Free Convection in the Turbulent Ekman Layer of the Atmosphere

H. Tennekes

Dept. of Aerospace Engineering, The Pennsylvania State University, University Park

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ABSTRACT

This paper deals with the problem of fully developed free convection in the atmospheric boundary layer. In free convection, the height of the Ekman layer is much larger than the absolute value of the Monin-Oboukhov length. The kinetic energy budget of the turbulence above the surface layer shows that the standard deviations of vertical velocity and of temperature are related to h/L by $\sigma_w/u_* \propto (-h/L)^{\frac{1}{3}}$ and $\sigma_\theta/\theta_* \propto (-h/L)^{-\frac{1}{3}}$. Because convection has no natural length scale, the height of the neutral Ekman layer ($h \propto u_*/f$) is used to explore the consequences of the proposed expressions for σ_w and σ_θ. The dissimilarity between the heat flux and the momentum flux is studied in terms of time- and length-scale ratios and in terms of a flux Richardson number. A definitive solution of the problem, however, cannot be formulated until an expression for the height of unstable Ekman layers, as a function of the time of day and the stability conditions at the top of the boundary layer, can be found.

1. Introduction

One of the outstanding features of atmospheric boundary layers with upward heat flux is the high intensity of turbulence produced by buoyancy. Casual observations confirm that in sunny, windy weather the wind tends to be gusty and airplane flight at low levels tends to be rather uncomfortable. However, the mean surface stress does not change by more than half an order of magnitude between neutral and unstable conditions with the same geostrophic wind speed (Kazansky and Monin, 1960; Businger, 1966; Monin, 1967). In unstable stratification, therefore, the intensity of turbulence appears to be much larger than the friction velocity u_* based on the surface stress. This suggests that mechanical production of turbulent kinetic energy may be neglected compared to buoyant production. This, in turn, implies that the budget of kinetic energy should be insensitive to the distributions of stress and wind shear. It is on basis of these arguments that the situation described above is commonly referred to as *free* convection.

One of the most controversial features of free convection is that it does not appear to have an intrinsic length scale. The Monin-Oboukhov length scale, which is often used in theories of mixed convection, is not a length characteristic of buoyant eddies, but a length associated with the *rate* at which heat flows through the boundary layer. In other words, the Monin-Oboukhov length relates to the *time* scale involved in buoyant heat transfer. We shall explore this interpretation carefully.

In the absence of an intrinsic length scale, it appears that the adoption of a length scale specified in some other way cannot be avoided. In this sense, the problem to be discussed is one of *mixed* convection.

For an unstable Ekman layer whose surface layer is in a state of forced convection, a length scale is specified by the dynamics of stress and wind shear in a flow field driven by the Coriolis acceleration. The available evidence indicates that this length scale is of order u_*/f (f is the Coriolis parameter), both for neutral and unstable conditions (Zilitinkevich *et al.*, 1967; Monin, 1967; Blackadar and Tennekes, 1968). In all of the current Russian literature on the subject this position is maintained without theoretical justification, but no sensible alternative seems available. We shall, therefore, use u_*/f as a characteristic height of Ekman layers, even though this procedure avoids the issue in question. This rather unsatisfactory state of affairs is characteristic of most, if not all, theories dealing with free convection. A solution of the dilemma must await future developments in the understanding of penetrative convection, of diurnal variations in the turbulent heat flux, and of the occurrence of inversions which may limit the vertical growth of Ekman layers.

A theory which pretends to deal with free convection will have to face the issue of the relation between the turbulent heat flux and the drop in potential temperature across the Ekman layer. Since most of the temperature drop occurs in the surface layer, the interaction between the surface layer and the outer part of the Ekman layer will have to be studied carefully. We shall find that this leads to results which imply the existence of more than one characteristic temperature.

2. The budget of kinetic energy

When free convection prevails, most of the kinetic energy of the turbulent motion arises from buoyant forcing; the mechanical production of kinetic energy can be neglected.

Observations indicate that a wide variation in the distributions of turbulent heat flux and of potential temperature may exist in conditions that are commonly classified as free convection. Therefore, it appears to be overly ambitious to attempt to formulate a theory which would predict such distributions. A somewhat more modest goal, however, might be attainable. For this reason, it is proposed to deal with a budget of kinetic energy which represents *average* conditions in the Ekman layer. This choice avoids the problems inherent in the determination of flux divergences; that is, by definition, no net flux of energy or heat enters or leaves the boundary layer. For the time being, a steady-state model with horizontal homogeneity will be considered adequate.

In typical daytime atmospheric conditions, the surface layer of the Ekamn layer is in a state of forced convection. Therefore, the *bulk* theory of free convection to be developed here specifically excludes the surface layer, and the results to be obtained are not applicable to the surface layer. The role of the surface layer will be studied in the second half of this paper.

We will restrict the analysis to fully developed convection in the afternoon hours. Here, the thickness of the momentum boundary layer and that of the thermal boundary layer are the same, and a quasi-steady analysis may suffice. Penetrative convection, which often occurs in the morning hours, is specifically excluded. In penetrative convection, the thickness of the thermal boundary layer is small relative to that of the momentum boundary layer (which adds another unknown length scale to the problem), and the thermal boundary layer grows fairly rapidly, so that a quasi-steady analysis may not be adequate. The analysis is further restricted to *dry* convection, because the release of latent heat often causes convective elements to penetrate to heights well beyond the top of the momentum boundary layer.

In fully developed, quasi-steady free convection in dry weather, the budget of kinetic energy may be approximated by

$$\left\langle \frac{g}{T_0}\overline{\theta w} \right\rangle = \left\langle \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} \right\rangle. \qquad (1)$$

In this equation, w represents the fluctuations in the vertical component of velocity, θ the temperature fluctuations and T_0 the temperature of an adiabatic atmosphere. The overbar denotes a time average and the angular brackets are averages taken across the entire boundary layer, but excluding the surface layer. All other symbols are conventional.

The mechanical production of kinetic energy is absent from (1). Because the surface layer has been assumed to be in a state of *forced* convection, (1) does not apply to the lowest 10 m. In the surface layer, the locally very high rate of dissipation is approximately balanced by mechanical production instead of buoyant production.

Let us define average amplitudes of w and θ by

$$\sigma_w^2 = \langle \overline{w^2} \rangle, \quad \sigma_\theta^2 = \langle \overline{\theta^2} \rangle, \qquad (2)$$

where the brackets again emphasize that σ_w and σ_θ are representative values throughout the Ekman layer, but excluding the surface layer. A relation between σ_w and σ_θ can be obtained by considering the budget of kinetic energy and the spectral transfer which provides the dissipative eddies with energy. For turbulence in an Ekman layer of height h and with kinetic energy of order σ_w^2, dimensional arguments suggest that spectral energy transfer toward dissipative scales of motion occurs at a rate proportional to σ_w^3/h (Batchelor, 1953). This rate is determined by the spectral dynamics of the large eddies; it should not depend crucially on the specific way in which the turbulence is maintained. The nature of the dissipative process is such that the dissipation rate adjusts itself to the available energy transfer rate. This, of course, implies that the right-hand side of (1) may be estimated as $c\sigma_w^3/h$, where c is a constant of order one. From the laboratory experiments on free convection by Deardorff and Willis (1967) one can estimate that an average value of c is about 2.4; we will use this value for lack of appropriate atmospheric data.

An estimate for the left-hand side of (1) can be obtained by recognizing that fluctuations of temperature and of vertical velocity should be highly correlated in conditions of free convection. We will assume that

$$\langle \overline{\theta w} \rangle = 0.2\sigma_w\sigma_\theta. \qquad (3)$$

The coefficient 0.2 is based on data given by Zsvang and Volkov (1967) and by Telford and Warner (1964).

On basis of these estimates for the heat flux and the dissipation rate, the kinetic energy budget (1) may be written as

$$0.2\frac{g}{T_0}\sigma_\theta\sigma_w = 2.4\frac{\sigma_w^3}{h}. \qquad (4)$$

This results in

$$\sigma_w^2 = 0.08\frac{gh}{T_0}\sigma_\theta. \qquad (5)$$

This expression, though barely more than a careful dimensional estimate, is rather appealing. The budget of kinetic energy appears to support the tempting assumption that the kinetic energy σ_w^2 should be proportional to the buoyant acceleration $(g/T_0)\sigma_\theta$, multiplied by the distance h over which the buoyancy force performs work.

The dependence of σ_w and σ_θ on the surface heat flux

H can be obtained as follows. Because we have assumed that the height of the thermal boundary layer and that of the momentum boundary layer are the same, the heat flux should vanish at the top of the boundary layer $(z=h)$. The average value of $\overline{\theta w}$ then may be estimated as

$$\langle\overline{\theta w}\rangle=0.5H/(\rho c_p). \tag{6}$$

Employing (3), (5) and (6), we obtain

$$\sigma_\theta=4.2\left(\frac{H}{\rho c_p}\right)^{\frac{2}{3}}\left(\frac{T_0}{gh}\right)^{\frac{1}{3}}, \tag{7}$$

$$\sigma_w=0.6\left(\frac{H}{\rho c_p}\right)^{\frac{1}{3}}\left(\frac{gh}{T_0}\right)^{\frac{1}{3}}. \tag{8}$$

These expressions are quite similar to those used in the *local* theory of free convection (Lumley and Panofsky, 1964). In local free convection near the surface, the height z above the surface is taken to be the characteristic length instead of the height h of the Ekman layer. Expressions similar to (7) and (8) then are obtained on dimensional grounds, without reference to the budget of kinetic energy.

3. Characteristic lengths and times

As indicated in the Introduction, this theory does not indicate in which way the height h of the boundary layer depends on convective processes, so that h must be specified by some other dynamical mechanism. A provisional solution to this dilemma is to assume that h is specified by the dynamics of stress and wind in the Ekman layer.

For Ekman layers in neutrally stratified conditions, the height h is given by (Csanady, 1967; Blackadar and Tennekes, 1968):

$$h=0.25u_*/f. \tag{9}$$

In the absence of reliable data on the effects of buoyancy on the thickness h of the momentum boundary layer, this relation will also be assumed to hold in fully developed free convection in dry air. Of course, penetrative convection, in which the length scales of the thermal and momentum boundary layers are not the same, cannot be treated if (9) is used. The position taken here is equivalent to the one adopted in the recent Russian literature (Kazanski and Monin, 1960; Monin, 1967; Zilitinkevich *et al.*, 1967). The data obtained by the Russians show that the wind distribution in unstable Ekman layers is very nearly the same as that in neutral Ekman layers; this supports the idea that the length scale of unstable Ekman layers is determined by the friction velocity u_* and the Coriolis parameter f. It should be mentioned explicitly that (9) is applicable only if the surface layer is in a state of *forced* convection, so that u_* is indeed the appropriate velocity scale (Blackadar and Tennekes, 1968).

During the afternoon, the thickness of the boundary layer slowly increases if convective conditions prevail. The numerical coefficient in (9) thus increases as time proceeds. Even so, convection appears to be a quasi-steady process, as we will see shortly.

Having adopted (9) as the length scale of the boundary layer, we now turn to the other length scale in the convection problem. The characteristic length associated with the heat flux is the Monin-Oboukhov length L, which is defined by

$$L=-\frac{\rho c_p T_0 u_*^3}{0.4gH}. \tag{10}$$

This length decreases as the heat flux increases; therefore, it is not representative of the physical dimensions of buoyant eddies. An interpretation of the role of L will be given later.

A characteristic temperature associated with the heat flux is θ_*, which is defined by

$$\theta_*=H/(0.4\rho c_p u_*). \tag{11}$$

Upon substitution of (10) and (11), (7) and (8) can be written as

$$\sigma_\theta/\theta_*=1.2(-L/h)^{\frac{1}{3}}, \tag{12}$$

$$\sigma_w/u_*=0.8(-h/L)^{\frac{1}{3}}. \tag{13}$$

Because u_* and θ_* are representative of fluctuation levels in velocity and temperature in the *surface layer* (Lumley and Panofsky, 1964), these equations shed light on the relations between the surface layer and the rest of the Ekman layer. In conditions of free convection, $-h/L \gg 1$; therefore, temperature fluctuations outside the surface layer are small compared to those inside the surface layer, and velocity fluctuations in the outer part of the Ekman layer are large compared to those in the surface layer. These observations imply that there is no similarity between the momentum flux and the heat flux. In particular, turbulence with kinetic energy of order σ_w^2 maintains a Reynolds stress of order u_*^2. According to (13), the stress is much smaller than σ_w^2 if $-h/L\gg1$; we must conclude that buoyant eddies support much less momentum transfer than their kinetic energy would seem to allow. In other words, buoyant eddies cause some increase in the momentum transfer (Businger, 1966), but this increase is much smaller than the increase in kinetic energy.

The results obtained above can also be interpreted in terms of time scales. The time scale associated with the wind distribution and the stress in Ekman layers is the inverse of the Coriolis parameter f. A suitable buoyant time scale is defined by

$$\frac{1}{t_b^2}=\frac{g\sigma_\theta}{T_0 h}. \tag{14}$$

The time scale t_b is like the inverse of an imaginary Brunt-Väisälä frequency. In the outer part of the Ekman layer, the mean potential temperature is very nearly constant, so that the mean potential temperature gradient $\partial\Theta/\partial z$ cannot be used to define a characteristic time. Also, according to (12), θ_* is not representative of temperature fluctuations in the outer part of the Ekman layer if $-h/L$ is large.

Substitution of (5) and (9) into (14) yields

$$(\sigma_w/u_*)^{-2}=190(t_b f)^2, \tag{15}$$

$$\sigma_w=0.28h/t_b. \tag{16}$$

Eq. (16), of course, is the only possible form permitted for turbulence generated in an environment with time scale t_b and length scale h. In free convection, the other available time scale, $1/f$, is so large that the corresponding velocity ($hf\propto u_*$) is negligible compared to σ_w.

From (13) and (15), a relation between the nondimensional groups $t_b f$ and h/L can be established. The result is

$$(t_b f)^{-1}=11(-h/L)^{\frac{1}{3}}. \tag{17}$$

This expression again illustrates that L is not associated with the dimensions of buoyant elements, but with the rate of heat transfer. The one-third power occurring in (17) suggests that, at a given value of f, t_b is fairly insensitive to the exact value of h/L. This implies that the estimates for L based on estimates of t_b are bound to be rather inaccurate.

A numerical example may be illustrative. Let us take $L=-50$ m as a representative value. Also, let $h=1000$ m and $f=10^{-4}$ sec^{-1}. Substituting these values into (17), we obtain $t_b=330$ sec, which is 5.5 min. Recalling that $1/f$ is 2.5 hr, we see that free convection is characterized by a small value of the ratio between buoyant and Coriolis time scales. With (14), we further obtain $\sigma_\theta=0.3$K, approximately, which is in rough agreement with the experimental data in the literature. If we substitute for h and t_b into (16), we obtain $\sigma_w=0.8$ m sec^{-1}, which again seems quite reasonable. Of course, the accuracy of the numbers involved is rather poor; many numerical coefficients had to be estimated before the coefficients in equations like (16) and (17) could be determined.

4. A flux Richardson number

In the preceding section, the existence of free convection was related to the values of the time-scale ratio, $t_b f$, and the length-scale ratio, L/h. A rather more conventional way to determine whether free convection prevails is to construct a Richardson number and to determine its value. A flux Richardson number applying to the boundary layer as a whole may be defined as

$$\mathrm{Rf}=-\frac{gHh}{\rho c_p T_0 u_*^2 G}. \tag{18}$$

In the construction of this equation, the total production of turbulent energy by the Reynolds stress was assumed to be proportional to u_*^2G, where G is the modulus of the geostrophic wind speed. The average mechanical production rate in the Ekman layer is thus proportional to u_*^2G/h.

It will be convenient to ignore for a moment the dependence of u_*/G on the surface Rossby number. If we use $u_*/G=1/30$ as an average value (Blackadar, 1962), we obtain from (10), (17) and (18)

$$\mathrm{Rf}=0.08h/L, \tag{19}$$

$$\mathrm{Rf}=-6\times10^{-5}(t_b f)^{-3}. \tag{20}$$

In the numerical example used before, $L=-50$ m and $h=1000$ m. According to (19), this corresponds to $\mathrm{Rf}=-1.6$. Since we prefer to think of free convection as being characterized by large negative Richardson numbers, the conditions of the example are indeed convective.

A conservative lower limit for the applicability of the theory may be put at $-h/L=10$, because it appears sensible to insist that $-L$ be smaller than the height of the Ekman layer. This corresponds to $\mathrm{Rf}=-0.8$ and $t_b f=4\times10^{-2}$, giving a value of σ_θ as small as about 0.16K. Hence, even if the absolute value of Rf is not large, the buoyant time scale is still substantially smaller than the Coriolis time scale. Apparently, buoyant transfer processes tend to be so rapid that the slow diurnal variations in the bulk properties of the boundary layer may be neglected for some purposes. In other words, a steady-state model should be adequate if $t_b f$ is small.

The extremely small coefficient occurring in (20) emphasizes that Rf is not a very suitable parameter. Free convection should be characterized by large negative values of Rf. However, in this context, -1 has to be considered a large number. It appears that $(t_b f)^{-1}$ is a more natural indicator. Free convection occurs when the buoyancy time scale t_b is much smaller than the Coriolis time scale $1/f$. The lowest possible value of $(t_b f)^{-1}$ which would be needed to obtain conditions of free convection may be put at 10, if we insist on at least one order of magnitude difference in time scales. This corresponds to $\mathrm{Rf}=-0.06$ and $h/L=-0.8$. The issue involved here is that buoyant production *does not need* to dominate mechanical production if the time scales of buoyant and mechanical components of the turbulence are different by more than an order of magnitude, because the interaction between the two components will be relatively weak.

The issues raised here need some further explanation. In free convection, buoyant eddies are quite energetic and have small characteristic times, but the mechanical eddies are weak and have large characteristic times. Because the mechanical eddies are weak and slow, they cannot affect the buoyant eddies and the heat flux significantly. On the other hand, the buoyant eddies are fast and energetic, so that it would seem that they could increase the momentum transfer by orders of magnitude. Clearly, this does not occur. The point is that eddies with small time scales have large vorticity and strain-rate fluctuations, which are associated with high frequencies. Now, it is one of the main assumptions of turbulence theory that high-frequency eddies do not interact strongly with low-frequency eddies. This assumption, for instance, is used to derive the turbulence spectrum in the inertial subrange and to explain local isotropy at small scales (Batchelor, 1953). Extending this concept to free convection, we have to conclude that the fast, energetic eddies associated with the heat flux cannot have an overwhelming effect on the mechanical eddies, which are associated with the momentum flux and the relatively slow mean shear, $\partial U/\partial z$. (Note that the mean shear is a frequency of order f if the momentum flux is of order ρu_*^2 and the height of the boundary layer is of order u_*/f.) Because the buoyant eddies have so much kinetic energy, their contribution to the momentum transfer is appreciable; however, it is not nearly as large as it might be if the buoyant and mechanical eddies were tuned to the same frequency.

This conclusion is essentially a negative one; an attempt to make a qualitative prediction of the increase in momentum flux will be made in Section 6. One point, however, deserves to be emphasized: the very fact that t_b and $1/f$ differ by at least an order of magnitude in free convection confirms the intuitive concept that the eddies which cause most of the heat flux are dynamically distinct from those associated with the momentum flux. In other words, in free convection two kinds of turbulence occur in the same boundary layer, even if the absolute value of the Richardson number is not large compared to unity.

5. The turbulent heat flux

The ultimate purpose of a theory describing turbulent heat transfer is to predict how the rate of heat transfer depends on the difference in potential temperature across the boundary layer. Let us use the symbol Θ for mean potential temperature, and let us define the drop in potential temperature by

$$\Delta\Theta=\Theta(z_0)-\Theta(h). \tag{21}$$

In analogy with the usual treatment of the wind profile near the ground, it would be tempting to equate $\Theta(z_0)$ to the surface temperature. However, there must be molecular heat transfer between the surface and the roughness height ($z=z_0$), so that there may be a difference between the surface temperature and $\Theta(z_0)$. The definition (21) avoids this problem, in exchange for the problem that $\Theta(z_0)$ cannot be measured easily. There is a real need for further study of these problems, but they are outside the scope of this paper.

It is well known that in convective conditions Θ drops very rapidly near the surface; it decreases more slowly in the region where the local theory of free convection is applicable and it tends to be roughly constant beyond the lowest 50 m. The temperature fluctuations of order σ_θ which are present throughout the Ekman layer are thus not generated by buoyant production of temperature variance but by energy flux out of the surface layer. The relation between $\Delta\Theta$ and the surface heat flux H thus depends only on what happens close to the surface. The frequent occurrence of a weak counter-gradient heat flux near the top of the boundary layer will be ignored here.

In the limit as $z/L\rightarrow 0$, Reynolds' analogy between heat and momentum transfer makes the temperature profile similar to the wind profile (Lumley and Panofsky, 1964). In differential form

$$-\frac{z}{\theta_*}\frac{\partial\Theta}{\partial z}=1. \tag{22}$$

For free convection in the vicinity of the surface, many investigators use

$$-\frac{z}{\theta_*}\frac{\partial\Theta}{\partial z}=0.32(-z/L)^{-\frac{1}{3}}. \tag{23}$$

The coefficient 0.32 was computed from a related coefficient given by Lumley and Panofsky (1964). A convenient interpolation formula is

$$-\frac{z}{\theta_*}\frac{\partial\Theta}{\partial z}=\frac{1}{1+3.1\zeta^{\frac{1}{3}}}, \tag{24}$$

where

$$\zeta=-z/L. \tag{25}$$

Eq. (24) can be integrated without effort; if we assume that $-z_0/L\ll 1$, we obtain

$$\Theta(z_0)-\Theta(z)=3\theta_*\ln\left[\frac{(z/z_0)^{\frac{1}{3}}}{1+3.1\zeta^{\frac{1}{3}}}\right]. \tag{26}$$

It would seem preposterous to extend the validity of this equation to $z=h$. However, at values of $\zeta>1$ we can ignore the first term in the denominator if we permit a small additive error in the logarithm of a large quantity. This approximation leads to

$$\Theta(z_0)-\Theta(z)=\theta_*\ln(-L/30z_0). \tag{27}$$

The right-hand side of this equation is independent of z, so that (27) can also be used for $z=h$. This yields

$$\Delta\Theta=\theta_*\ln(-L/30z_0). \tag{28}$$

The validity of this equation is clearly limited to situations where $-h/L\gg 1$ and $-z_0/L\ll 1$.

Before the implications of (28) are considered, we want to show that the functional relationship involved here does not depend on the exact form of the interpolation formula (24). Instead of (24), Businger (1966) uses

$$-z\frac{\partial\Theta}{\partial z}=\theta_*(1+\beta\zeta)^{-\frac{1}{2}}. \tag{29}$$

Integration of (29), after making approximations consistent with $-h/L\gg 1$ and $-z_0/L\ll 1$, yields

$$\Delta\Theta=\theta_*\ln(-L/2\beta z_0). \tag{30}$$

Businger states that β may range between 15 and 20; apparently, the distinction between (24) and (29) is irrelevant as far as the final expression for $\Delta\Theta$ is concerned.

The potential temperature at $z=z_0$ is not often measured in field work. However, (28) and (30) also apply if the lower limit of integration is taken at any other value of z such that $z\ll -L$. In particular, the potential temperature drop between $z=h$ and $z=2$ m depends on the logarithm of $-L/60$ if L is given in meters.

If (28) and (30) have any validity at all, they are very powerful indeed, because they assert that $\Delta\Theta$ is not related to or dependent on the height of the boundary layer as long as $-h/L\gg 1$. This again illustrates the problem of choosing a length scale for unstable Ekman layers. Because the potential temperature is approximately constant above $z=-L$, there seems to be no way in which the heat flux can determine the boundary-layer height h. Free convection has no natural length scale, and the assumption that h should be proportional to u_*/f, as in (9), seems unavoidable until a definitive theory can be formulated.

Eqs. (28) and (30) emphasize that, next to h/L, the length-scale ratio z_0/L associated with the surface layer has a very distinctive role. These relations also imply that there is no similarity between momentum and heat transfer in free convection. The ratio G/u_* is roughly proportional to $\ln(h/z_0)$ according to Blackadar and Tennekes (1968), but the ratio $\Delta\Theta/\theta_*$ is proportional to $\ln(-L/z_0)$, and it decreases as the heat flux increases.

A few numbers may be helpful. We found before that for $L=-50$ m, $\sigma_\theta=0.3$K. Using (12), we obtain $\theta_*=0.7$K if $h=1000$ m. If we select $z_0=1$ cm, (28) yields $\Delta\Theta=4$K. All of these numbers seem quite reasonable for convective conditions.

Other nondimensional forms of (28) are

$$0.16\frac{g\Delta\Theta L}{T_0u_*^2}=\ln\left(-\frac{L}{30z_0}\right), \tag{31}$$

$$0.4\frac{g\Delta\Theta}{T_0Gf}=\frac{gH}{\rho c_pT_0u_*Gf}\ln\left(-\frac{L}{30z_0}\right). \tag{32}$$

If we ignore the dependence of G/u_* on h/z_0 and use $G/u_*=30$ as an average value, these expressions become

$$\frac{g\Delta\Theta h}{T_0G^2}=0.007\frac{h}{L}\ln\left(-\frac{L}{30z_0}\right), \tag{33}$$

$$\frac{g\Delta\Theta}{T_0Gf}=75\frac{gH}{\rho c_pT_0G^2f}\ln\left(-\frac{L}{30z_0}\right). \tag{34}$$

The nondimensional temperature drop in (33) is similar to one used by Businger (1966); it is seen that it depends both on h/L and on z_0/L, making interpretation of field data extremely difficult. However, the approximation $G/u_*=30$ implies that $\ln(h/z_0)$ is approximately constant. For $h/z_0=10^4$ and 10^5, $\ln(h/z_0)=9.2$ and 11.5, respectively, so that a crude approximation would be $\ln(h/z_0)=10$, say. Using this in (33), we obtain

$$\frac{g\Delta\Theta h}{T_0G^2}=0.007\frac{h}{L}\left[\ln\left(\frac{h}{z_0}\right)-\ln\left(\frac{30h}{-L}\right)\right], \tag{35}$$

$$=0.007\frac{h}{L}[6.6-\ln(-h/L)]. \tag{36}$$

It should be possible to test this relation by making field experiments.

The nondimensional temperature drop and heat flux in (34) are similar to those used in the current Russian literature (e.g., Orlenko, 1967). However, (34) indicates that this relation is also affected by z_0/L, so that no universal relation between these parameters can exist.

6. Unsolved problems

A recapitulation of the major equations of this paper is in order. We have found that

$$\sigma_\theta/\theta_*=1.2(-L/h)^{\frac{1}{3}}, \tag{12}$$

$$\sigma_w/u_*=0.8(-h/L)^{\frac{1}{3}}, \tag{13}$$

$$\Delta\Theta/\theta_*=\ln(-L/30z_0). \tag{28}$$

The major unsolved problem, of course, is the effect of convection on the wind profile and the geostrophic drag coefficient u_*/G. There is no evidence in favor of the validity of Reynolds' analogy, but *if* any similarity between turbulent stress and heat flux existed, (28) could be transposed to give

$$0.4G/u_*=\ln(-L/30z_0). \tag{37}$$

This is meant to be an approximate formula, which does not take into account that the Ekman layer has a wind *spiral.* Thus, Eq. (37) is for demonstration purposes only. An equation similar to (37) was proposed by Gill (1968). The factor 0.4 (von Kármán's constant) enters into (37) due to the difference between the definition of u_* and θ_*.

It is implied by (37) that the modulus of the mean wind vector does not change above $z/L\approx -1$. This is in clear contradiction with most field observations. Also, if $30z_0=1$ m, the right-hand side of (37) is 2.3 if $-L=10$ m, and 4.6 if $-L=100$ m. This indicates that (37) implies that G/u_* should depend very strongly on L. Experimental results, however, show that G/u_* does not change substantially for $10\text{ m}<-L<100\text{ m}$ (Businger, 1966), even though the value of u_*/G in unstable conditions is about 1.5 times its value in neutral conditions. Therefore, we cannot escape the conclusion that (37) is not valid and that no similarity exists between heat and momentum transfer.

This result forces us to return to a subject discussed earlier. The equations for the mean flow in Ekman layers do not depend explicitly on the heat flux, and the value of u_*/G remains approximately constant if the absolute value of the Monin-Oboukhov length is less than the height of the boundary layer. Therefore, the dependence of u_*/G on the surface Rossby number, $G/(fz_0)$, in unstable conditions must be similar to the relation valid in neutral conditions, i.e.,

$$0.4G/u_*=\ln(h/z_0), \tag{38}$$

where h is defined by Eq. (9). Eq. (38) is a crude form of the exact expressions developed recently (Csanady, 1967; Blackadar and Tennekes, 1968); however, it is adequate in the present context.

The heart of the matter, apparently, is the striking dissimilarity between (28) and (38). In other words, why do the momentum and heat fluxes go separate ways? The nature of this problem can be studied conveniently in terms of characteristic times. In the outer part of Ekman layers, the buoyant time scale t_b is much smaller than the Coriolis time f^{-1}. This implies that the vorticity in buoyant eddies is $(t_bf)^{-1}$ times as large as the vorticity associated with the wind shear [the latter is of order $u_*/h=O(f)$]. If there is a large gap between vorticities, there is good reason to assume that there is no strong interaction between the processes involved (Lumley, 1967). In particular, if

$$-\overline{uw}=\gamma\sigma_w h(\partial U/\partial z), \tag{39}$$

then γ is probably proportional to the vorticity ratio t_bf [cf. other examples given by Lumley (1967)]. If we put $\gamma\propto t_bf$, $\sigma_w\propto u_*(t_bf)^{-1}$ and $h\propto u_*/f$ [by virtue of (15) and (9), respectively], and if $\partial U/\partial z\propto f$, we find that $-\overline{uw}\propto u_*^2$. This supports the observations which show that, in free convection, the kinetic energy σ_w^2 is much larger than the Reynolds stress, and that there is no continuing increase of u_*/G with increasing heat transfer.

In the *surface* layer, on the other hand, the temperature and velocity scales are θ_* and u_*, respectively, while the length scale is z. The buoyancy time scale is thus like the square root of $T_0z/(g\theta_*)$ and the inverse of the wind shear is like z/u_*. Using (10) and (11), we find that the ratio of time scales is $(-z/L)^{\frac{1}{2}}$. If $-z/L\ll1$, the buoyancy time scale is the larger of the two, in contrast with the situation in the outer part of the Ekman layer. However, if $-z/L=O(1)$, the two time scales are comparable, so that strong interaction between thermal and mechanical turbulence must be expected. This line of thought suggests that the cause of the rather abrupt change in u_*/G between neutral and unstable conditions should be located in a layer in the vicinity of $-z/L\approx1$. This is the regime studied in *local* theories of free convection. However, much of the dynamics of this layer remains unclear at present.

The layer near $-z/L$ may also hold the key to another problem. If the gradient of potential temperature is essentially zero above $-z/L\approx1$, the temperature variance above that level must be maintained by the turbulent flux, $\overline{\theta^2w}$, out of the convective layer near the surface. This flux would have to balance the dissipation of temperature variance throughout the Ekman layer, which is probably of order $\sigma_\theta^2\sigma_w$. A prediction of the flux based on the dynamics of the convective layer near the surface may help to explain how the height of the Ekman layer changes in convective conditions.

It should be clear that the problem of free convection in Ekman layers is far from solved. Penetrative convection into a medium with known stratification, in conditions with zero wind, is fairly well understood (e.g., Deardorff and Willis, 1967), as are the stress and the wind distribution in neutral Ekman layers. However, the effect of the heat flux on the stress and the wind in unstable Ekman layers, and the resulting changes in the geostrophic drag coefficient and the height of the boundary layer will require much work and effort beyond the stage of exploratory studies such as the one reported here.

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