Dependence of the Highly Truncated Spectral Vorticity Equation on Initial Conditions

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ABSTRACT

The analytic solutions in time to the highly truncated (low-order) spectral vorticity equation, involving the nonlinear interaction of one planetary wave with an arbitrary zonal flow, have been investigated for a wide variety of initial conditions which span the range of atmospheric observations. These conditions include both a characteristic mean wintertime zonal jet, a simulated double jet, all planetary waves from wavenumbers 1-12, and various wave configurations for wavenumber 3. The results show wide variability in solutions from configurations which are almost independent in time to those which describe highly elliptic oscillations. There is no systematic dependence of nonlinearity on total energy amplitude of the system nor do small initial perturbations necessarily imply quasi-linearity. The solutions described for this simple model exemplify the extreme complexity of large-scale atmospheric turbulence.

1. Introduction

Whereas the details of energy transfer among different scales of motion in the atmosphere are obscured by the inherent nonlinearity of the physical system and its mathematical representation, a highly simplified model has been developed which has exact solutions. This model, the truncated form of the spectral barotropic vorticity equation, was first considered by Lorenz (1960) in an unbounded channel and later by Platzman (1962) in a spherical surface. The nature of the solution wherein an arbitrary zonal flow interacts with a single planetary wave has been presented by the author (1970) in terms of elliptic functions.

Availability of the exact solutions in time permits a discussion of the nonlinear energy transfers as they depend on the initial conditions. Such conditions may be represented by the total energy in the system (a conservative property), the relative energy initially in the wave, the latitudinal distribution of the zonal flow, and the profile and scale of the planetary wave. Initial conditions are chosen to include a wide range of possible atmospheric configurations and the consequent solutions show large variability. Although atmospheric wave energy is only infrequently adequately described by a single planetary wave, the solutions shown in the following discussion should indicate the complexity of the more general turbulent exchange.

2. The truncated system

The highly truncated form of the spectral vorticity equation, frequently referred to as a low-order system, evolves from an expansion of the streamfunction $\psi$ in terms of surface solid harmonics, substitution of the series into the barotropic vorticity equation,

$$ \frac{\partial}{\partial t} \mathbf{\nabla} \psi = J(\mathbf{\nabla} \psi + f, \psi), $$

and integration over the spherical surface. The resulting equations for the time-dependent coefficients—nonlinear coupled ordinary first-order differential equations in time—have been presented by Platzman (1960). The truncation used to reduce (1) to low-order form allows for an arbitrary zonal flow $\psi(\mu, t)$ and two complex wave components involving only one planetary wavenumber ($l$) and two different latitudinal wavenumbers ($n_\alpha, n_\beta$), where $n$ represents the ordinal number of the solid harmonics (the degree of the opolynomial). Thus, we have

$$ \psi = \psi_a(t, l) + \psi'/(\mu, \lambda, \mu, l) $$

$$ \psi' = \sum_\gamma \psi_{\gamma}(l) Y_\gamma(\mu) $$

$$ \psi_a = \psi_a(l) Y_a(\lambda, \mu) + \psi_{\beta}(l) Y_\beta(\lambda, \mu) $$

$$ \alpha = n_\alpha + il, \quad \beta = n_\beta + il $$

where $\mu = \sin \phi$ (where $\phi$ is the latitude), $\lambda$ the longitude, and $\gamma$ a zonal wave vector similar to $\alpha$ and $\beta$ but for $l=0$. The solid harmonics $Y_a(\lambda, \mu)$ are normalized over the unit sphere and have been discussed in many references (see for example, Hobson, 1955). They satisfy the differential equation

$$ \nabla^2 Y_a = -c_a Y_a, $$

where

$$ c_a = n_\alpha(n_\alpha + 1). $$

Following the procedure outlined above with the expansion (2), the low-order equations which develop
from (1) are
\[
\psi_n = 2a_n \text{Im} \psi \psi^* \\
\psi_\alpha = -i \phi_\alpha \psi_\alpha + i g_\alpha \psi \psi_\alpha ^* + i g_\alpha \psi \psi_\alpha ^* \\
\psi_\beta = -i \phi_\beta \psi_\beta + i g_\beta \psi \psi_\beta ^* + i g_\beta \psi \psi_\beta ^*.
\] (3)

In Eqs. (3) the coefficients \((a, \phi, \psi, \beta)\) all depend on the choice of wave components \((\alpha, \beta)\), the dot represents time differentiation, and the asterisk conjugation. Furthermore, \(n\) is selected as a specific value of \(\gamma\) since it can be shown that all the zonal coefficients \(\psi_\gamma\) are linearly related to one another and hence to \(\psi_n\). If the streamfunction coefficients are represented in terms of amplitude and phase, i.e.,
\[
\psi_n = B_n \cos \omega t, \quad \psi_\alpha = 2^{-1} B_n e^{i \theta},
\]
the solution to (3) is given in terms of the function \(B_n\) as
\[
B_n = B_n \sin \omega t,
\] (4)
which depends on the elliptic sine function, the frequency \(\omega\) and the initial conditions. The nonlinear period is given as \(T = 4K/\omega\) where \(K\) is the complete elliptic integral of the first kind. Moreover, since the other dependent variables \((B_n, B_\beta, \theta_n, \theta_\beta)\) are known functions of \(B_n\), the solution is completely specified. For complete details on the solutions given by (4), the reader is referred to Baer (1970).

3. Initial conditions and truncation

Characteristic properties of the solution (4), and, consequently, the behavior of the system (3) as it reflects the true atmosphere, will be described in terms of the initial configuration of the system to be studied, since (3) is represented as an initial value problem. However, since we are dealing with a highly truncated system, the character of the truncation applied will also affect the solution. We may therefore state that the following parameters are necessary as initial conditions for solution of (3):

- Total available kinetic energy
- Kinetic energy in the wave
- Latitudinal distribution of zonal flow
- Wavenumber and profile

These parameters may be varied, thereby yielding solutions to (3) for different atmospheric conditions. Aside from the specification of these initial parameters, however, it is necessary—as indicated above—to establish truncation by determining the wave vectors \(\alpha, \beta\) and the allowed range of \(\gamma\). For consistency in establishing different solutions, we shall always use for truncation the wave vectors (including zonals) given by the initial representation, together with a zero initial phase angle for both wave components \(\alpha\) and \(\beta\). The latter condition may be made completely general if the initial specification of the zonal field includes components which may be active but begin with zero amplitude.

The wave truncation is directly related to the initial specification (the fourth parameter) and cannot be altered.

Let us now consider the initial representation in somewhat more detail. Following the presentation used by the writer (1964), we will define the stream field as
\[
\psi_\gamma = \frac{\gamma F_\gamma (\mu) \cos \lambda}{2(2 - \delta, \beta)} \left\{ \begin{array}{l}
\sum_\gamma \psi_\gamma P_\gamma (\mu), \quad l = 0 \\
= 2(\psi_\alpha \psi_\alpha + \psi_\beta \psi_\beta) \cos \lambda, \quad l \neq 0
\end{array} \right.
\] (5)
where \(g_l\) represents an amplitude factor for the zonal motion or the wave, \(F_\gamma (\mu)\) describes the initial latitudinal distribution, and \(P_\gamma (\mu)\) represents the normalized associated Legendre polynomial for wave vector \(\alpha\). The stream coefficients in (5) may be determined by applying the orthogonality condition for the Legendre polynomials with the result that
\[
\psi_\gamma = \frac{g_\gamma A_\gamma}{2(2 - \delta, \beta)} \int_{-1}^{1} P_\gamma (\mu) d\mu,
\] (6)
where \(\gamma\) will represent \(\alpha, \beta\) for \(l \neq 0\). The kinetic energy in the zonal field and the wave may be written
\[
K_l = \left\{ \begin{array}{l}
\sum_\gamma c_\gamma \psi_\gamma^2, \quad l = 0 \\
2c_\alpha \psi_\alpha^2 + 2c_\beta \psi_\beta^2, \quad l \neq 0
\end{array} \right.
\] (7)
Finally, if the energy and truncation are specified, the amplitude factors may be computed as
\[
ge_0 = \frac{2}{\sum_\gamma c_\gamma A_\gamma^2} \left( \frac{K_0}{\sum_\gamma c_\gamma A_\gamma^2} \right)^{1/2},
\] (8)
\[
ge_l = \frac{2}{c_\alpha A_\alpha^2 + c_\beta A_\beta^2} \left( \frac{K_l}{c_\alpha A_\alpha^2 + c_\beta A_\beta^2} \right)^{1/2}.
\]
In terms of (5)–(8), we may now discuss the details of the initialization. The process outlined is chosen so that systematic variation of parameters is easily achieved. It should be noted, however, that the numerical specification of all the required stream coefficients is sufficient as initial conditions.

a. Total available energy

We have chosen to represent the total energy available to the system by assuming that all the initial energy resides in the zonal flow described by a specified profile and an amplitude such that
\[
u(\mu = 0) = \frac{\partial}{\partial \gamma} G(\mu).
\] (9)
The distribution \(G\), in effect, determines the factors \(A_\gamma\) from the relationship between the stream field and the
zonal wind, i.e.,

$$F_0(\mu) = \int_0^\infty (1-\mu^2)^{-1}G(\mu)d\mu,$$

and the second of Eqs. (6). If we now set \( a_0 = -\bar{a}_0 \), we may compute the initial zonal stream coefficients from the first of (6) and the zonal kinetic energy (here the total energy) from the first of (7). Thus, we state that the kinetic energy of the system will be given formally as

$$K = K(\bar{a}_0, G),$$

(10)

following the calculation procedure outlined above. Depending on the function \( G \), some of the energy computed from (10) may not be available for energy exchange if there exist some \( \psi_\gamma \) for which \( a_\gamma = 0 \). Thus, we may define the initial available kinetic energy as

$$K_a = K - K_u,$$

$$K_u = \sum_\gamma c_\gamma \psi_\gamma^2, \quad \gamma \text{ only for } a_\gamma = 0.$$

(11)

b. Kinetic energy in the wave

With the determination of the total available energy computed from the specification of the zonal wind amplitude \( \bar{a}_0 \) and profile \( G \), we may describe the kinetic energy in the wave relative to the total energy by the parameter \( \rho \), where

$$\rho = \frac{K}{K_a}.$$

(12)

For fixed \( K \) the energy may be partitioned initially between the zonal flow and the wave, differently for different values of \( \rho \). Once \( \rho \) is specified, \( K_1 \) is easily calculated from (12). Given the wave profile \( (A_a, A_\gamma) \) we may then calculate the wave amplitude from (8) and finally the wave stream coefficients from (6). Furthermore, since the zonal kinetic energy is given from the total energy and that in the wave,

$$K_0 = K(\bar{a}_0, G) - K_1,$$

the zonal amplitude \( \bar{a}_0(G) \) and the zonal stream coefficients are computed in a manner identical to the calculation of the wave coefficients, using the known values of the \( A_\gamma \).

c. Latitudinal distribution of zonal flow

The latitudinal distribution of zonal flow implies a specification of the function \( G(\mu) \) as may be seen from (9). There are a number of ways in which this may be accomplished, but perhaps the most general form would be a polynomial in \( \mu = \sin \phi \). Since, however, we wish the zonal field to be symmetric about the equator (an even function) and since we furthermore wish to have a series which is easily represented by a finite series of Legendre polynomials, we choose the form

$$G(\mu) = (1-\mu^2)^{1/2} \sum_{\ell=0}^M \hat{a}_\ell \mu^{2\ell},$$

(13)

where the coefficients \( \hat{a}_\ell \) may be chosen so that the series represents the desired zonal flow data. Integrating \( G \) as indicated in the equation following (9), we find

$$F_0(\mu) = \sum_{\ell=0}^M \frac{\hat{a}_\ell}{2\ell+1} \mu^{2\ell+1}.$$

Now since \( \mu^{2\ell+1} \) may be represented in terms of a series of normalized Legendre polynomials (see Platzman, 1960), the function \( F_0 \) may also be represented as a series of Legendre polynomials of the form

$$F_0(\mu) = \sum_{j=0}^M b_j P_{2j+1}^{2j+1},$$

(14)

where

$$b_j = 2^{2j+1}(4j+3)^{1/2} \sum_{i=j}^M (2i)! \left[ \frac{(i+j+1)!(i-j)!(2i+2j+3)!}{} \right] \hat{a}_i.$$ 

Finally we have from (6) that the parameters which specify profile are given as

$$A_{2j+1} = 2b_j.$$

(15)

Two zonal profiles have been studied and both apply for the barotropic problem. One has been designed to conform to the observed normal wind profile for January at 500 mb, and the other to a split-jet. The observed jet and the representations used are shown in Fig. 1. To establish the jet corresponding to the observed, we have used the representation

$$G(\mu) = (\sin^4 3^0 \phi + \sin^2 2^0 \phi - \alpha \cos^2 \phi) \cos \phi,$$

(16)

where the conversion of (16) to the form (13) may be achieved by direct expansion. The distribution (16) allows for a jet with maximum around 30N, and a choice of \( \alpha = 0.2 \) will establish the equatorial easterly with magnitude one-fifth that of the maximum westerly jet, comparable to the observed profile. The coefficients \( \hat{a}_\ell \) for this profile (denoted A-J) are listed in Table 1.

The double jet (D-J) profile has been chosen from the representation

$$G(\mu) = (\sin^4 4^0 \phi + 2\sin^2 \phi) \cos \phi,$$

(17)

and is also described in Fig. 1. The coefficients \( \hat{a}_\ell \) for this profile are listed in Table 1.

d. Wavenumber and profile

The constraint imposed by the low-order system allows interaction between only one planetary wave and a zonal flow. However, the selection of the planetary wavenumber is arbitrary and must be set as an initial condition. We must furthermore specify the profile...
parameters $A_\alpha$ and $A_\beta$ either directly or through a profile function $F_l(\mu)$; their interpretation has already been made evident. The profile function may be interpreted physically as the latitudinal distribution of the meridional wind component at the longitude where the wave zonal wind component vanishes.

It is immediately apparent that the wave representation is considerably more constrained than that for the zonal wind field. One may, however, adjust profiles by a combination of the $\alpha$ and $\beta$ contributions. Consider, for example, the profile

$$F_l(\mu) = \sin^4 \phi \cos^2 \phi,$$

a representation used by the author in a previous study (Baer, 1964). Such a profile, with specification of $r_l$ and $q_l$, may be easily varied for different planetary wave numbers. On the assumption that we wish to investigate the nonlinear interaction of the longest allowed waves in the latitudinal direction (given the planetary wave number $l$), that the wave should interact with the lowest active component of the zonal field ($\gamma=3$), and that, furthermore, (18) can indeed be represented by no more or less than two associated Legendre polynomials, we find from the definition of the polynomials (Jahnke and Emde, 1945) that the following restrictions must be imposed:

$$r_l + q_l = l + 2s_l,$$
$$2r_l + q_l = l + 3,$$

where $s_l$ may take on the values 0 or 1, as desired. Solving for $r_l$ and $q_l$ from (19) in terms of the known quantities $s_l$ and $l$, we find for the profile function that

$$F_l = 2^{(l-2s_l)(1-\mu^2)^{1/2}}[s_l + (-)^s l^2].$$

Expressing (20) in terms of $P_\alpha$ and $P_\beta$ and substituting into the second of (6), the integration yields

$$A_\alpha = p_l \left[ (2l+5)s_l + (-)^s 3 \right] \left[ \frac{3(2l+7)}{l+1} \right]^{-1},$$

$$A_\beta = (-)^s 6p_l,$$

$$p_l = 2^{(l+5)} \left[ \frac{(l+1)!}{2l+5} \right] \left[ 6(2l+7)(2l+3)! \right]^{-1}.$$

To understand the limitations placed on the profile (20), several other profiles have been tested by barotropic calculation. Fig. 2 describes various profiles for wave $l=3$, including $F_l$ [Eq. (20)] for $s_l=0$, and variable
coefficients for \( n_a = 4 \), \( n_B = 8 \). The amplitudes of the coefficients are listed on the figure and need not be duplicated in the text. It is noteworthy, however, that (20) describes the longest wave—as intended—and that other allowed profiles can be remarkably different, possibly leading to different solutions of (3). Other profiles have also been used for calculation and will be discussed subsequently.

4. Some barotropic calculations

The exact variation of system (3) will be described only by applying specific initial conditions to the solution outlined in Section 2. There are, however, an infinite variety of initial conditions which could be considered. We consequently confine our attention to a discussion of barotropic flow with a limited range of initial conditions. Those conditions, as we shall see, confine the energy of the system to the range of atmospheric possibility, the zonal wind configuration to realistic flows, and the wave configurations to simple distributions for the long- and medium-scale waves. In all calculations, we shall investigate the changes in the system due to variations in \( \rho \) (the relative energy in the perturbation) while maintaining the other initial conditions invariant. We may then expect to observe, over the range of atmospheric conditions, the variability of the model solutions.

In the calculations to be discussed, five different amplitudes \( (\tilde{a}_0) \) were used, three with the A-J (see Section 3 for definitions) and two with the D-J. These amplitudes have been used together with the jet profiles to give an average energy \( K \) [see (10)] for a layer of 50 mb in units of \( 10^4 \) J m\(^{-2}\). The choice of these units was established to compare with the values given by Kung (1966a) for winter, summer and the annual mean of 1962, all of which have been listed in Table 2. It is evident from Table 2 that the range of energy chosen for calculation effectively straddles the values observed in the atmosphere. One could reasonably extend the range of the calculations, but the results from the values of energy actually used may be meaningfully compared to atmospheric events.

The zonal profiles used in the computations have already been described in Section 3 and depicted in Figs. 1 and 2. The A-J is clearly applicable to the real atmosphere by definition, whereas D-J was chosen more for simplicity of representation than for reality. A more realistic D-J would require too much resolution in the zonal coefficients and consequently work counter to the low-order concept. Nevertheless, the computations with the selected D-J should be indicative of the interactions which might take place in atmospheric flow with a double jet.

Having established the variations on the zonal flow, we now consider the initial wave profiles. The wave energy will be considered, as suggested previously, by allowing \( \rho \) to vary over its entire range \((0 < \rho \leq 1)\). We first select the simple profile given by (20) and allow \( \ell \) to vary over the range \( 1 \leq \ell \leq 18 \); this profile has the virtue of involving only the lowest allowed modes in the latitudinal direction. From calculations with this function we are able to establish how the nonlinear properties of our system vary with regard to wave-number. Let us generalize on these solutions, however, we have chosen to investigate a number of different wave profiles for a given wavenumber \((\ell = 3)\) to determine the variability of solutions in terms of wave profiles. These profiles allow for arbitrary specification of \( A_{\alpha}, A_{\beta}, n_{\beta}, \) and \( n_{\alpha} \) (where we require, with complete generality, \( n_{\alpha} > n_{\beta} \)). Some of these profiles have been presented in Fig. 2, together with \( F_1 \) of Eq. (20) for \( \ell = 3 \).

The low-order truncation for the initial configurations discussed above may be conveniently represented on an \( n-\ell \) diagram in which the active components (represented by \( \gamma, \alpha, \beta \) values) are described by circles. Fig. 3 includes a number of such diagrams and we shall denote the allowed (circled) components in each diagram as a spectral configuration. Reference to Table 1 indicates

| Table 2. Mean kinetic energy \((10^4 \text{ J m}^{-2}) \) used in calculations and observed atmospheric values (Kung, 1966a) for a 50-mb layer. Observed values have been averaged between 900–100 mb. |
|---|---|---|
| \( \bar{a}_0 \) | Jet | \( K \) |
| 0.075 | A-J | 0.340 |
| 0.135 | A-J | 1.10 |
| 0.275 | A-J | 4.57 |
| 0.067 | D-J | 1.34 |
| 0.133 | D-J | 5.33 |

Observed values: Kung (1966b)

<table>
<thead>
<tr>
<th>Season</th>
<th>( K )</th>
</tr>
</thead>
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<tr>
<td>Summer</td>
<td>0.527</td>
</tr>
<tr>
<td>Winter</td>
<td>1.46</td>
</tr>
<tr>
<td>Annual mean</td>
<td>0.948</td>
</tr>
<tr>
<td>Layer extremes</td>
<td>0.130–3.01</td>
</tr>
</tbody>
</table>

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![Fig. 2. Wave profiles vs latitude for wave \( \ell = 3 \) including different combinations of the polynomials \( P^a, P^b \), presented on a relative scale.](image-url)
that the A-J has eight zonal components whereas the D-J has only four. Columns one and two of Fig. 3 show some possible configurations with either of the allowed zonal jets, using the wave profile given by (20) for several different wavenumbers. The last two columns of the figure show the configurations for the A-J with different wave profiles of wave $l=3$.

We shall now consider some of the features of the low-order system which result from the solutions for specific initial conditions and truncation. Foremost among these features is the nonlinear energy exchange period. Since the conventional linear theory cannot predict nonlinear periods, their prediction by the lower-order equations should add significantly to our understanding of the more general system. Nevertheless, unless the atmosphere is constituted as a highly truncated system (not a frequent occurrence), one should not anticipate—nor do we find—exact periodicity in nonlinear flow. There are indications that some quasi-periodicity may exist in the atmosphere (Namias, 1954; Eliasen, 1958) although data limitations must be considered in evaluating such observations. Furthermore, integrations of less truncated models (Baer, 1964) also indicate periodicities, although the number of such calculations is severely limited. We must therefore interpret the observation of exact periods as representing atmospheric behavior with a mixture of optimism and suspicion.

Fig. 4 describes the nonlinear exchange period $T$ as defined in Section 2, in days, for various energy levels, both jets, and a selection of wavenumbers ($l=1, 3, 6, 12$) for the wave configuration given by (20); the periods are plotted against $\mu$ (ordinate). We first observe that for all realistic values of $\mu$ (values of $\mu>0.8$ are rarely, if ever, observed in the atmosphere) the periods are relatively insensitive to the initial partitioning of energy between the zonal flow and the wave. For all waves there appears to be a systematic decrease in the period with increasing total energy: the curves for those waves not described in the figure show no erratic behavior and may be interpolated from the given data. The periods for the D-J appear somewhat longer than those for the A-J, but this variation is not of great significance. We also note a maximum in the period for wave three, with periods gradually becoming shorter for shorter wavelengths. The magnitudes of the periods, however, which run from less than one to about four days are not in close agreement with previous calculation or observation. Whereas the three-day period for wavenumber 1 is in agreement with a previous calculation by the writer (Baer, 1964), the value for wavenumber 3 does not compare favorably (previously calculated as $\sim 6$ days), nor does it correspond with the observed value of 6-7 days (Namias, 1954). Although direct observations or calculations are not available for shorter waves, it is unlikely that waves in the primary energy input region ($l=5-7$) would undergo periodicities as short as four days or less.
Lest one conclude that the low-order system will yield periods only of the type described by Fig. 4, we consider the nonlinear periods generated by alternate wave profiles. In Fig. 5 we describe the periods for four different wave truncations for wavenumber 3 with four independent choices of the profile parameters $A_\alpha$ and $A_\beta$ for each truncation, with periods plotted against $\rho$. The first two truncations do not show striking differences from the results of Fig. 4; however, the latter two, especially for truncation $n_\alpha=6$, $n_\beta=8$, show remarkably different periods, which appear to depend considerably more on the truncation than on the selection of profile parameters. Whereas our previous observation suggested that the nonlinear periods are only weakly dependent on $\rho$, for the latter two truncations shown on Fig. 5 the period decreases rapidly with increasing $\rho$. We furthermore note that the periods may be as long as 15 days.

The existence of a nonlinear period does not, of course, give an indication of the nonlinear activity of the system, measured by the amount of energy actually exchanged during the period. Linear theory indicates that for small $\rho$ the exchange must also be small, provided that the initial configuration is stable (a condition which is generally satisfied in our calculations).

Long periods for small values of $\rho$ (little energy initially in the wave) therefore indicate a very slowly and weakly changing system. The amount of energy exchanged per unit time under this condition must indeed be small since it will be proportional to $\rho$ normalized by the period.

Since we know all the dependent variables as a function of the zonal variable $B_n$, that $B_n$ is periodic in time [Eq. (4)], that the energy components have their extrema at the same times as $B_n$, and from the selection of initial conditions that $B_n$ has extrema at $t=0$, $T/2$, we may define the maximum energy exchange for each component, say $E$, as

$$\Delta E = E(0) - E(T/2).$$

(22)

In (22), $E$ may represent $K_\alpha$ (zonal energy), $K_\alpha$ or $K_\beta$, the latter two representing the energy in the $\alpha$ and $\beta$ waves, respectively.

We now present the energy ranges as defined by (22) but normalized by the available energy $K_n$ [see Eq. (11)]. Fig. 6 describes the maximum energy exchange on a scale $-1 \leq \Delta K/K_n \leq 1$ vs $\rho$ for the total zonal energy, the lowest active zonal component energy ($\gamma = 3$), the $\beta$-wave energy and the $\alpha$-wave energy. Our presentation

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**Figure 4.** Nonlinear energy exchange periods for different initial configurations for a variety of wavenumbers, plotted against $\rho$ (ordinate).
is limited to the results for \( l = 1, 3, 6 \), since the variations become small for shorter waves. The initial conditions applicable for the calculations described in the figure are those for which the periods were shown in Fig. 4. The purpose of presenting the \( \gamma = 3 \) zonal mode is to determine the importance of the lowest mode in the zonal field. For the conditions represented in Fig. 6, it is evident, on comparing the upper two diagrams for each wave, that the \( \gamma = 3 \) mode dominates the exchange for the total zonal energy. This observation is consistent with previous work (Baer, 1964) for a less truncated model. Moreover, from an observational point of view, it is reasonable to expect the least variable (in latitude) part of the zonal field to vary most with time in order to maintain the moderately smooth zonal profiles which are actually measured.

A second pronounced feature of the exchange as described in Fig. 6 is the compensation between the two wave components. In all cases, except for very large and unrealistic values of \( \rho \), when the \( \beta \)-energy is increasing the \( \alpha \)-energy is decreasing; this results in a smaller exchange of the zonal energy. The activity of the system is therefore not uniquely defined simply by the exchange between the zonal flow and the wave, since the wave may undergo significant modifications without involving the zonal energy. One obvious exception to the above conclusion is the behavior of the \( D-J \) profile for \( l = 1 \), wherein both the \( \alpha \)- and \( \beta \)-wave energies change with the same sign. The \( D-J \) profile seems, furthermore, to be almost completely inactive for the case \( l = 3 \), and shows behavior similar to the \( A-J \) profile for shorter waves.

Although the exchanges are not large for reasonable values of \( \rho \), a somewhat surprising feature is the influence of the total energy on the exchange process. Whereas for wavenumber 1 the activity of the system increases for increased energy amplitude, as might be anticipated, this tendency is reversed as the waves get shorter. Thus for wavenumber 6, as the energy amplitude in the system increases, the exchange process is inhibited.

The variety of energy exchange which may occur for a single wave (here chosen as \( l = 3 \)) due to modifications
in wave truncation and profile parameters is exemplified by Fig. 7. Many possible exchange properties may be noted. In one case \( (n_a=6, n_\beta=8) \) the exchange is small and not strongly dependent on \( \rho \); in the others there is a strong dependence on \( \rho \) with a tendency for the exchange of zonal energy during the period to change direction as \( \rho \) increases. In one case \( (n_a=4, n_\beta=8) \), where the zonal energy does not have any range (\( \rho \approx 0.7 \)), the wave components are reasonably active; in case \( n_a=4, n_\beta=10 \), however, at \( \rho \approx 0.3 \), the entire system appears to be inert. There does seem to be an increase of activity with the addition of the component \( n_\beta=10 \); however, in one case the \( \alpha \)-wave energy is almost completely inactive whereas in the other it shows significant activity. The inevitable conclusion from this figure is clearly the great variety of exchange possibilities for the low-order system depending on truncation.

The previous discussion has dealt with properties of the flow independent of their detailed time characteristics. Since the exact time variations are known, we now present the time fluctuations of the normalized energy components during an entire period. Fig. 8 shows the solutions in time for the initial configuration given by the A-J and the wave profile taken from (20) for selected values of \( \rho \). Since each value of \( \rho \) constitutes a separate problem, the solutions are shown graphically with \( \rho \) increasing to the right. For each \( \rho \) value, two charts are shown; the upper one describes the behavior of the zonal energy and its individual components, and the lower one describes the total wave energy and the \( \alpha \)- and \( \beta \)-wave components. The upper set of charts applies for wave \( l=10 \) and energy amplitude \( \tilde{u}_\theta=0.135 \), whereas the lower set applies for \( l=3 \) and \( \tilde{u}_\theta=0.275 \). Since both initial states are stable by linear analysis, the lack of activity with time for small values of \( \rho \) is to be anticipated. We see for the case \( l=10 \) that the individual wave components are quite active, but their activity tends to cancel such that the zonal energy does not have large time fluctuations. This tendency for the short waves has already been indicated from Fig. 6. On the contrary, for the \( l=3 \) case, there is little tendency for cancellation between the wave components and the consequent zonal energy variations are large. Since the \( \beta \)-wave energy is reasonably uniform throughout the period for all \( \rho \) values, the dominant exchange is carried on by the \( \alpha \) wave. We also note the dominant influence of the \( \gamma=3 \) zonal component in the total zonal wave energy.
Fig. 7. Maximum energy exchange for wave $l=3$ with different wave configurations and profile parameters, including normalized energy in the zonal flow and both wave components, plotted against $\rho$ (ordinate).
The changes in time variation which arise due to the choice of different profile parameters and/or different wave truncations are made evident from Fig. 9. We have selected to describe three different truncations, with three different sets of profile parameters for each truncation. In all cases, the initial conditions involve interaction of wave $l=3$ with the A-J, for an energy amplitude of $\tilde{u}_{a}=0.135$ and for initial wave energy given by $\rho=0.6$. Several interesting features appear in this figure. Because of the interaction of the wave components, we note that a double period in the total zonal energy (also total wave energy) evolves. At least one of the cases is almost entirely inactive, a remarkable result for a specification with more than half of the energy in the wave initially. Finally, the elliptic nature of the solution is clearly evident in the case $n_{a}=8, n_{b}=10$ and $A_{a}=4, A_{b}=1$.

Although we do not plan to discuss the linear stability properties of system (3) in this report, it is a simple matter to linearize Eqs. (3) and establish the stability criteria (see Baer, 1968). The results of one calculation involving initial conditions near the stability point but on the stable side are shown in Fig. 10. The initial conditions for this calculation of the interaction of wave $l=3$ with the A-J are listed in the figure. The important feature to note from this calculation is that whereas the system behaves linearly for very small $\rho$ as may be expected from linear theory, nonlinearity becomes pronounced while $\rho$ is still small ($\rho \approx 0.1$) and reaches maximum exchange for $\rho < 0.2$. Thus, contrary to expectation from linear theory, we find initial conditions which lead to pronounced nonlinearity for small amounts of energy in the perturbation.

5. Conclusion

To bring the details of the nonlinear energy exchange process among different scales of motion into focus, we have calculated for a variety of initial conditions the periodic behavior of the exchange of a single planetary wave with an arbitrary zonal flow. The exchange process takes place in an atmosphere, described by the "low-order" or highly truncated spectral vorticity equation.
Fig. 9. Time variations of the different energy components during a complete nonlinear period for \( l = 3, \rho = 0.6 \), shown for various wave profiles and profile parameters.

The most evident observation from these calculations is the variability of the solutions. Depending on the initial specification, the solutions may range from a completely inactive system to one with strongly elliptic variations. They may show highly variable wave components which may or may not interact to cancel the variations in the total wave energy. The response of the system does not necessarily depend on its total energy amplitude. When the initial energy in the wave (perturbation) is chosen small, the nonlinearity as exemplified by the energy exchange may nevertheless be large. In order to make a satisfactory comparison of the low-order system to the atmosphere, therefore, it is essential to know very precisely the configuration of the atmosphere. Should the atmosphere not be in a configuration which is representable by the low-order truncation, any comparison between reality and the model may be seriously questioned. The model and the representative calculations herein described nevertheless give an indication of the variety of barotropic motions which might be experienced in the atmosphere.

Since we are interested in the exchange problem for the general atmosphere, an obvious extension to the
study described herein is to include more waves in the interactions. A study which includes an additional wave has been performed and indicates that under certain conditions the exchange properties between each wave and the zonal flow are not affected by the other wave. This is not universally true, however, and multiple periodicities do arise. These results will be the subject of a future report. Finally, since a baroclinic model may also be described by low-order spectral equations with exact solutions (Baer, 1970), energy transformations may be discussed together with the scale transfers which have been considered here for the barotropic atmosphere.

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