l_1=500$ m).

While the various theories do not give any significant differences in the calculated velocity profiles, there are important differences in the variation of the surface shear stress with $x$, the salient parameter involved in calculations of turbulent heat and mass transport. The usual procedure for obtaining these quantities involves the measurement of mean velocity $U$ at different values of $z$. These are then plotted as a function of $\ln z$ and a straight line fitted through the plotted points. The value of $\tau$ can then be calculated from the slope of this line, and the roughness height $z_0$ from its intersection with the $\ln z$ axis. This procedure is only valid when the heights at which the velocity is measured are smaller than $h'$.

When the plotted points do not fall on a straight line, a zero-plane displacement $D$ is often introduced, so that the points fit the following modified form of Eq. (1):

$$U=\tau_0^{\frac{1}{2}}k^{-1}\ln[(z-D)/z_0]. \tag{42}$$

Again the assumption that the measured values belong to the adapted part on the boundary layer is tacitly introduced. In addition it is assumed that the layer $0<z<D$ does not contribute to the roughness of the surface and therefore does not influence the airflow. It should be noted that the applicability of Eqs. (1) and (42) is confined to $z$ values for which $z/z_0\gg 1$ and $(z-D)/z_0\gg 1$, respectively.

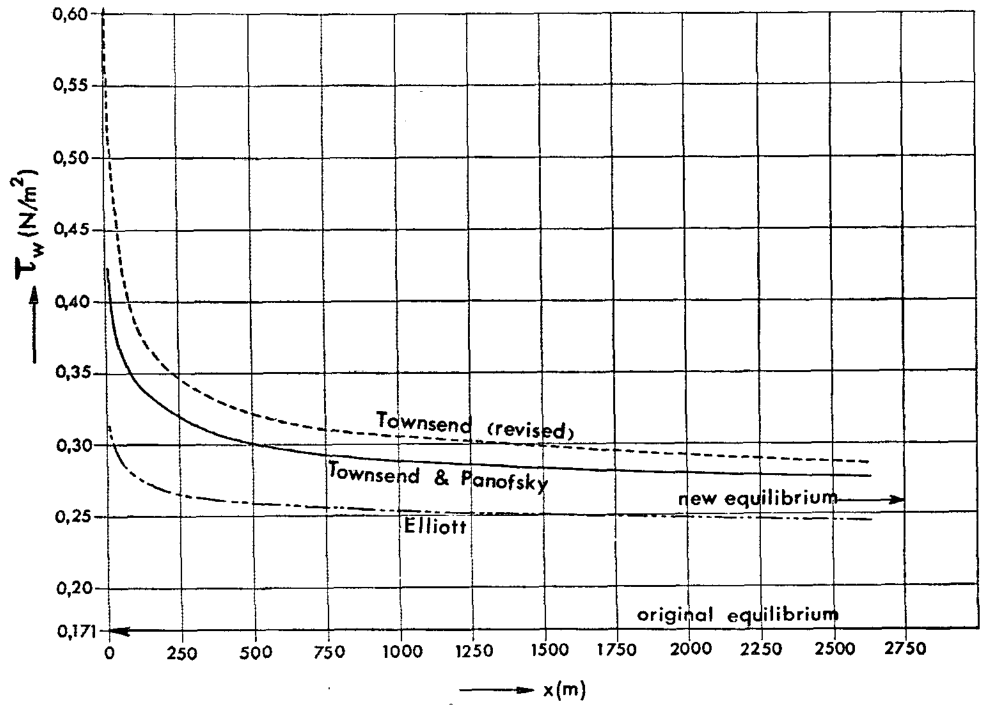


FIG. 4. The change of surface shear stress downwind of a single abrupt change in surface roughness.

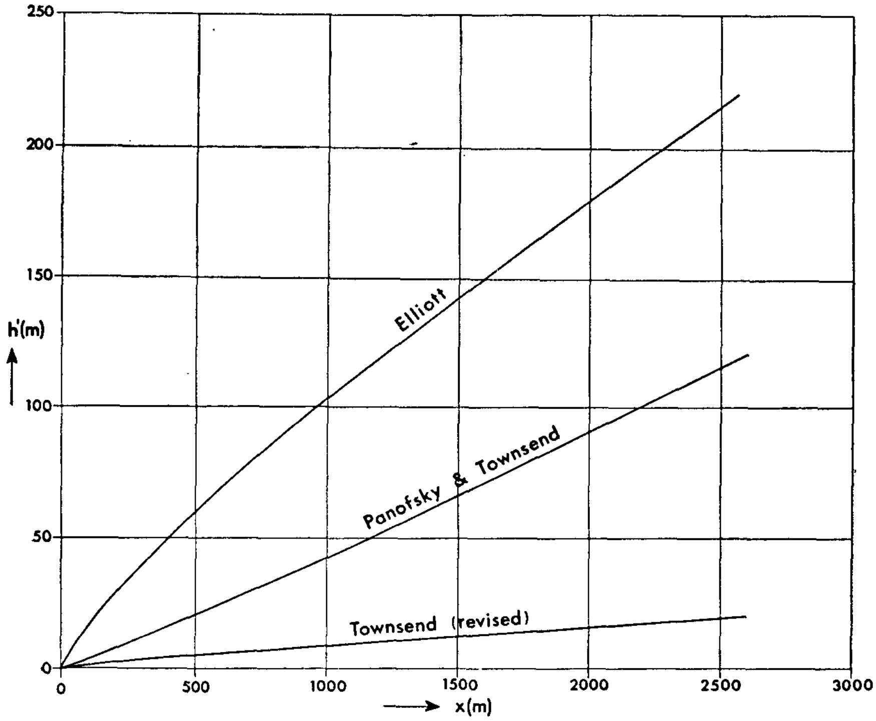


FIG. 5. The growth of the adapted layer with distance downwind of a single abrupt change in surface roughness.

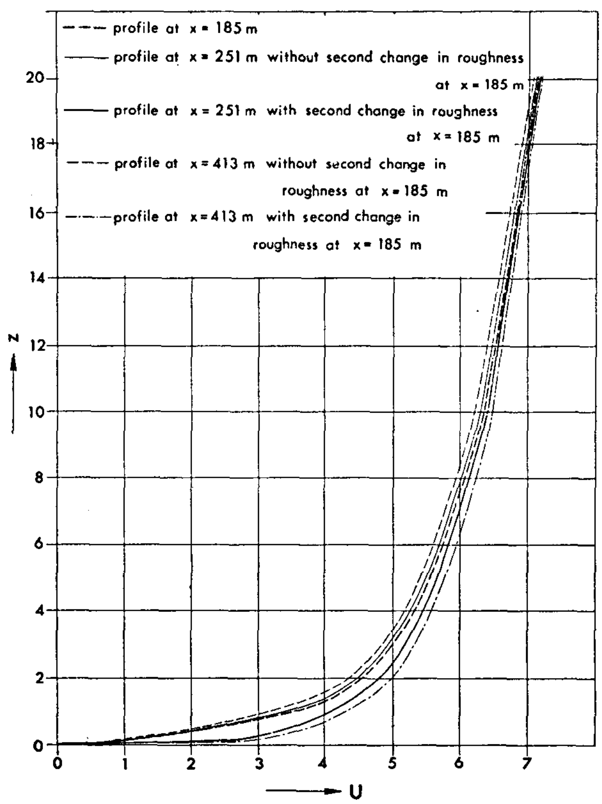


FIG. 6. Some calculated velocity profiles for two subsequent abrupt changes in surface roughness.

Our results, however, throw considerable doubt on the above procedures for finding $\tau$, $z_0$ and $D$, as it appears, for instance, that for our calculated profiles one can still draw reasonably good straight lines through the points for which $z > 0.1l_1$ in a plot of $U$ vs $\ln z$. Such a line, however, gives values for $z_1$ and $\tau_1$ which are too low. From Townsend's theory and its extension given here it appears that one must be very careful in determining the surface roughness height and the surface shear stress from measured velocity profiles. In each case a careful analysis should be made of the possible influence of changes in upwind surface roughness.

*b. Two subsequent abrupt changes in surface roughness*

In this case we assume the same conditions for $x$ given in the previous example for all $x < L$. For the distance $L$ to the second abrupt change of surface roughness we choose the value of 185 m, while the roughness height $z_2$ for $x > L$ is the same as $z_0$, i.e., 10 mm. The resulting velocity profiles (shown in Table 4 and Fig. 6) give the following results at $x = 251$ m: $l_2 = 4.17$ m, $C_{0,2} = 0.453$, $u_2 = -0.1327$ m sec$^{-1}$; and at $x = 413$ m: $l_2 = 11.98$ m, $C_{0,2} = 0.217$, $u_2 = 0.1210$ m sec$^{-1}$. Other profiles given in Table 4 are those for $x < 0$ and $x = 185$ m ($x = L$) as well as that for $x = 413$ m with no change of roughness at $x = 185$ m.

The surface shear stress is given in Table 5 for values of $x$ between 185 and 413 m, the values being shown as a function of $x$ in Fig. 7.

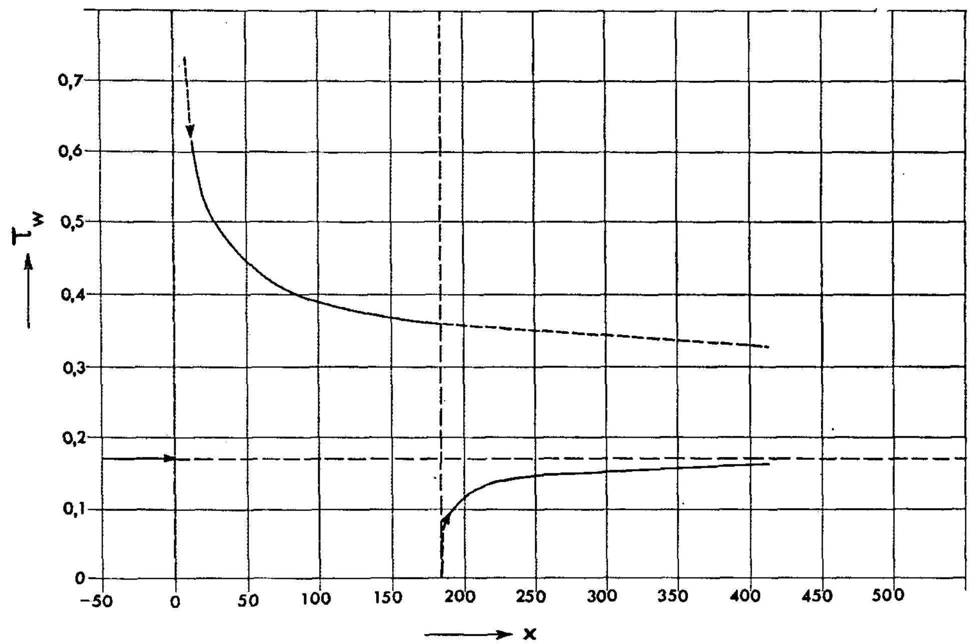


FIG. 7. The change of surface shear stress due to two subsequent abrupt changes in surface roughness.

As was to be expected, the surface shear stress behind the second change drops below the new equilibrium value for $\tau_{2,w}$ of 0.1710 N m². At $x=251$ and 413 m the thicknesses of the layers for which the calculated velocity profile [Eq. (41)] differs by less than 1% from the adapted velocity profile [Eq. (20)] are of the order of 1 and 3 m, respectively. This means that the adaption of the boundary layer downwind of the second abrupt change in surface roughness is of the same order of magnitude as the adaptation downwind of the first abrupt change in roughness height. This is not surprising since a self-preserving development has been assumed for both changes in the boundary.

In the case of the abrupt changes of surface roughness we can then divide the turbulent boundary layer into the following four regions, from the surface upward:

1) A layer, adjacent to the surface, with height $h_2'$ adapted to the underlying surface roughness $z_2$.
2) A layer, with $h_2'<z<O(l_2)$, where the velocity profile is determined by the roughness heights $z_1$ and $z_2$.
3) A layer, with $O(l_2)<z<O(l_1)$, where the velocity profile is fully determined by the influence of the first abrupt change in roughness on the original profile for $x<0$, i.e., it depends on the roughness heights $z_0$ and $z_1$.
4) A layer $z>O(l_1)$, where the velocity profile equals the original velocity profile for $x<0$.

The above given analysis can be extended to the case of three or more subsequent abrupt changes in surface roughness. In this case the turbulent boundary layer

TABLE 4. Calculated velocity profiles (m sec⁻¹) at various downwind distances for two abrupt changes in surface roughness. See text for details.

| $z$ (m) | Original equilibrium profile for $x<0$ | Profile at $x=185$ m | Profile at $x=251$ m | Profile at $x=413$ m | Profile at $x=413$ m without change of roughness at $x=185$ m |
|---|---|---|---|---|---|
| 0.1 | 2.17 | 0.40 | 2.02 | 2.12 | 0.40 |
| 0.2 | 2.83 | 1.35 | 2.63 | 2.75 | 1.30 |
| 0.3 | 3.21 | 1.90 | 3.00 | 3.15 | 1.84 |
| 0.5 | 3.69 | 2.59 | 3.45 | 3.61 | 2.50 |
| 1.0 | 4.35 | 3.52 | 4.09 | 4.26 | 3.40 |
| 1.5 | 4.73 | 4.06 | 4.48 | 4.62 | 3.91 |
| 2 | 5.00 | 4.42 | 4.75 | 4.89 | 4.28 |
| 3 | 5.38 | 4.94 | 5.13 | 5.28 | 4.80 |
| 5 | 5.87 | 5.58 | 5.68 | 5.79 | 5.45 |
| 10 | 6.52 | 6.38 | 6.41 | 6.44 | 6.27 |
| 20 | 7.18 | 7.13 | 7.14 | 7.14 | 7.06 |
| 50 | 8.04 | 8.03 | 8.03 | 8.01 | 8.01 |

TABLE 5. Calculated length and velocity scales, and surface shear stresses, at various downwind distances for two abrupt changes in surface roughness. See text for details.

| $x$ (m) | $l_1$ (m) | $l_2$ (m) | $u_1$ (m sec⁻¹) | $u_2$ (m sec⁻¹) | $\tau_{2,w}=\rho\tau_2$ (N m⁻²) |
|---|---|---|---|---|---|
| 185 | 10 | 0 | 0.1416 | 0 | 0.3573 |
| 191 | 10.3 | 0.622 | 0.1408 | −0.0924 | 0.0976 |
| 206 | 11 | 1.66 | 0.1391 | −0.1317 | 0.1240 |
| 251 | 13 | 4.17 | 0.1350 | −0.1327 | 0.1469 |
| 296 | 15 | 6.49 | 0.1317 | −0.1286 | 0.1548 |
| 342 | 17 | 8.71 | 0.1288 | −0.1251 | 0.1590 |
| 389 | 19 | 10.89 | 0.1265 | −0.1224 | 0.1616 |
| 413 | 20 | 11.98 | 0.1253 | −0.1210 | 0.1626 |
| ∞ | | | | | 0.1710 |

downwind of the last abrupt change in surface roughness, depending on the distance to this last change and the distances between the previous abrupt changes, can be divided into several layers, in which the velocity profile is determined by the influence of the individual abrupt changes of the original velocity profile. This means that one must be very careful in interpreting a velocity profile measured downwind of several abrupt changes in surface roughness, because all these changes can have an influence on the measured velocity profile.

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## APPENDIX A

### Derivation of the Relationship Between $\delta(z)$ and $V_2(z)$

In the equation of continuity we compare the amount of air flowing through a plane $x=\text{constant}>L$ with the amount which would have passed the same plane if no abrupt change in surface roughness at $x=L$ had occurred. Thus, we have

$$\int_0^z U(\zeta)d\zeta=\int_0^{z+\delta} U_2(\zeta)d\zeta, \qquad \text{(A1)}$$

which, if $\delta/z\ll 1$, can be written as

$$\int_0^z (U-U_2)d\zeta=\delta U_2(z). \qquad \text{(A2)}$$

The combination of Eqs. (21) and (A2) then gives

$$\int_0^z [V_2(\zeta)-\delta\partial U/\partial\zeta]d\zeta = -\delta[U-\delta\partial U/\partial z+V_2(z)]. \qquad \text{(A3)}$$

In Eq. (A3) we must now substitute the velocity distribution $U(z)$ according to Eq. (10). In doing so, we arrive at mathematically very intractable expressions. However, we can approximate the integrand in Eq. (A3) by its value for small values of $z$, since the latter provide the main contribution to the integral. Thus, for small values of $z$ Eq. (10) can be approximated by

$$U=u_0^*k^{-1}\ln(z/z_0)+u_1k^{-1}[(\ln\eta+\gamma)\times(1+P_1)-P_1], \qquad \text{(A4)}$$

and

$$\partial U/\partial z=u_0^*(kz)^{-1}+u_1(kz)^{-1}(1+P_1). \qquad \text{(A5)}$$

Substitution of (A4) and (A5) in (A3) then leads to

$$\int_0^z \{V_2(\zeta)-\delta(k\zeta)^{-1}[u_0^*+u_1(1+P_1)]\}d\zeta = -\delta\{u_0^*k^{-1}\ln(z/z_0)+u_1k^{-1}[(\ln\eta+\gamma)(1+P_1)-P_1] -\delta(kz)^{-1}[u_0^*+u_1(1+P_1)]+V_2(z)\}. \qquad \text{(A6)}$$

We now let $z$ extend to infinity and, neglecting $\ln(z/l_1)$ with respect to $\ln(z/z_0)$, we arrive at the following relationship between $V_2(z)$ and $\delta_2=\lim_{z\to\infty}\delta$:

$$\int_0^\infty V_2(\zeta)d\zeta=-u_0^*\delta_2k^{-1}[\ln(l_2/z_1)-C_{0,2}], \qquad \text{(A7)}$$

where

$$C_{0,2}=\lim_{z\to\infty}\left\{\int_0^z \delta(\delta_2u_0^*\zeta)^{-1}[u_0^*+u_1(1+P_1)]d\zeta -\ln(z/l_2)+M-u_1\gamma u_0^{*-1}(1+P)+u_1u_0^{*-1}P\right\}. \qquad \text{(A8)}$$

Before $C_{0,2}$ can be evaluated, however, we must find an expression for $\delta/\delta_2$.

As a result of the definition of $V_2$ we have

$$V_2=-U\partial\delta/\partial z, \qquad \text{(A9)}$$

which with the use of (A4) leads to

$$V_2\simeq-u_0^*k^{-1}[\ln(z/z_0)+u_1k^{-1}\gamma(1+P_1) -u_1u_0^{*-1}P_1]\partial\delta/\partial z. \qquad \text{(A10)}$$

By combining (A10) and (A6), and integrating partially, we obtain

$$\int_0^z V_2\{1-[u_0^*+u_1(1+P_1)]\ln(\zeta/l_2) \times[u_0^*(\ln\zeta/z_0)+u_1\gamma(1+P_1)]^{-1}\}d\zeta =-u_1\delta k^{-1}[u_0^*u_1^{-1}\ln(l_2/z_0)+(1+P_1)\ln(l_2/l_1) +\gamma(1+P_1)-P_1+V_2ku_1^{-1}]. \qquad \text{(A11)}$$

If the upper boundary in the integral in (A11) is taken to be infinite, the equation remains the same except that $\delta$ changes to $\delta_2$ while $\lim_{z\to\infty}V_2=0$. Thus, with $\ln(l_2/z_0)\gg\ln(z/l_2)$, we have

$$\delta/\delta_2\simeq\int_0^z V_2(z)d\zeta\Big/\int_0^\infty V_2(\zeta)d\zeta. \qquad \text{(A12)}$$

Substitution of this result in (A8) appears to give double integrals, which are difficult to cope with. To avoid this difficulty we subtract $\lim_{z\to\infty}$ (A11) from Eq. (A7); after elimination of $\delta_2$ using $\lim_{z\to\infty}$ (A11), we have

$$C_{0,2}=-u_1u_0^{*-1}\Big\{[u_0^*u_1^{-1}\ln(l_2/z_0) +(1+P_1)\ln(l_2/l_1)+\gamma(1+P_1)-P_1] \times\int_0^\infty V_2Qd\zeta\left[\int_0^\infty V_2(1-Q)d\zeta\right]^{-1}+u_0^*u_1^{-1}M -(1+P_1)\ln(l_2/l_1)-\gamma(1+P_1)+P_1\Big\}, \qquad \text{(A13)}$$

where

$$Q=[u_0^*+u_1(1+P_1)]\ln(\zeta/l_2)[u_0^*\ln(\zeta/z_0)+u_1\gamma(1+P_1)-u_1P_1]^{-1}. \quad \text{(A14)}$$

With the use of the same approximations used to obtain (A12), Eq. (A13) reduces to

$$C_{0,2}=-u_1u_0^{*-1}\Big\{[u_0^*u_1^{-1}+(1+P_1)]\times\int_0^\infty V_2\ln(\zeta/l_2)d\zeta\Big[\int_0^\infty V_2(\zeta)d\zeta\Big]^{-1}+Mu_0^*u_1^{-1}-(1+P_1)\ln(l_2/l_1)-\gamma(1+P_1)+P_1\Big\}. \quad \text{(A15)}$$

The needed relation between $V_2$ and $\delta$ is formed by combining (A7) and (A12); thus,

$$u_0^*\delta k^{-1}=-[\ln(l_2/z_1)-C_{0,2}]^{-1}\int_0^z V_2(\zeta)d\zeta. \quad (22)$$

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