NOTES AND CORRESPONDENCE

Approximate Formulas Fitted to the Davis-Sartor-Schafrir-Neiburger Droplet Collision Efficiency Calculations

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ABSTRACT

Two approximate interpolating formulas suitable for rapid computation are given that fit a combined version of the Shafrir-Neiburger and Davis-Sartor collision efficiencies for uncharged spherical cloud droplets. The method of fitting is indicated for use in approximating future improvements in the theory.

The most reliable theoretical predictions of the collision efficiencies for uncharged spherical cloud droplets are those of Shafrir and Neiburger (1964) as recently modified by Davis and Sartor (1967). These results are needed for computer programs concerned with the history of droplet spectra in various cloud circumstances, and it is desirable to have a relatively simple and fast, if approximate, formula that will allow smooth interpolation to be carried on and will maintain the appropriate limiting characteristics of these theories. This note reports two such formulas and the methods by which they were obtained. While the two methods give nearly the same quality of fit to the Davis (1967) graphs, the active state of research in droplet collision theory makes it likely that new results will be shortly available, and it will not be known until then which of our fitting methods will continue to be useful. Hence both are included and enough details given to facilitate such future applications.

Our work started with the discovery several years ago that a quite simple formula can fit the Shafrir-Neiburger (SN) results to a precision that is undoubtedly within the accuracy of their theory. Letting the linear collision efficiency (the ratio of the radius of the cylinder swept out by the large droplet of radius a_L to the radius of the cylinder within which the center of the small droplet of radius a_S must lie if a collision is to occur) be Y_c, and the ratio a_S/a_L be x, we have

$$Y_c=\max\{0,\ 1+x-B/[x(1-x)]\}, \qquad (1)$$

where negative values of Y_c are replaced by zero and

$$B=34.3a_L^{-1.7},\ \text{with } a_L \text{ in microns.} \qquad (2)$$

For large a_L and small B, Y_c approaches the geometrical limit, $1+x$, and for decreasing a_L and increasing B, there comes a cut-off beyond which no positive values of Y_c exist, corresponding to the well known 19 μ cut-off of Hocking's (1955) original theory. Fig. 1 shows the extent of agreement between Eq. (1) and the SN results.

We follow here the graphical results of Davis (1967), which are that Y_c is positive for x between 0 and 1, even for a_L as small as 10 μ. Davis' curves for $a_L=10$, 20, 25 and 30 μ all show $Y_c=0$ at $x=0$ and $Y_c=0.25$ at $x=1$. We assume these end point values to hold for larger a_L as well, which amounts to assuming the SN theory incorrect at these end points and the Davis-Sartar theory correct. For intermediate results the curves are similar to the SN curves, but with maxima at different heights.

Our first effort at fitting involved adjusting (1) to provide the Davis end-point values at $x=0$ and 1 and making provision for varying the location of the maxima for several values of a_L. Eq. (1) was modified to give

$$Y_c=1+x-\frac{B_1}{[x^m+x_1{}^m]^{1/m}}-\frac{B_2}{[(1-x)^n+u_2{}^n]^{1/n}}, \qquad (3)$$

where x_1 and u_2 are obtained from

$$\left.\begin{aligned} 0&=1-\frac{B_1}{x_1}-\frac{B_2}{[1+u_2{}^n]^{1/n}}\\ 0.25&=2-\frac{B_1}{[1+x_1{}^m]^{1/m}}-\frac{B_2}{u_2}\end{aligned}\right\}, \qquad (4)$$

or equivalently,

$$\left.\begin{aligned} x_1&=\frac{B_1}{1-B_2[1+u_2{}^n]^{-1/n}}\\ u_2&=\frac{B_2}{1.75-B_1[1+x_1{}^m]^{-1/m}}\end{aligned}\right\}. \qquad (5)$$

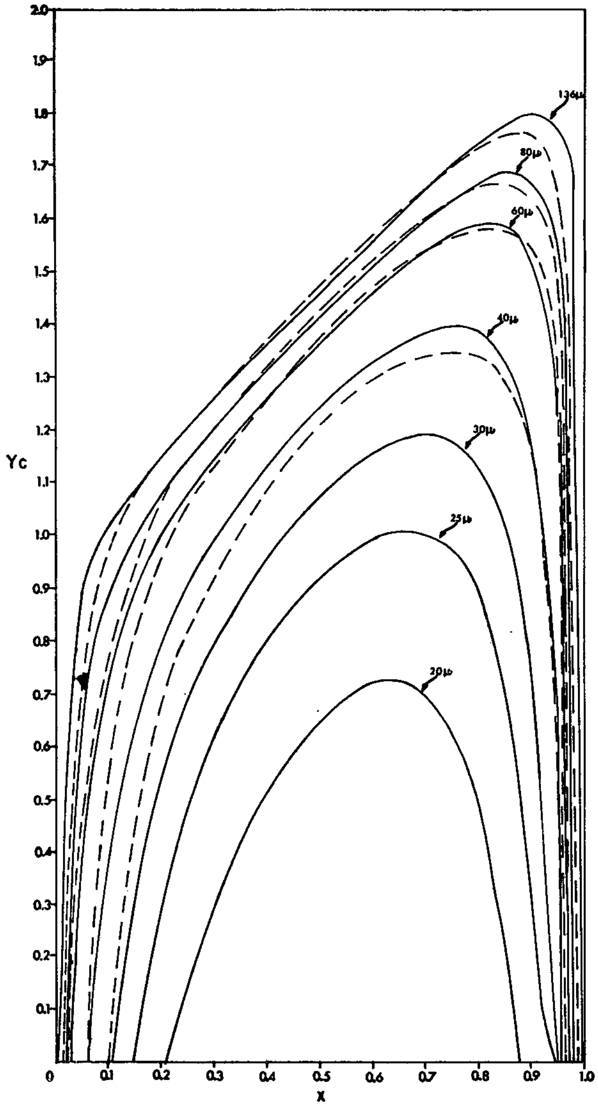

FIG. 1. The extent of agreement between Eq. (1), solid lines, and the *SN* result, dashed lines. For $a_L=20$, 25, and 30 μ, the difference is insignificant.

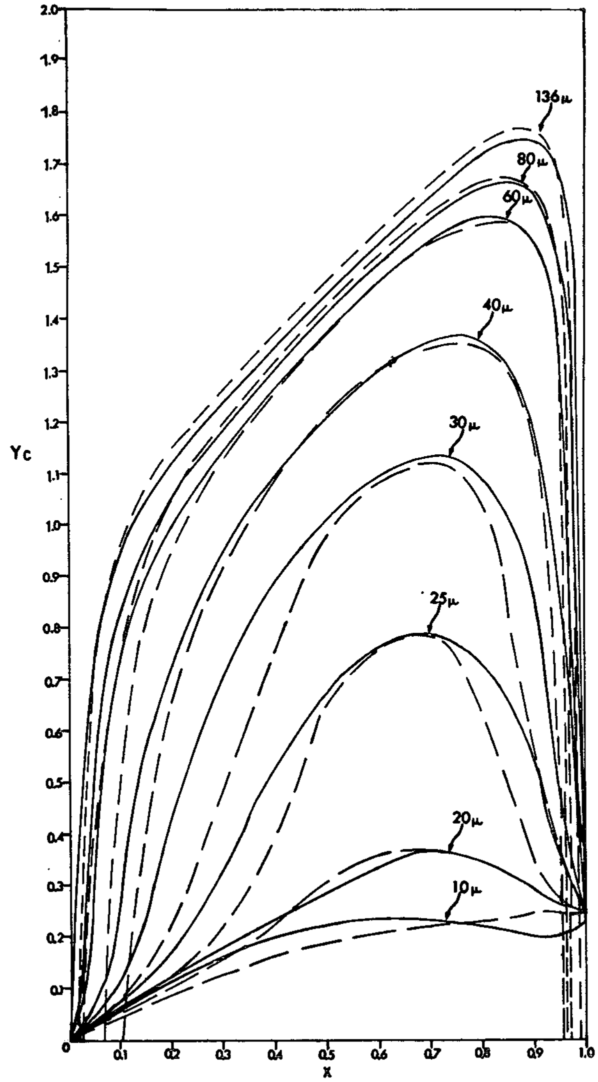

FIG. 2. First approximate formula, Eqs. (3)–(8), solid lines, compared with SN ($a_L \geq 40\,\mu$) and Davis ($a_L \leqslant 30\,\mu$) curves, dashed lines.

A rapid solution of these equations is obtained by successive approximation once B_1, B_2, m and n are known, by starting with $x_1=u_2=0$ on the right side of (5).

The values of m and n were chosen by having the computer print out graphs of Y_c vs x for chosen values of m and n and of B_1 and B_2 and comparing the shapes of the curves with the SN results for $a_L \geq 40$ and with the Davis curves for smaller a_L. The values that seemed to be the best choice obtainable are given by

$$\left.\begin{matrix} m=6 \\ n=1.5 \end{matrix}\right\}. \tag{6}$$

Trial and error was then used with the same graphing print-out to find values of B_1 and B_2 that made the maxima of the curves agree with the SN values for $a_L=40$, 60, 80 and 136 μ, and with the Davis values for $a_L=10$, 20, 25 and 30 μ. The first set of values found involved quite small differences between B_1 and B_2; a slight adjustment of each allowed them to be set equal, $B_1=B_2=B$. The results are shown below:

$a_L(\mu)$	10	20	25	30	40	60	80	136
B	0.486	0.365	0.222	0.125	0.0756	0.0337	0.0253	0.01345

The next task was to find a formula for B as a function of a_L. The first method used started with a plot of $Z=40 \ln B$ vs a_L, yielding a curve with the appearance of a sinusoidal oscillation about the straight line, $Z=-16.587-1.1413\,a_L$. The curve crossed this line at $a_L=10$, 20 and 136 μ. The function

$$(a_L-10)\cos\left[\frac{\pi}{2}\left(\frac{a_L-78}{58}\right)\right]$$

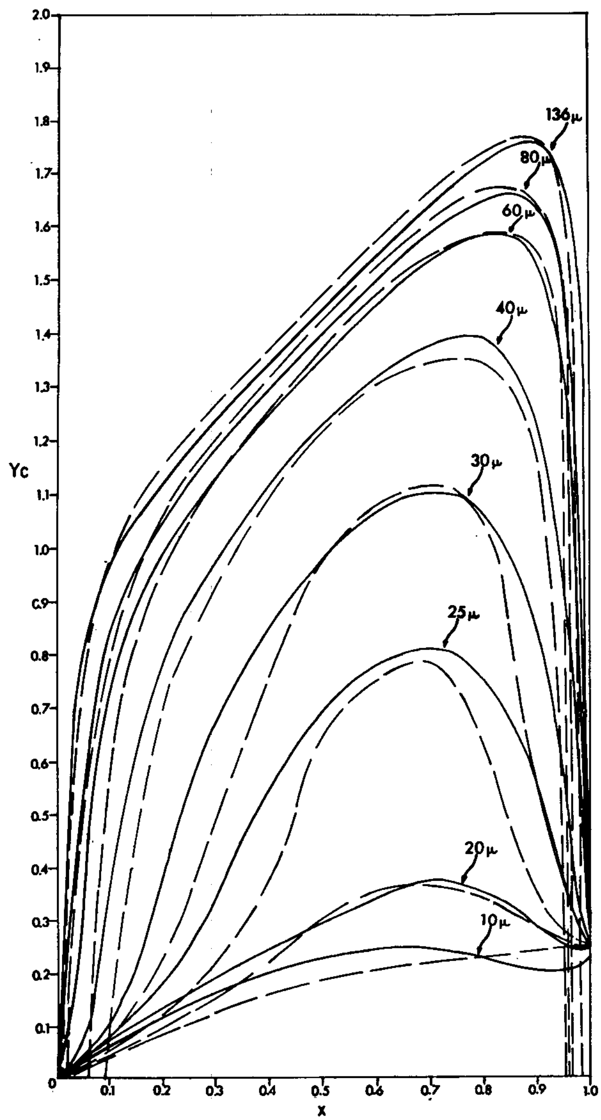

FIG. 3. Second approximate formula, Eqs. (3)–(6) and (9), solid lines, compared with SN ($a_L \geqslant 40\ \mu$) and Davis ($a_L \leqslant 30\ \mu$) curves, dashed lines. Solid line: second formula.

vanishes at $a_L = 10$, 20 and 136 μ, so this function multiplied by a polynomial $g(a_L)$ was added to the straight-line expression for Z. The polynomial was then chosen to fit the other values of a_L, yielding

$$g(a_L) = -4.5039 - 1.1135a_L + 0.061475a_L^2 - 0.0010388a_L^3 + 5.65\times10^{-6}a_L^4. \quad (7)$$

Thus, we finally find our first formula

$$B_1 = B_2 = B(a_L) = \exp\left\{\left[-16.587 - 1.1413a_L + (a_L - 10)g(a_L)\right]\cos\left[\frac{\pi}{2}\left(\frac{a_L - 78}{58}\right)\right]\Big/40\right\}, \quad (8)$$

with $g(a_L)$ given by Eq. (7).

The joint use of Eqs. (3), (5), (6), (7) and (8) represents a good interpolation of the efficiency of theories up to a $a_L \approx 200\ \mu$, but for larger values the fourth power term in $g(a_L)$ will produce a divergence.

The second method of fitting was carried out by plotting Ba_L^2 vs a_L and observing that the curve had the appearance of a damped sinusoidal "wave" superimposed on a straight line of positive slope. The zeroes of the "wave" came at approximately $a_L = 10$, 73 and 136 μ, and the damping factor was easily established. The resulting second formula is

$$B = 1.587a_L + 32.73 + 344(20/a_L)^{1.56} \times \exp[-(a_L - 10)/15]\sin[\pi(a_L - 10)/63]/a_L. \quad (9)$$

The damped oscillation and inverse powers of a_L in (9) make formulas (3), (5), (6) and (9) reasonable interpolator for all values of $a_L \geq 10\ \mu$.

Figs. 2 and 3 show graphs computed by these formulas with the original results of SN and Davis as comparison. Applications to droplet-growth computations by the Telford (1955) and Berry (1967) methods will be given at a later date.

As further, more accurate, theoretical calculations of collision and collection efficiencies become available, methods similar to those outlined here can undoubtedly be used to find corresponding approximate representations with relatively little effort.

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