On Planetary Boundary Layer Flow under Conditions of Neutral Thermal Stability

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ABSTRACT

A wind spiral model, similar to that proposed by Blackadar, is used to represent the flow above a surface of uniform roughness in the planetary boundary layer (extending up to ~1 km). An attempt is made to determine the applicability of the mixing length model used and to evaluate an empirical parameter used in the model. This attempt, using existing experimental observations of surface shear stress and wind direction, is inconclusive and leads us to suspect that surface inhomogeneity has played a role in some of the experimental data.

1. Introduction

The planetary boundary layer, involving the wind spiral or the approach to the geostrophic wind, has been studied by many authors since G. I. Taylor (1915) gave a solution using a constant eddy viscosity. Recently, Lettau (1962) and Blackadar (1962, 1965) have obtained numerical results based on mixing length models while Estoque (1967) has given an approximate analytic solution using Blackadar's form for mixing length. For flow above a uniformly rough surface either an eddy viscosity or a mixing length model may be used, but for considerations of flow above a change in surface roughness, which is the topic of a companion paper (Taylor, 1969), hereafter referred to as B, only the Prandtl mixing length model is suitable.

The mixing length form adopted in the present work is almost identical in form to that proposed by Blackadar (1962) or Estoque (1967) and may be written as[2]

$$\ell=\frac{k(z+z_0)h}{z+h}, \tag{1}$$

where z_0 is the local roughness length, k von Kármán's constant (taken equal to 0.4), and h a variable height parameter. If we nondimensionalize with respect to z_0 and consider just the uniform roughness case, we may rewrite (1) in dimensionless form as

$$\ell'=\frac{k(z'+1)}{(1+kz'/\lambda')}, \tag{2}$$

where, following Blackadar, λ' is the value of ℓ' given by (2) as $z'\rightarrow\infty$. This form is chosen as a simple model that behaves like $k(z'+1)$ close to the ground, and so gives rise to a logarithmic velocity profile in the approximately constant stress region, and approaches a constant value (λ') at large heights.

Assuming that (2) gives a reasonable form for the mixing length, we may expect the velocity and shear stress profiles obtained by using it in the equations of motion to be approximately correct provided the parameter λ' is chosen correctly.

If we assume that the overall flow is determined by the roughness length z_0, the Coriolis parameter f, and the geostrophic wind G, then by dimensional reasoning we may expect $\lambda(=\lambda' z_0)$ to be of the form

$$\lambda=\frac{G}{f}E(\mathrm{Ro}), \tag{3}$$

where Ro $(=G/fz_0)$ is the roughness Rossby number. The particular form (3) was chosen so that E has a relatively small variation over a wide range of values for Ro. In nondimensional form

$$\lambda'=\mathrm{Ro}E(\mathrm{Ro}). \tag{4}$$

Blackadar (1962) assumes a constant value of E, while in a later paper (Blackadar, 1965) he uses the relation

$$\lambda=u_*H/G, \tag{5}$$

where H is a measure of the boundary layer thickness and u_* is the surface friction velocity. Inasmuch as u_* and H will themselves depend on the external parameters G, f and z_0, and on the value chosen for λ, there seems little merit in this particular form at this stage. It may, however, be useful to have some dependence on boundary layer thickness in the case of flow above a change in roughness where the thickness H would change gradually in the downstream direction.

An alternative way of looking at (3) is that λ will be determined primarily by H, which in turn will depend on G/f and weakly on z_0.

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[2] Definition of all symbols are given in the Appendix.

One of the aims of the work described here is to determine the form of the empirical relationship $E(\mathrm{Ro})$. There are unfortunately very few detailed observations of velocity profiles in the planetary layer, especially under the special conditions of constant density considered here. There is, however, a reasonable amount of data on surface shear stress and wind direction, most of which is given in Blackadar's papers, and we may use these observations in an attempt to determine $E(\mathrm{Ro})$.

2. Equations and boundary conditions

Under conditions of neutral thermal stability and assuming a constant density throughout the layer, the governing equations of motion for equilibrium flow above an infinite plane are well established (see, for example, Haltiner and Martin, 1957, p. 219), and may be written in the nondimensional form

$$\frac{d\tau_x'}{dz'} = -V'/\mathrm{Ro}, \qquad (6)$$

$$\frac{d\tau_y'}{dz'} = (U'-1)/\mathrm{Ro}, \qquad (7)$$

where the nondimensionalization is with respect to G and z_0, and the x axis is taken parallel to the geostrophic wind direction. The mixing length relation may be written as

$$\tau_x' = \nu_T' \frac{dU'}{dz'}, \qquad (8)$$

$$\tau_y' = \nu_T' \frac{dV'}{dz'}, \qquad (9)$$

where $\nu_T' = \tau'^{\frac{1}{2}}\ell'$ and $\tau'^2 = \tau_x'^2 + \tau_y'^2$. Substituting in (8) and (9) for ν_T' and putting in a form suitable for numerical solution gives

$$\frac{dU'}{dz'} = \tau_x'/[(\tau_x'^2+\tau_y'^2)^{\frac{1}{4}}\ell'], \qquad (10)$$

$$\frac{dV'}{dz'} = \tau_y'/[(\tau_x'^2+\tau_y'^2)^{\frac{1}{4}}\ell']. \qquad (11)$$

The boundary conditions we wish to apply are $U' = V' = 0$ on $z' = 0$, and $U' \to 1$, $V' \to 0$ as $z' \to \infty$.

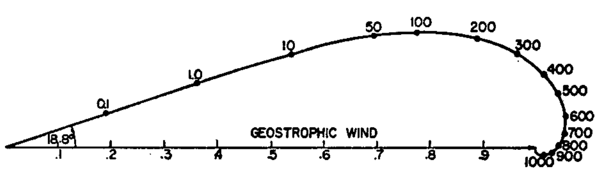

FIG. 1. Wind hodograph for $\mathrm{Ro}=10^7$, $E=0.0004$. Heights on curve are in meters for $z_0=1$ cm.

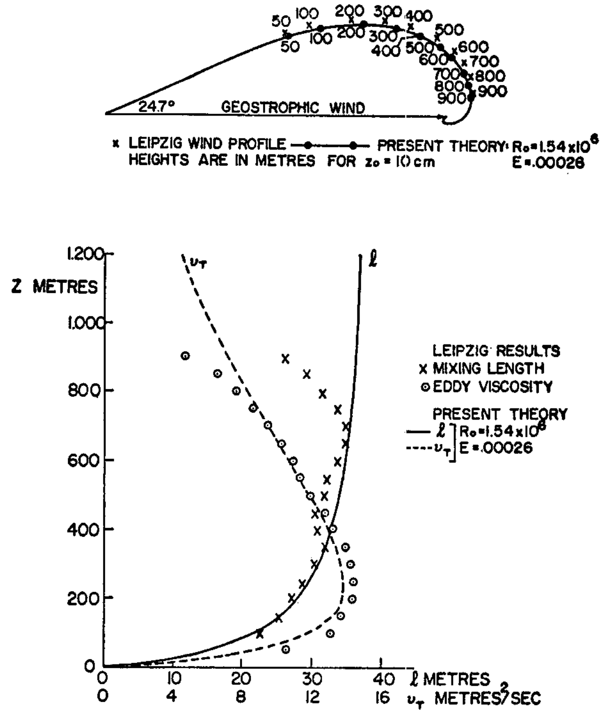

FIG. 2. Comparison of theoretical results with the "Leipzig wind profile."

Now since Eqs. (8)–(11) form a set of first-order nonlinear ordinary differential equations and we have a boundary value problem to solve, we must resort to an essentially trial and error method in order to obtain a numerical solution satisfying all the boundary conditions. The method employed is to choose values of τ_x, τ_y at $z=0$, solve the initial value problem by one of the standard methods (Kutta-Merson was used in this case) and modify the initial choice of τ_x, τ_y until the boundary condition $U \to 1$, $V \to 0$ as $z \to \infty$ is approximately satisfied. Blackadar and Ching (1965) describe a technique for doing this which would appear to be essentially the same as that used in obtaining the present results.

3. Numerical results for the flow above a surface of uniform roughness

Numerical results for velocity and shear stress profiles have been obtained using the mixing length form given in (2) for a range of values of Ro and E. The hodograph for a typical case $\mathrm{Ro}=10^7$, $E=0.0004$ is shown in Fig. 1, and results for the Leipzig wind profile case are compared with Lettau's (1950) analysis of the experimental results in Fig. 2. The parameters used in this case were $\mathrm{Ro}=1.54\times10^6$ and $E=0.00026$. The value for Ro is based on Lettau's values for the Coriolis parameter ($0.000114\ \mathrm{sec}^{-1}$) and the geostrophic wind, together with a value of 10 cm for z_0 based on assuming a logarithmic profile up to the first experimental point

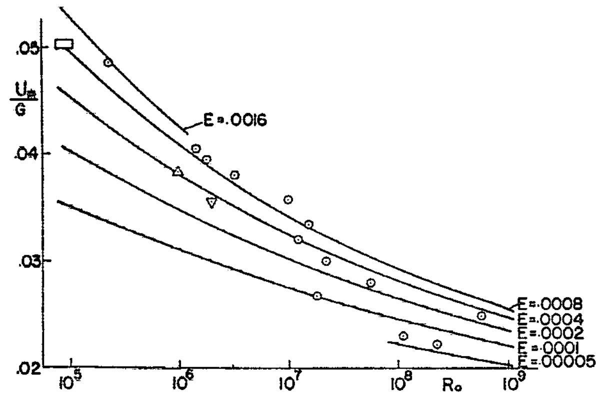

FIG. 3. Variation of surface friction velocity with surface Rossby number, using experimental results from Blackadar (1965): ⊙ Lettau's collection; □ Brookhaven, July 1950; △ Brookhaven, November 1950; ▽ Hanford 1959.

at 50 m. The E value was chosen to match the experimental determination of ℓ. The theoretical and expermental results are in quite good agreement. The results for surface friction velocity u_*' and the angle α_0 between the surface and geostrophic wind directions are as follows:

	u_*'	α_0
Leipzig	0.0373	26.1°
Theory	0.0351	24.7°

Having set up the model and computer program, one can obtain a whole series of results for velocity, shear stress, eddy viscosity profiles, etc., for different values of the parameters Ro and E. Sample results, together

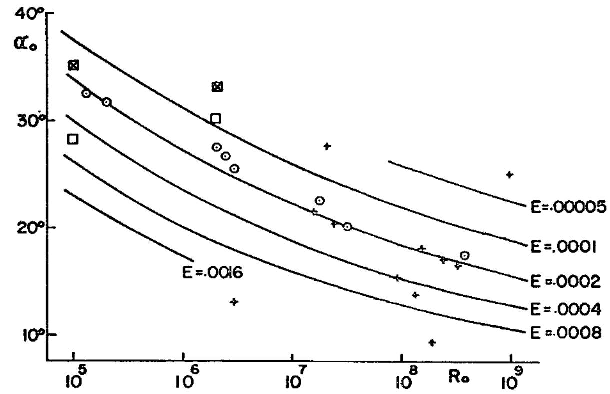

FIG. 4. Variation of surface wind direction with surface Rossby number, using experimental results from Blackadar (1965): ⊙ neutral stability, baroclinicity unknown; ⊠ barotropic, stability unknown; + baroclinicity and stability unknown; □ baroclinic, stability unknown. Data sources are given by Blackadar (1965, p. 20).

with a description of the numerical methods used, are given in Taylor (1967).

We now turn to the investigation of the empirical function E(Ro). Surface friction velocity and wind direction have been calculated on the present theory for a range of values of Ro and E. These results, together with boundary layer thicknesses H_1 and H_2, are given in Table 1; in Figs. 3, 4 they are compared with the experimental observations given by Blackadar (1962, 1965). The experimental results for friction velocity show rather less scatter than those for α_0 largely due

TABLE 1. Dependence of overall parameters on Ro and E(Ro).

Ro		E(Ro) 0.000025	0.00005	0.0001	0.0002	0.0004	0.0008	0.0016
10^5	u_*'			0.0350	0.0402	0.0452	0.0495	0.0528
	α_0			37.4	33.8	30.0	26.2	22.9
	H_1			250	357	493	640	831
	H_2			480	730	1050	1500	2000
10^6	u_*'			0.0309	0.0347	0.0380	0.0407	0.0425
	α_0			31.2	27.2	23.4	20.1	17.3
	H_1			220	300	402	530	672
	H_2			460	660	920	1300	1740
10^7	u_*'			0.0275	0.0303	0.0325	0.0341	
	α_0			25.9	22.2	18.8	15.9	
	H_1			190	264	349	445	
	H_2			410	610	830	1120	
10^8	u_*'		0.0222	0.0246	0.0266	0.0282	0.0292	
	α_0		25.7	21.9	18.3	15.4	12.9	
	H_1		110	174	232	304	387	
	H_2		270	380	530	750	1000	
10^9	u_*'	0.0182	0.0203	0.0222	0.0236	0.0247	0.0255	
	α_0	26.2	22.4	18.8	15.6	12.9	10.8	
	H_1	80	110	154	207	272	340	
	H_2	170	250	370	498	700	910	

u_*'—dimensionless surface friction velocity $=[\tau_x^2(0)+\tau_y^2(0)]^{\frac{1}{2}}/G$.
α_0—angle between surface and geostrophic wind directions, in degrees.
H_1, H_2—boundary layer thicknesses, in meters, for $G/f=10^7$ cm.
H_1—height at which wind speed first reaches geostrophic value.
H_2—height at which wind direction is first parallel to geostrophic direction.

to the difficulty in determining the geostrophic wind direction accurately.

In spite of the scatter it is possible in both cases to detect a definite trend relative to the curves for constant E. The trends in the two cases are unfortunately different. For the friction velocity results the experimental points would suggest that E should vary from about 0.001 at $\mathrm{Ro}=10^5$ to 0.00005 at $\mathrm{Ro}=10^9$, while the surface wind direction results suggest a constant E of about 0.0002. Blackadar (1965) has implied that we could expect λ to be loosely related to some boundary layer thickness H, and we may anticipate λ' increasing with H' for decreasing Ro. This in turn suggests that the form for $E(\mathrm{Ro})$ predicted by the u_* results is more likely to be correct than that given by the α_0 results.

While many explanations of the discrepancy in predicted forms for $E(\mathrm{Ro})$ are possible, among the most likely are that we have chosen a poor form for the mixing length in the present analysis, or that the experimental observations were of cases in which baroclinicity or non-neutral thermal stability were present, or where there was some upstream surface inhomogeneity.

Tests using modified forms for $\ell(z,z_0)$ have been carried out and indicate, provided the general shape of the curve is the same and incorporates $\ell \simeq k(z+z_0)$ near the ground, that the predictions of surface values and velocity profiles are not altered significantly. It is also possible that the use of the mixing length theory is unjustified but the errors due to this factor would probably be too small to explain the differences described. The effects of baroclinicity and thermal stability have been investigated by Blackadar (1965) and Blackadar and Ching (1965) but a comparison of their results with the solutions obtained here indicate that the discrepancy is too large to be explained by these factors.

We now consider the possibility of spatial variations in the assumed steady state situation and, in particular, the effect of a change in surface roughness upstream of an experimental site. It is shown in B that wind direction is much slower to respond to roughness changes than is surface shear stress; thus, if roughness changes are present the effect will be to give incorrect values for α_0 (for equilibrium conditions at the local Rossby numbers), while u_* may be approximately correct. It is also shown in B that a fetch of approximately $z_0\mathrm{Ro}$ is required for the wind directions to reach their equilibrium values. For $\mathrm{Ro}=10^7$ and $z_0=1$ cm this would give a required fetch of 100 km which was probably not available for some of the observations. For large (10^8–10^9) values of Ro the experimental points are mainly derived from observations over the sea and there should thus be a sufficiently large fetch to eliminate roughness change factors. Some of the results for Ro in the region 10^5–10^7 are, however, taken from experimental sites with relatively small fetches. In general, the area around the site would have a larger roughness length than the site itself and the results for α_0 would correspond to the rougher surface. If this were the case, as is quite possible, it would tend to correct the divergence in predicted values of E between the two sets of results.

4. Conclusions

The attempt to determine $E(\mathrm{Ro})$ has not been entirely satisfactory. On the basis of the surface friction velocity results we may expect that E will decrease slowly as Ro increases varying by a factor of O(1) as Ro varies by a factor of 10. The basic form chosen for mixing length [Eq. (1)] appears to be satisfactory but until more and better experimental observations are available little further progress can be made.

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APPENDIX

(Primes are to used denote nondimensional quantities, where nondimensionalization is with respect to z_0 and G.)

$E(\mathrm{Ro})$	parameter in mixing length model, Eq. (8)
f	Coriolis parameter
G	geostrophic wind speed
h	parameter in mixing length model, Eq. (6)
H	planetary boundary layer thickness
k	von Kármán's constant (0.4)
ℓ	Prandtl mixing length
Ro	roughness Rossby number ($=G/fz_0$)
$\mathbf{U}$	mean velocity vector
U,V	horizontal velocity components
u_*	surface friction velocity ($=\tau_0^{1/2}$)
x,y	horizontal distance coordinates
z	vertical height above ground
z_0	surface roughness length
α_0	angle between surface and geostrophic wind directions
λ	maximum value of mixing length
ν_T	eddy viscosity
$\boldsymbol{\tau},\tau_x,\tau_y$	kinematic shear stress vector and horizontal components
τ	magnitude of kinematic shear stress, equal to $(\tau_x^2+\tau_y^2)^{1/2}$
τ_0	surface shear stress

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