Convective Plumes in a Convective Field

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ABSTRACT

The theory of isolated turbulent plumes given earlier is used to study a field of plumes. Each plume is immersed in the turbulent downdraft which comprises the return flow. The field of flow is specified by three parameters: the heat flux into the atmosphere at the surface, the depth of the convecting layer, and the intensity of turbulence at the surface (where turbulence is steadily generated by the wind) and where a plume element which leaves the surface returns there in a downdraft after a period of the order of 10^6 sec. The change in air properties during this period is of the essence of the problem. Since the process is driven by the changing density resulting from heating, the equations describing the field must be time-dependent in this essential respect. Wind is neglected, and the horizontal pressure gradient assumed to be the same at all heights.

The derived plume properties—size, temperature excess, upward velocity and turbulent intensity—are in agreement with observation. The formulation predicts a maximum possible depth for convection in the form of a field of plumes, depending on the magnitudes of the heat flux and surface turbulence. As a result, it is suggested that the theory of a field of plumes could lead to a prediction of the onset of a different form of convection, such as on a larger scale, resulting from instabilities in the convecting layer as a whole.

1. Introduction

The surface boundary layer has been extensively studied over the past 20 years (Priestley, 1959; Lumley and Panofsky, 1964), and through such works considerable insight has been gleaned concerning the nature of its turbulent motion. Particular stress has been placed on adequately describing the variation of wind speed and temperature with height, the so-called “profiles” of the layer. From the beginning it was noted that the warmer air tended to occur in patches with a fluctuating temperature which was elevated relative to a uniform base temperature in the intervening regions. In fact, this pattern exists throughout the convecting layer, except for the first few meters above the ground, where the temperature fluctuates everywhere.

Most of the work has been done with ground-based towers or balloons, although Bunker and others were working on airborne measurements from 1955. Warner and Telford (1967) published an article summarizing their previous work and analyzed the details of the structure of the warm air parcels. While they did not study the wind because no good continuous measurements were available from the aircraft, the temperature structure in both the vertical and horizontal, and the vertical velocity of the air, were recorded and analyzed with instruments of improved accuracy and frequency response. The mobility of airborne observational platforms such as gliders and aircraft has made obvious the similarity between the warm patches of air and the blobs of warm air which form clouds, and, furthermore, has drawn attention to the departures of the atmosphere from the stationary, horizontally uniform state, which is sometimes assumed.

The following features of convection need to be clearly recognized: 1) the heat flux obviously diminishes with height since it must have a vertical gradient in order to distribute heat into the air; 2) there is a continually rising upper limit to the height of convection, which changes greatly during the day; 3) when we take a horizontally averaged temperature, the convecting region does not appear to be unstable above ~100 m; and 4) the horizontal temperature gradient in the air (Warner and Telford, 1965) often contributes a term in the rate of change of temperature at a fixed station comparable to that contributed by the heat flux which has inherently determined the convective pattern.

The study of convective blobs has developed simultaneously with the boundary layer studies. Morton et al. (1956) laid the foundations for a long line of experimental and theoretical studies. Experimental work has proceeded both in the field, with studies of cloudy convection and chimney plumes, and in the laboratory, where simulated atmospheric convection has been studied with water solutions in tanks.

It is the purpose of this paper to draw on these ideas in an attempt to explain some of the observations about convective elements. Previous work on isolated plumes [Telford (1966), hereafter referred to as I] will be extended to a field of plumes. Account is taken of the interaction between the updrafts and downdrafts. Since the horizontal momentum equations are not included at this stage, no information is found about the wind
profile. However, wind does enter indirectly into the formulation since the surface turbulence, which relates to the wind, does exert a strong influence on the field.

To put this paper into context it would seem advisable to remark on the approach taken here as compared to the usual approach via the Navier–Stokes equations. Since these latter equations are not at present soluble for turbulent convective plumes, they are usually simplified by discussion using steady and fluctuating components, profile assumptions, and physical order-of-magnitude arguments. These arguments are used to delete product terms, involving the steady values and fluctuating components of the density and three velocity components, and their volumetric integrals, and do not always differ greatly from arbitrary assumptions.

The present paper does not start from these fluid flow equations but sets up the hypothesis that a plume element with a sharp boundary and a statistically uniform interior is an adequate basis for a mathematical description of real convection. While this approach may be unconventional in this subject, it is a standard physical method, i.e., that of exploring the simplest hypothesis (not necessarily the most obvious). Since the simplicity has the great virtue in this instance of providing a framework for unambiguous quantitative conclusions, it thus allows and invites direct confrontation with measurements.

2. The nature of the field of plumes

a. General description

Plumes rise in a field of descending air which itself was rising a few minutes earlier. To clarify this, let us discuss first a hypothetical field in which the rising air is prevented from mixing back and forth with the downdraft by an impenetrable film surrounding each plume. This simple picture helps to illustrate qualitatively most of the features of the plumes in a convective field. In later sections a quantitative picture will be developed, based on the theoretical concepts related to mixing which are given in I. Mixing between the rising and descending air is included in these later sections, and in the model which then results the size of the plumes is calculable, together with temperature excess and all the other properties of the field which are not related directly to the wind.

b. The plumes in a convective field without mixing

Imagine a recirculating convective field wherein the warm air rises in plumes from the surface to some height above which the overlying air is stable and warmer than in the field below. At this height the warm air is no longer buoyant so as the circulation continues it must turn around and subside back to the surface. At the surface the subsiding air is heated by an amount depending on the surface heat flux. It then enters the updraft again, and thereby completes the circulation cycle. The heat must be carried by the air moving upward in the plumes and must result from the temperature change which occurs as the air leaves the downdraft and enters the updraft at the ground. Radiative exchanges are negligible at the heating rates considered here. Since the recirculation is a continuing process, mixing being excluded for the moment, air at the surface in the downdraft will have the same temperature as when it started upward in the updraft some minutes earlier. Thus, the air in the updraft at any given fixed height increases in temperature at a rate equal to the surface temperature excess in the updraft relative to the downdraft, divided by the time for a complete circulation cycle. Similarly, every part of the circulating field increases in temperature at this rate.

We may also make some deductions concerning the vertical gradient in temperature. If the radii of the plumes are constant with height and no mixing occurs, the vertical potential temperature gradient in both the rising and descending air is linear. At any instant, the rising air is potentially cooler at greater heights, since, after a particular higher altitude element left the surface, the surface temperature rose. Similarly, the surface air in the downdraft is cooler than the air above it because a longer time has elapsed since it was last heated at the surface. In the updraft the potential temperature gradient is linear and unstable, while in the downdraft it is linear and stable.

If the area in the updrafts is smaller than in the downdrafts, the updraft velocity is higher than the downdraft velocity, and the vertical temperature gradient in the updraft is less. The smaller area and smaller gradient in the updraft means that the gradient, when taken in the vertical, of the horizontally averaged potential temperature is stable. Thus, this argument suggests there is an upward transfer of heat from lower levels with a cooler average temperature to warmer air above, and, indeed, this phenomenon is a feature of the observations above the first few hundred meters and has aroused comment in the past. However, surface temperature appears to be always higher than anywhere else in the region of convection.

Before we formulate the equations mathematically, let us qualitatively look at the effects of mixing between the upward moving plume and the downdraft. The effect of this air interchange is always to cool the updraft and warm the downdraft. Thus, because the effect of mixing is greater when the temperature difference is greatest, the updraft will be more unstable and the downdraft less stable near the surface. The average will thus be less stable near the surface, and, indeed, the calculations show the effect is sufficient to always make the horizontally averaged potential temperature gradient in the vertical unstable near the surface. At higher levels the mixing is much less effective because the temperature excess in the rising plume over its surrounding descending air is much less, and the stable gradient
(which results from a higher updraft velocity than downdraft) dominates. The instability of the average temperature near the ground has been well known for a long time from observational work.

3. The equations of motion for the plumes in a convective field

We will examine theoretically the behavior of a representative plume rising through the descending airstream of its own return flow. Each example of an updraft and its surrounding downdraft will be examined while occupying a fixed area of the field so that the variables which change at different heights will be the vertical velocities \( \vec{w}_u \) and \( \vec{w}_d \), turbulence velocities \( \vec{i}_u \) and \( \vec{i}_d \), densities \( \rho_u \) and \( \rho_d \) (this represents the temperature of the air) for both the updraft and downdraft (subscripts \( u \) and \( d \), respectively), and, in addition, the changing radius \( \beta \) of the updraft, and the varying vertical pressure gradient \( \frac{dp}{dz} \). It should be mentioned that the concept of “bouyancy” often used in the past in discussing plumes is based on the idea that the vertical pressure gradient in the surrounding fluid is hydrostatic and that it controls the vertical pressure gradient in the plume. As in former work, it is assumed that the horizontal pressure gradient is the same at all levels. Thus, the vertical pressure gradient is the same in the rising and descending air. However, inertial terms cannot be neglected anywhere in the field, and as a consequence, the simplifying idea of buoyancy is not relevant. Thus, it is necessary to keep track of volume as well as mass, since these two equations can no longer be combined to give a buoyancy equation in place of one of them. Basing our physical analysis on a “conservation envelope” surrounding the test section of the plume, equations are needed to describe its volume change with time due to entrainment and detrainment through the envelope, as well as to describe changes in mass, momentum and energy. Thus, when similar equations are written for the subsiding air, there are eight ordinary nonlinear differential equations to be solved simultaneously.

4. The equations

In I, the physical approximations used in setting up the equations for the isolated plume were explicitly stated. There is, however, one approximation we can no longer ignore. In a field of plumes where the air is circulating in a closed path, turbulence will increase without limit unless it is removed by dissipation.

To include dissipation, we will make the usual assumption that the time rate of decay of turbulence is

\[
\frac{d}{dt} \vec{v}^2 = -\frac{A}{l} \vec{v}^2
\]

where \( \vec{v} \) is the rms turbulent velocity, \( A \) a constant and \( l \) a typical length.

Experimental evidence from wind tunnels suggests the constant of proportionality \( A \) is \( \sim 1 \) (see Batchelor, 1960, p. 106). We will set \( A = 1 \), taking the length scale \( l \) to be the diameter of the plume. The turbulent decay, in the form we need, is then

\[
\frac{d}{dt} (2E) = \frac{d}{dt} (\rho \vec{v}^2) = -A \rho \vec{v}^2
\]

where \( E \) is the kinetic energy per unit volume, and \( b \) the plume radius.

Another equation we will need is

\[
\frac{\partial H}{\partial \vec{v}} = \rho C_p \frac{\partial \theta}{\partial t}
\]

where \( \theta \) is the potential temperature, \( H \) the heat flux, \( \rho \) the air density, \( C_p \) the specific heat of air at constant pressure, \( z \) height, and \( t \) time.

The potential temperature is associated with the usual assumption, also used here, that the real atmosphere will be aptly described by deductions based on an incompressible fluid which matches air at the surface. This comparison should be valid if we use potential temperature and density wherever normal temperature and density usually occur.

If \( z_0 \) is the total depth of the convecting layer and \( H \) is now the heat flux entering the bottom, then

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t} \frac{H}{\rho C_p z_0}
\]

We will assume that the entrainment velocity through the bounding surface between the ascending and descending air is proportional to the rms turbulent velocity of the region into which mixing is occurring. Thus, although a turbulent volume appears to mix into an adjacent quiescent region, we describe mathematically the changes in the boundary envelope as due to a flow through it from the quiet to the turbulent side. The factor of proportionality is \( a \) and is given by \( a = (0.6)^{10} = 0.1(0.6)^{12} \) as deduced previously in I.

The derivation of the equations is essentially that employed in Appendix A of I, except now we must consider changes in variables with time. In I, time was not a variable, so that this distinction was unnecessary. We now need to consider a slice of the plume with continuing material identity and to follow its changes as it rises. This is typified by the derivation of Eq. (A2) from that paper, which describes the changing momentum in a slice of thickness \( h \), where \( h \) depends on the updraft velocity; thus, since continuity must be maintained in the plume, \( h = rw_u \), where \( r \) is a small constant time. This equation was

\[
\frac{d}{dz} \left( m_\omega - m_{\omega 1} \right) = -\pi b^2 \rho \frac{d}{dz} \left( h \frac{d}{dz} \right) - 2\pi b \rho \frac{d}{dz} \left( \rho w_u \right)
\]

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where \(m_2w_2\) is the momentum (the product of mass and velocity) of the parcel after time \(\delta t\), when it was \(m_1w_1\) at the beginning of this interval; subscript \(p\) always refers to the rising air in the plumes and subscript \(s\) to subsiding air in the surroundings; \(dp/dz\) is the vertical gradient of pressure which is assumed constant everywhere at a given level; and \(g\) is the acceleration due to gravity. The plume is in downward moving surroundings so air entrained into the plume now brings momentum with it. This means we must add a term, \(2\pi b_2 h_2 a_i \delta t\), to the right side of the previous equation, where \(w\) is again vertical velocity (n.b., \(w_s < 0\) always).

Thus, substituting for \(h\), we get

\[
\delta (mw_p) = \pi b_2 \left[ -\pi b_2^{2} \frac{dp}{dz} - \pi b_2^{2} \rho_p \right. \\
+ \left. 2\pi ab (\rho_s w_s \delta t - \rho_p w_p \delta t) \right].
\]

Since the momentum is

\[mw_p = \pi b_2^{2} \rho_p w_p h = \pi b_2^{2} \rho_p w_p^2,
\]

we therefore have

\[
\frac{d}{dt} (b_2^2 \rho_p w_p^2) = -b_2^2 \frac{dp}{dz} - \pi b_2^{2} \rho_p w_p + 2ab (\rho_s w_s \delta t - \rho_p w_p \delta t).
\]

This is the total derivative describing changes in a material test slice of continuing identity. Thus,

\[
\frac{d}{dt} (b_2^2 \rho_p w_p^2) = \frac{\partial}{\partial t} (b_2^2 \rho_p w_p^2) + \frac{\partial}{\partial t} (z) \frac{\partial}{\partial z} (b_2^2 \rho_p w_p^2).
\]

Now, \(dz/dt = w_p\), since this describes the total movement of the test slice with time.

It is now necessary to introduce the assumptions for a quasi-stationary solution by saying that \(\partial b/\partial t\) and \(\partial w_p/\partial t\) are small enough to be omitted; at the same time, however, \(\partial \rho_p/\partial t = \rho\) exerts a dominating influence on the motion.

This is a good approximation, assuming as it does that \(b\) and \(w_p\) do not change with time at a particular height. However, with the upper boundary conditions we have chosen there is a top to the convecting region; it thus has a definite depth, and because the air is continually expanding with time due to the increasing temperature, this depth must continually increase. As an alternative, if the air is imagined to be cooled evenly by radiation at the same rate as the heat flux heats it, expansion does not occur and exact formulation is possible. This alternative model has been numerically solved (see Telford, 1968, Appendix A) and the solutions show very little difference (<0.1%) from those given here.

Thus, accepting the physical similarity of the two formulations, we can proceed with confidence, knowing that our approximations in regard to the omitted time derivatives have not invalidated any of the basic physics by omitting terms of importance.

Thus, the momentum equation is

\[
\frac{\partial}{\partial z} (b^2 w_p^2 \rho_p) = -b_2^2 \frac{dp}{dz} - g \rho_p \left[ -2ab (i_p w_p \delta t - i_s w_p \delta t) - b_2^2 w_p \delta t \right].
\]

The other three equations for volume, mass and turbulent kinetic energy can be derived in the same way.

The four equations for the updraft are as follows:

\[\text{Volume}\]

\[
\frac{\partial}{\partial z} (b^2 w_p^2 \rho_p) = 2ab (i_p - i_s)
\]

\[\text{Mass}\]

\[
\frac{\partial}{\partial z} (b^2 \rho_p w_p) = 2ab (i_p - i_s) - b_2^2 w_p \delta t
\]

\[\text{Momentum}\]

\[
\frac{\partial}{\partial z} (b^2 w_p^2 \rho_p) = -b_2^2 (\rho) - g \rho_p \\
+ 2ab (i_p w_p \delta t - i_s w_p \delta t) - b_2^2 w_p \delta t
\]

\[\text{Turbulent kinetic energy}\]

\[
\frac{\partial}{\partial z} (b^2 w_p^2 i_p^2) = 2ab \left[ \rho_s i_p \left[ (w_p - w_s)^2 + i_s^2 \right] - \rho_p i_p i_s^2 \right] \\
- b_2^2 i_p^2 \delta t - \frac{A}{b_2^2 n_e i_p^2}
\]

Partial derivatives of all variables, other than density, are zero with respect to time.

The last term in (4) is the dissipation term and it is derived as follows:

\[
\frac{d}{dt} (b^2 \rho_p A i_p^2) = 2ab \left[ \rho_s i_p \left[ (w_p - w_s)^2 + i_s^2 \right] - \rho_p i_p i_s^2 \right] \\
- (b^2 \rho_p A) i_p^2
\]

where the rate of decay of turbulent energy per unit mass is \(A_i_p^2/(2b)\).

We now need an additional set of four equations to describe the changes in a slice of downdraft. These will then allow a complete description of the updraft by specifying the conditions which modify it, and vice versa.

The downdraft can be given a new radius, but it is simpler to say it surrounds the updraft and the combined concentric pair have a radius \(s\). This radius determines the holding area of a plume and its downdraft, or
the density of plumes with area in the field. Clearly, it must be constant with height. Thus, the downdraft area will be \( \pi (c^2 - b^2) \) while the updraft area will be \( \pi b^2 \).

The volume of a slice of downdraft is \( \pi (c^2 - b^2)h' \), with a surface area common to the updraft of \( 2\pi bh' \), where \( h' = r'w_s \), and \( h' \) is the thickness of the slice corresponding to \( h = rw_p \) for the updraft. By making these changes in the appropriate places in the derivations for the plume, and changing subscripts appropriately, we are led to the following four equations for the downdraft:

\[
\text{Volume} \quad \frac{\partial}{\partial z} [(c^2 - b^2)w_s] = 2ab (i_s - i_p) \tag{5}
\]

\[
\text{Mass} \quad \frac{\partial}{\partial z} [(c^2 - b^2)w_{dp_s}] = 2ab (i_p \rho_p - i_p \rho_s) - (c^2 - b^2) \rho \tag{6}
\]

\[
\text{Momentum} \quad \frac{\partial}{\partial z} [(c^2 - b^2)w_{dp_p}] = -(c^2 - b^2)(\frac{\partial}{\partial z} \cdot g(c^2 - b^2)\rho_s) + \rho_s i_p i_s^3 + 2ab (i_p w_p - i_p w_s) - (c^2 - b^2)w_{dp_s} \tag{7}
\]

\[
\text{Turbulent kinetic energy} \quad \frac{\partial}{\partial z} [(c^2 - b^2)w_{dp, i_s^3}] = 2ab (\rho_p i_s [w_s - w_p]^2 + i_s^3) - \rho_s i_p i_s^3 - (c^2 - b^2)i_s^3 \rho - \frac{A}{2} (c^2 - b^2) \rho_p i_s^3 \tag{8}
\]

The last term in (8) is the decay term for the turbulence in the downdraft based on the scale length of \( 2(c^2 - b^2) \).

Since \( c \) is constant, we do not need an equation to specify it, and we could thus determine \( \rho \) if it were useful. However, only \( \partial \rho / \partial z \) has been evaluated for use during the calculation in the other equations.

Eqs. (1) and (5) give an immediate integral,

\[
b^2w_p + (c^2 - b^2)w_s = \text{constant} = 0. \tag{9}
\]

The constant is zero if there is no average volume flow at some particular level, say the upper boundary.

Eqs. (2) and (6) also give an integral,

\[
\frac{\partial}{\partial z} [b^2w_{dp_p} + (c^2 - b^2)w_{dp_s}] = -c^2 \rho, \tag{10}
\]

from which we have

\[
b^2w_{dp_p} + (c^2 - b^2)w_{dp_s} = -c^2 \rho \Delta z + \text{constant} = -c^2 \rho \Delta z. \tag{11}
\]

Again the constant is zero if we take the origin of \( z \) at a zero mass flux surface. The upper boundary is such a surface since here \( \rho_p = \rho_s \), so the left-hand side of (10) is zero by (9).

Rearranging (9), we have

\[
w_s = -b^2w_p / (c^2 - b^2), \tag{12}
\]

and substituting (9) in (10) and rearranging,

\[
\rho_p = \rho_s - c^2 \rho / (b^2 - c^2). \tag{13}
\]

Thus, differential equations are not needed for \( w_s \) or \( \rho_p \).

We chose \( \rho_p \) rather than \( \rho_s \) since \( \partial \rho_s / \partial z \) is specifically required to determine the lower boundary condition. We derive this explicitly from (5) and (6) as follows:

\[
\frac{\partial}{\partial z} [(c^2 - b^2)w_{dp_s}] = (c^2 - b^2)w_s - \frac{\partial}{\partial z} w_s \tag{14}
\]

\[
+ \rho_p - [(c^2 - b^2)w_s],
\]

\[
\frac{\partial}{\partial z} [(c^2 - b^2)w_{dp_s}] = - \rho_s 2ab (i_s - i_p) + 2ab (i_p \rho_p - i_p \rho_s) - (c^2 - b^2) \rho,
\]

\[
\frac{\partial}{\partial z} \rho_p = \frac{2ab (i_p \rho_p - i_p \rho_s)}{(c^2 - b^2)w_s} - \frac{\rho}{w_s}. \tag{15}
\]

Similarly, using (2) and (3),

\[
\frac{\partial}{\partial z} [b^2w_{dp_p}] = b^2w_{dp_p} + \frac{\partial}{\partial z} [b^2w_{dp_p}] = -b^2\left( \frac{\partial}{\partial z} \cdot g(b^2w_p) + \frac{\partial}{\partial z} (b^2w_{dp_p}) \right)
\]

\[
\frac{\partial}{\partial z} [b^2w_{dp_p}] = -b^2\left( \frac{\partial}{\partial z} (b^2w_p - b^2w_{dp_p}) \right) + 2ab (i_p w_p - i_p w_s) - b^2w_{dp_s}
\]

\[
\frac{\partial}{\partial z} (w_p) = \frac{1}{w_p \rho_p \Delta z} (p - \frac{g}{w_p} (w_p - w_s)). \tag{16}
\]

To go further, we must evaluate \( \partial \rho / \partial z \). This can be done by using (2) and (3) to give \( \partial w_{dp} / \partial z \) [as in (14)], (6) and (7) to give \( \partial w_s / \partial z \), and a combination of (1) and (5) as follows:

From (1) we have

\[
\frac{\partial}{\partial z} [b^2w_p] = b^2\frac{\partial}{\partial z} (w_p) + 2ab \frac{\partial}{\partial z} (b^2w_p) = 2ab (i_p - i_s),
\]
while (5) gives
\[
\frac{\partial}{\partial z}[(c^2-b^2)w_s] = (c^2-b^2)\frac{\partial}{\partial z}(w_s) - 2bw_e\frac{\partial}{\partial z}(b)
\]
\[
= -2ab(i_p-i_s).
\]
Thus, dividing the two previous equations by \(w_p\) and \(w_s\), respectively, and adding to eliminate \(\partial b/\partial z\), we have
\[
\frac{b^2}{w_p}\frac{\partial}{\partial z}(w_p) + \frac{(c^2-b^2)}{w_s}\frac{\partial}{\partial z}(w_s) = 2ab(i_p-i_s)\frac{1}{w_p}\frac{1}{w_s}.
\]
This elimination of derivatives other than \(\partial w_p/\partial z\) and \(\partial w_s/\partial z\), possible only because \(c\) is constant, illustrates how the evaluation of \(\partial p/\partial z\) depends on this fact.

Using (6) and (7), we obtain
\[
\frac{\partial}{\partial z}[(c^2-b^2)w_p] = w_p\frac{\partial}{\partial z}[(c^2-b^2)w_p] + (c^2-b^2)w_p\frac{\partial}{\partial z}(w_p),
\]
\[
(\frac{c^2-b^2}{w_p})(w_p) = -(c^2-b^2)(\frac{\partial}{\partial z}(p) - g(\frac{c^2-b^2}{w_p})(\rho_s) + 2ab(i_s\rho_p - i_p\rho_s) - (c^2-b^2)w_p\frac{\partial}{\partial z}(w_p).
\]
\[
= -(c^2-b^2)(\frac{\partial}{\partial z}(p) - g(\frac{c^2-b^2}{w_p})(\rho_s) + 2abi_s\rho_p(w_p - w_s),
\]
\[
\frac{\partial}{\partial z}(w_p) = -\frac{1}{w_p}\frac{\partial}{\partial z}(p) - \frac{g}{w_p}\frac{\partial}{\partial z}(\frac{2abi_s\rho_p}{w_p}(w_p - w_s)).
\]
Substituting values for \(\partial w_p/\partial z\) [from (14)] and \(\partial w_s/\partial z\) [from the above equation] in the immediately preceding relationship between them, we obtain
\[
\frac{b^2}{w_p}\frac{\partial}{\partial z}(w_p) + \frac{(c^2-b^2)}{w_s}\frac{\partial}{\partial z}(w_s) = \left[\frac{b^2}{w_p^2\rho_p} - \frac{(c^2-b^2)}{w_s^2\rho_s}\right]\frac{\partial}{\partial z}(p) + g\left[\frac{b^2}{w_p^2} - \frac{(c^2-b^2)}{w_s^2}\right]
\]
\[
+ 2ab(w_p - w_s)\left(\frac{b^2\rho_p}{w_p^2\rho_p} + \frac{b^2\rho_s}{w_s^2\rho_s}\right) = 2ab(i_p-i_s)\frac{1}{w_p}\frac{1}{w_s}.
\]
Thence, taking the previous equation, multiplying through by \(w_p^2\rho_p^2\rho_s\), rearranging and factoring, we have
\[
[b^2\rho_p^2\rho_s + (c^2-b^2)w_p w_s](\frac{\partial}{\partial z})(p)
\]
\[
= -g\rho_p\rho_s[b^2w_s^2 + (c^2-b^2)w_p^2] + 2ab(w_p - w_s)(i_s\rho_p w_p - i_p\rho_p w_s) + 2abi_s\rho_p w_p (i_p - i_s)(w_p - w_s),
\]
\[
= -g\rho_p\rho_s[b^2w_s^2 + (c^2-b^2)w_p^2] + 2ab(w_p - w_s)(i_s\rho_p w_p - i_p\rho_p w_s) - i_p\rho_s w_p + i_p\rho_s w_p - i_p\rho_s w_p - i_p\rho_s w_p + i_p\rho_s w_p - i_p\rho_s w_p,
\]
\[
= -g\rho_p\rho_s[b^2w_s^2 + (c^2-b^2)w_p^2] + 2ab(w_p - w_s)(i_s\rho_p w_p - w_p\rho_s) + i_s\rho_s w_s (w_p - w_s),
\]
\[
= -g\rho_p\rho_s[b^2w_s^2 + (c^2-b^2)w_p^2] + 2ab(w_p - w_s)(w_p \rho_s - w_s \rho_p) (w_s \rho_p + w_p \rho_s).
\]

Thus,
\[
\frac{\partial}{\partial z}(p) = -g\rho_p\rho_s[b^2w_s^2 + (c^2-b^2)w_p^2] + 2ab(w_p - w_s)(w_p \rho_s - w_s \rho_p) (w_s \rho_p + w_p \rho_s).
\]

We now need equations for \(b\), \(i_p\) and \(i_s\) to complete the set. From (1) we have
\[
\frac{\partial}{\partial z}(b^2 w_p) = 2bw_p^2 (\frac{\partial}{\partial z}(b)) + b^2 (\frac{\partial}{\partial z}(w_p)),
\]
from which we obtain
\[
\frac{\partial}{\partial z}(b) = -\frac{a}{w_p} (i_p - i_s) - \frac{b}{2w_p} (\frac{\partial}{\partial z}(w_p)).
\]
Using (2) and (4), in a manner similar to that used previously with (6) and (7), we find that

$$\frac{\partial}{\partial z}(b^2 w_\rho p z^2) = i_\rho \frac{\partial}{\partial z}(b^2 w_\rho p) + 2b^2 w_\rho p z^2 \frac{\partial}{\partial z}(ip),$$

$$2\beta w_\rho p z^2 \frac{\partial}{\partial z}(ip) = 2ab \{p_i i_s [w_\rho - w_s z^2 + i_\rho z^2] - p_i i_s [w_\rho - w_s z^2 + i_\rho z^2] - b^2 i_\rho z^2 - \frac{A}{2} [2ab (i_\rho p_s - i_\rho p) - b^2 i_\rho z^2],$$

$$\frac{\partial}{\partial z}(ip) = -\frac{2ab i_\rho i_s [w_\rho - w_s z^2 + i_\rho z^2] - \frac{A}{2} b^2 i_\rho z^2}{4b^2 i_\rho z^2}. \tag{17}$$

Again for $\partial i_s / \partial z$, using (6) and (8), we have

$$\frac{\partial}{\partial z}[(c^2 - b^2) w_\rho p z^2] = i_s \frac{\partial}{\partial z}[(c^2 - b^2) w_\rho p] + 2(c^2 - b^2) w_\rho p z^2 \frac{\partial}{\partial z}(i_s),$$

$$2(c^2 - b^2) w_\rho p z^2 \frac{\partial}{\partial z}(i_s) = 2ab \{p_i i_s [w_\rho - w_s z^2 + i_\rho z^2] - p_i i_s [w_\rho - w_s z^2 + i_\rho z^2] - (c^2 - b^2) i_s \rho - \frac{A}{2} (c^2 - b^2) p_i i_s^2,$$

$$-i_s [2ab (i_s p_s - i_s p) - (c^2 - b^2) i_s^2]$$

$$\frac{\partial}{\partial z}(i_s) = \frac{2ab i_s [w_\rho - w_s z^2 + i_\rho z^2] - \frac{A}{2} (c^2 - b^2) i_s z^2}{(c^2 - b^2) w_\rho p}.$$ \tag{18}

Thus, Eqs. (11) (18) describe the field. The partial derivatives are, of course, now full derivatives effectively since time does not enter into the formulation. The dependent variables chosen here are $p_\rho, w_\rho, b, i_\rho$ and $i_s$ and these are integrated with a standard Runge-Kutta subroutine; $\rho$ is not evaluated. In the evaluation of the differential equations, the order in which the equations are used is (11), (12), (13), (15), (14), (16), (17) and (18). The value of $\partial p / \partial z$ from (15) is needed for substitution into (14), and (14) into (16). The quantity from (13), $\partial p / \partial z$, is referenced independently of the integration to provide one of the lower boundary conditions.

5. The boundary conditions

We start the integration at the top of the layer and integrate downward. At the top, since surrounding descending air has just turned over from the updraft, its density and turbulence will be the same as in the updraft. Thus, to start the integration, we choose a radius for the plume, a radius for the total updraft and downdraft combined which is fixed during the integration, an updraft velocity for the plume, and a value for the turbulence. We set as fixed the air density and the rate of change of density or temperature with time. The integration is then carried down through the depth of the layer. At the bottom, the updraft has a new radius and velocity; for the turbulence in both the updraft and downdraft we have new values and we have a value for the density gradient in the descending air.

At the bottom, the downdraft reverses very close to the surface. In these few meters, density differences have little effect and the downward momentum of the descending air must be destroyed; at the same time the upward momentum of the rising air must be generated by a vertical pressure gradient slightly in excess of the hydrostatic value. Under the assumption that the horizontal pressure gradient is the same at all levels, the momenta destroyed and generated per unit area and time must be equal. Thus, at the lower boundary, the speeds, and by continuity also the areas, must be the same for the upward and downward moving air. Since the turbulence in the forced convection surface layer is specified, the turbulence in the departing upcurrent and arriving downcurrent must have the same value—that specified for the surface layer. Physically this means that surface turbulence feeds into the updrafts and returns back into the surface layer from the downdrafts equally. If turbulence values are low, it is postulated, the surface friction is able to increase the turbi-
lence as the air transfers from the downdraft back into the updraft, thus increasing the overall turbulence. Eventually, the increasing dissipation in the surface layer balances this increasing intensity and equilibrium is reached. Similarly, the turbulence diminishes for the case when temporary turbulence exceeds equilibrium values.

Finally, we assume that the vertical potential temperature gradient in the descending air is zero at the surface. If it were unstable, it would become irregular in temperature structure, as in the updraft, and be indistinguishable from it. On the other hand, a positive potential temperature gradient in the descending air at the surface implies the air is moving up and down faster than need be since, if it were slower, the effect of mixing would reduce the stable gradient nearer to zero. Because we can reasonably expect the process to operate with the minimum possible kinetic energy, a neutral vertical density gradient in the descending air at the surface is a required condition for a viable solution.

We therefore need to adjust the four initial conditions at the start of the integration at the top of the layer to provide these required conditions at the surface. This can be done by appropriate iterative techniques.

Thus, by setting in the depth of the convective layer the surface turbulence and the heat flux (or $\partial \rho / \partial t$ which is equivalent), we can obtain a quantitative solution to describe the properties of these plumes.

6. Some solutions for typical conditions for strong convection

a. General remarks

In a theoretical analysis was made of an isolated plume immersed in air with zero mean motion, but turbulent. The results were compared with observations from aircraft of temperature pulses and velocity fluctuations. This treatment rested on two main assumptions: 1) that the rising air forms vertically contiguous plumes or updrafts which are statistically uniform in the horizontal with distinct boundaries, and 2) that entrainment into them can be described as a transfer velocity through this boundary proportioned to the random turbulent velocity within the plume. Detrainment can be similarly described by an outward velocity from the plume proportional to the environmental turbulence.

The present model recognizes not only the need in a field for the environmental air to possess a mean downward motion, but also a need to account for the dissipation of turbulence. This is represented by a standard formula which can be derived dimensionally and has been tested empirically.

In Paper I six independent parameters described unambiguously the initial conditions of an isolated plume. These were temperature excess, radius, updraft velocity, internal turbulence, external turbulence and environmental lapse rate. It was shown, given reasonable values based on the measurements, that the theoretical change of radius and velocity with height seemed consistent with the observations.

The present treatment describes the interaction of both updrafts and downdrafts. Only three independent parameters are needed: the surface turbulence, the rate of change of temperature resulting from the heating, and the depth of the convecting layer. The solutions yield absolute values for radius, updraft velocity, temperature excess, turbulence and average temperatures, as well as their height dependence. Two features of the previous model are again evident. The updraft is almost constant in radius with height after the initial contraction, and the updraft velocity initially increases with height and then diminishes. These points are illustrated in Fig. 1, which also shows the velocity excess of the rising plume relative to its sinking surroundings. This difference varies much less with height than the updraft speed itself. The fact that magnitudes of the radius and other numerical values given in Fig. 1 are derived from the theory shows that it is capable of yielding quantitative estimates which can be compared with measurements.

b. A case of strong convection

The surface turbulence has been taken as 1.0 m sec$^{-1}$ and the rate of change of potential temperature as 1.5K hr$^{-1}$. In Fig. 1 results are plotted which have been calculated for a layer 400 m in depth. The theory predicts that measurements taken when flying at 200 m in these conditions will show pulses averaging 165 m in length with an average temperature excess of 0.18K. This average length is based on circular plumes 105 m in radius which gives an average traverse length of 105$m/2$ m. Observations such as those of Warner and Telford (1967) give values of 200–300 m for the average pulse length. Although the computed figure for the plume size is in good agreement, it is a little small and it should be pointed out that lower heating rates give larger plumes as discussed below. Since the heating rate of 1.5K hr$^{-1}$ used in the theory is rather high and no estimates are available to show that the bulk of the measurements were in such strong convection, the difference is just what should be expected.

In this calculation the radius $r$ of the circle containing both the updraft and its surrounding downdraft was found to be 191 m. Assuming that the length of the pulses observed are proportional to the areas of the updraft and downdraft, the space pulse ratio is 2.3. Warner and Telford (1967) report a value of 1.2 for this ratio, which is the only real discrepancy between the theory and measurements, but Vuifson (1961) is in better agreement (discussed below).

The maximum temperature excess given in Fig. 1 is an average value for the temperature pulse at the bottom of the plumes. The estimates from the measurements, indicating $\sim$1K, have often been peak values,
but Table 1 of Warner and Telford (1967) gives "top hat" temperatures \( \lesssim 0.5K \). This is in reasonably adequate agreement with the computed value of 0.37 in view of the present accuracy of the data. The linear decrease of temperature excess with height is a direct consequence of nearly equal turbulence inside both the rising and the descending air. Eq. (12) shows that \( \Delta \theta \) (i.e., \( \Delta \rho = \rho_i - \rho_p \)) is linearly proportional to \( z \) if \( \partial \rho / \partial z \) is constant. Eq. (1) shows this will be constant if \( i_p = i_s \). Thus, in these solutions \( \partial \rho / \partial z \) is almost constant, and the erosion from the rising air approximately equals the entrainment into it.

The velocity excess is also in acceptable agreement with measured values. However, it is not likely to be profitable to press the comparison further until better observational data are available. More data are needed to get better sampling accuracy. In particular, we need data on the rate of change of temperature with time, surface turbulence, vertical and horizontal temperature structure, and wind.

The example discussed above was chosen as representative of clear sunny mid-morning convection in summer at mid-latitudes. More typical of unselected observations would be a lower rate of change of temperature.

c. Moderate convection

The case, \( \partial \theta / \partial t = 0.375K \text{ hr}^{-1} \), needs no further calculation. This case is analogous to the previous example at 1.5K hr\(^{-1} \) if all velocities are kept unchanged and all lengths doubled. Thus, at twice the height from the surface, the width from the surface, the width from the width and intervening spaces are doubled. To find the radius of the new plumes at 200 m, we need the radius at 100 m from Fig. 1. This radius is only \( \sim 5\% \) larger than at 200 m, so the new solution for \( \partial \theta / \partial t = 0.375 \) about doubles the radius (within 5\%) and results in a space-pulse ratio which remains practically unaltered at 2.3.

Some comment on the interpretation of the measurements is relevant. At present there is no technique for measuring directly the average updraft velocity used in this theory, and we are completely dependent on the temperature trace to identify plumes. The temperature trace is divided into an almost featureless base level with interspersed regions of raised and fluctuating temperature. In this theory, the plume is, fundamentally, the region which ultimately progresses to the higher levels. Air recently eroded from the plume could still well be warmer than the uniform base-level characterizing, for the most part, the descending air. Thus, our simple classification of the plume as the region of raised temperature could be misleading and cause some overestimation of the relative area of the ascending air. Some of the discrepancy between the observation and the present theory could perhaps be due to this.

Another important point arises from the views of some authors, who have reported that the number of plumes per unit area diminishes with increasing height. The present theory is based on the claim that all plumes leaving the surface continue on to the upper boundary. Thus, unless plumes coalesce on the way up their numbers should be the same at all levels. Probably the most detailed and comprehensive study is that of Vulfson. The following discussion is intended to show that alternative explanations should be considered partly because more information is now available in regard to vertical velocity. This work shows beyond doubt that the ther-
mals are found only with convection; however, the assumptions of the statistical analysis are open to some question. The analysis appears to be based on the assumption that the plumes are in outline either perfect circles or ellipses. The actual complexity of the situation is shown by Warner and Telford (1963, their Fig. 4) in which temperature records are given from three thermometers positioned across the aircraft span. These temperature contours graphically demonstrate that the plume is embellished with detail, contour incursions often extending more than halfway through the plume. Frequently within the plumes near the surface, hot fragments appear to be separated by regions of lower temperatures. Thus, the classification as an individual plume of every region above an arbitrary temperature level will give large numbers of very small plumes near the surface (see, e.g., p. 31 of Vul'son). At higher levels the temperature structure is less complex.

If one accepts the increase in vertical velocity in the plumes with height, as found by Warner and Telford (1967), one is led to the conclusion that if all small low-level disturbances are classified as plumes, then these plumes must, as the height increases, coalesce into fewer single entities of smaller total area.

Table 7 of Vul'son implies that if, of the 306 "jets" at 10 m, only 40 "jets" survive to 500 m, then these remaining "jets" must have increased in area individually by a factor of 3–4, since the total area in the updraft has diminished only by a factor of 2 (relative areas being given as 0.50 and 0.27). Since observations, not available to Vul'son, show an increase in upward speed of almost twofold, a more reasonable explanation is to suggest that most of the 306 "jets" near the surface have coalesced to give the 40 "jets" at 500 m. In this way the reduction in total area matches the increase in speed. Thus, the decrease in the number of plumes per unit area with height does not mean some plumes are coming to rest at intermediate levels. This interpretation would also change Vul'son's Fig. 42 so the plumes were largest at the surface and decreased to a near constant value with height. His Fig. 42 at present shows an increase in size up to a height of about 300 m because the small surface "jets" are coalescing.

The assumption that the statistics deal with perfectly circular plumes also leads to a considerable reduction in the estimated plume size. Vul'son (p. 74) adds small plumes to those actually intercepted to restore the deduced size distribution. Thus, an average interception width of 92 m is presented as a "jet" diameter of 62 m; furthermore, he states, including only those "currents" actually recorded, the equivalent diameter is 117 m. It is, therefore, not surprising that his plume sizes are smaller than those which Warner and Telford report.

The number of plumes per unit area will be similarly affected and the deduced behavior with changing height further modified (because this bias is greater in the more variable structure near the surface) toward constant numbers at different heights. This further reduces the force of any suggestion that some plumes perish on the way up. Of course, at the top of the forced convection region where free convection is just starting, some fragments of warm air probably sink back again, but this is not likely to occur above, say, 50 m, where our discussion becomes relevant. These comments do not in any way weaken Vul'son's conclusion that the relative area in the thermals is only 0.27 at 500 m altitude. This is in direct conflict with Warner and Telford (1967) and since no explanation is apparent, further investigation is needed on this point.

d. Limits on possible layer depths

Two other theoretical solutions, with \( \frac{\partial T}{\partial t} = 1.5 \ \text{K hr}^{-1} \), \( i_s = 1.0 \ \text{m sec}^{-1} \), \( Z_0 = 200 \) and \( 500 \) m are shown in Fig. 2.

As the depth \( Z \) decreases, the radius of the plumes tends to diminish but less than in proportion; at some stage, the variation in radius with height changes from a decrease to an increase. This happens in this example at about \( Z = 250 \) m and thereafter the solutions do not appear to be viable, since the velocity (and hence the kinetic energy) of the rising air at the upper boundary is less than that of the descending air. This suggests there may be a minimum depth for which viable solutions can be found.

The changes in the convective field as the depth of the layer increases are illustrated by the second set of graphs in Fig. 2, for a layer depth of 550 m. In comparison to the example for 400 m given in Fig. 1, the radius has only increased from 105 to 120 m. The holding area radius has, however, increased from 191 to 352 m with the average number of plumes per square kilometer decreasing from 8.7 to 2.5 (i.e., despite the fact that the average plume area has only increased by \( \sim 30\% \)). These plumes are now carrying about twice the temperature excess at about twice the velocity; hence, the smaller number can transport the heat necessary to give the same rate of temperature increase.

Increasing the depth through which the convection must carry the heat gives fewer, hotter, faster plumes of about the same size. In this case, when the depth is about 590 m these tendencies become unbounded, and there is no real solution; at this stage the infinitely separated plumes have infinite temperature excesses. Thus, there are no solutions to the steady-state convection field beyond a particular depth determined by the given conditions.

c. Temperature gradients

Previous theories have predicted temperature and velocity profiles with height. Fig. 3 shows the results of the present theory, the temperature profiles for the subsiding air, and for the horizontal space-average temperature. When appropriately scaled, the three cases shown are representative of all conditions in which this
theory predicts physically viable solutions. They are in agreement with the well-established observations of many workers for the temperature profiles.

7. Concluding remarks

The theory of an isolated plume in still air has been extended to provide a complete picture of rising and descending air in a field of plumes. The quantitative predictions of the theory agree fairly well with the available observations.

The changes found on increasing the layer depth give a clear indication that no solutions are available beyond a certain maximum, depending on heat flux and turbulence. If valid, this is a most significant result. Some remarks are therefore appropriate in regard to the general fitness of the model to describe the actual physical conditions. The most encouraging feature of the theory is that it predicts plumes of about the right size. Actual plumes are about 200 m in width, and the model yields this size plume when the total depth of the heated layer is only 400 m. Thus, in moving only a small fraction of its width vertically upward from the ground, the mixing process results in the plume following the same behavior pattern which theoretically has been attributed, in the model plume, to horizontal uniformity. A turbulent decay based on the scale of the plume has been assumed. At first sight, the width of the plumes...

Fig. 3. Temperature profiles for the three cases shown in Figs. 1 and 2. The subsiding air (potential temperature $\theta_0$) shows the neutral and then stable temperature profiles with increasing height. The average temperature $\theta_{av}$ is at first unstable near the surface, then neutral, and slightly stable with increasing height. These three cases, when appropriately scaled, are representative of most practical circumstances.
and the assumption of uniformity appear quite inconsistent. It seems surprising, for example, that there can be sufficient horizontal transport in a few meters of disorganized surface air to feed the bottom of a plume a hundred times this size at more than 1 m sec\(^{-1}\). This appears, however, to be what actually occurs at the surface. A possible explanation is that the plume can be fed by moving around over the thin surface layer, gathering the hotter air from in front of it and leaving behind cooler air from the downdraft. Thus, the hot surface layer is peeled away to rise up in the plumes, and the base of the plume, therefore, can be expected to move horizontally at several times its vertical velocity. Since the surface air is appreciably hotter than in the updraft, some of the downdraft air can return to the updraft without entering in the lowest layers; only some of the updraft air experiences the maximum heating. Since the position where the new hot air enters the plume at the base is always moving away from the position where the earlier air began to rise, the base of the plume must always move relative to the air in the forced convection layer. Aloft, the plumes must move more slowly than the average wind since they must transport upward the horizontal momentum arising from the wind drag at the surface. Thus, the surface layer peels off and flows into the plume on the upwind side.

Mixing of the descending air into the plume probably occurs mostly on the hotter upwind side because it does, in fact, lean with the wind. Thus, small buoyant elements rising through the plume itself break through this side and promote mixing. Such small elements certainly carry much of the heat flux (Telford and Warner, 1964). A large part of the buoyant energy that goes to drive the turbulence is likely to be released where the mixing occurs, thus encouraging faster mixing over this top surface. Thus, the overall effect may be the same as if this turbulence were evenly released over the whole area and resulted in entrainment around the whole periphery. In a similar way erosion from the plume probably occurs at a high rate from only a part of the boundary. There are likely to be cooler pieces on the lower downward side which erode, and here detachment is enhanced because these pieces are left below the underside as the more buoyant portions migrate upward. Thus, again, as for entrainment, the overall effect appears to agree with the model where detachment occurs equally in all directions.

Theoretical models such as this serve a useful purpose by providing a framework for the design and interpretation of observations, but the limitations of the theory must be borne in mind, and discussion of details is of limited relevance except in a conceptual framework.

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