Some Effects of Stratification on Rotating Fluids

ANTS LEETMAA
Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge

(Manuscript received 8 September 1970)

ABSTRACT

A simple example of the steady motion of a rotating, stratified fluid is studied. The solution which is uniformly valid for all values of the stratification, $\sigma s\delta = \nu\alpha g D\Delta T/(\kappa f^2 L^2)$, is presented. The transitions in the dynamics from the homogeneous limit to strong stratification are illustrated in detail. The motion is driven by a stress. Consequently, Ekman suction is weaker than in cases where the driving force is a moving boundary, and Ekman layers are important until a stratification of O(1) at which point they combine with Lineykin layers to form the thermal equivalent of the Stewartson $E^{\frac{1}{3}}$ layer.

1. Introduction

The works of Barcilon and Pedlosky (1967a,b) and Veronis (1967a,b) examine some of the effects of stratification upon the steady linear dynamics of a rotating fluid. Because of their general nature, these studies did not present solutions which were uniformly valid for all values of the stratification.

In the present paper a simple example of a rotating, stratified fluid is studied in which the motion is driven by an applied stress rather than a moving boundary as in the works quoted above. In Section 2 the equations of motion are nondimensionalized and scaled. In Section 3 the solution to an example is presented. Section 4 contains a thorough discussion of this solution and its convergence to the homogeneous limit. The major difference between this and previous work is that because of the stress driving, the Ekman layer suction is much weaker and the critical values of the stratification are much greater than in the previously quoted works. Section 5 considers some aspects of thermal forcing, while Section 6 mentions a possible geophysical application of the ideas.

2. Formulation of equations

Consider the steady motion of an incompressible Boussinesq fluid in a coordinate frame rotating with constant angular velocity $f/2$ about the vertical. The equations of motion are

$$\mathbf{u}'\cdot\nabla\mathbf{u}'+f\hat{\mathbf{k}}\times\mathbf{u}'=-\frac{1}{\rho}\nabla p'-g\hat{\mathbf{k}}+\nu\nabla^2\mathbf{u}', \qquad (2.1)$$

$$\nabla\cdot\mathbf{u}'=0, \qquad (2.2)$$

$$\mathbf{u}'\cdot\nabla T'=\kappa\nabla^2 T', \qquad (2.3)$$

$$\rho'=\rho_0[1-\alpha(T'-T_0)], \qquad (2.4)$$

where $\mathbf{u}',p',\rho',T',\nu,\kappa$ are, respectively, the velocity, pressure, density, temperature, constant kinematic viscosity, and constant thermometric conductivity; and $\hat{\mathbf{k}}$ is the unit vector in the vertical. The equation of state is assumed to be linear, where α is a constant thermal expansion coefficient and ρ_0 and T_0 are reference values of density and temperature. Centrifugal accelerations are assumed to be small compared to gravity and have been neglected.

The simplest model in which thermal effects are important is considered. The heat equation is linearized by imposing an external temperature gradient. The fluid is contained between two horizontal surfaces separated by a distance D. The upper surface is maintained at a temperature $T_0+\Delta T$, the lower surface at T_0 where $\Delta T>0$. In the absence of any motion the equilibrium temperature is given by

$$T_e=(T_0+\Delta T)+\Delta T z'/D, \quad 0\geqslant z'\geqslant -D. \qquad (2.5)$$

The problem is assumed to be independent of y, i.e., $\partial/\partial y=0$. The equations are nondimensionalized as follows:

$$\left.\begin{aligned} &(u',v')=U(u,v); \quad \omega'=(D/L)Uw\\ &x'=Lx; \quad z'=Dz\\ &T'=T_e+\frac{UfL}{g\alpha D}T\\ &p'=p_0-\rho_0 gDz+\rho_0 g\alpha\Delta T z(1+z/2)+\rho_0 fULp \end{aligned}\right\}, \qquad (2.6)$$

where u,v,w,T,p are nondimensional variables. Note that T and p represent deviations of the temperature and the pressure from their equilibrium values. The nondimensional equations are as follows:

$$R_0(uu_x+\omega u_z)-v=-p_x+E\nabla^2 u, \tag{2.7}$$

$$R_0(uv_x+\omega v_z)+u=E\nabla^2 v, \tag{2.8}$$

$$\delta^2 R_0(u\omega_x+\omega\omega_z)=-p_z+T+\delta^2 E\nabla^2 w, \tag{2.9}$$

$$u_x+\omega_z=0, \tag{2.10}$$

$$\sigma R_0(uT_x+\omega T_z)+(\sigma s\delta)w=E\nabla^2 T, \tag{2.11}$$

where

$$\left.\begin{array}{l}\sigma\equiv\text{Prandtl number}=\nu/\kappa\\ E\equiv\text{Ekman number}=\nu/fD^2\\ R_0\equiv\text{Rossby number}=U/fL\\ s\equiv g\alpha\Delta T/f^2L\\ \delta\equiv\text{aspect ratio}=D/L\end{array}\right\}, \tag{2.12}$$

and

$$\nabla^2=\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

In the following, $\sigma s\delta$ will be called the stratification.

A right-handed coordinate system is used where z is positive upward. The velocities u, v, w are in the x, y, z directions, respectively. The notation $(\ )_x$ is shorthand for $\partial/\partial x(\ )$. Since the motion will be driven by surface stresses, the velocity amplitude U is chosen to be $\tau_0/(\nu^{\frac{1}{2}}\rho_0 f^{\frac{1}{2}})$ where τ_0 is the amplitude of the stress.

We assume that $\sigma=O(1)$ and $R_0\ll E\ll O(1)$. Neglecting terms of $O(R_0)$, the equations become

$$-v=-p_x+E\nabla^2 u, \tag{2.13}$$

$$u=E\nabla^2 v, \tag{2.14}$$

$$0=-p_z+T+\delta^2 E\nabla^2 w, \tag{2.15}$$

$$0=u_x+\omega_z, \tag{2.16}$$

$$(\sigma s\delta)\omega=E\nabla^2 T. \tag{2.17}$$

When $\delta=O(1)$, this set of equations is the same as that considered by Barcilon and Pedlosky. However, further simplification occurs if we assume that $\delta\ll O(1)$. Thus, neglecting terms of $O(\delta^2 E)$ the equations become

$$-v=-p_x+Eu_{zz}, \tag{2.18}$$

$$u=Ev_{zz}, \tag{2.19}$$

$$0=-p_z+T, \tag{2.20}$$

$$0=u_x+\omega_z, \tag{2.21}$$

$$(\sigma s\delta)\omega=ET_{zz}. \tag{2.22}$$

This was the set of equations examined by Stommel and Veronis (1957). From the above we form a single equation for pressure:

$$E^2P_{8z}+P_{4z}+(\sigma s\delta)P_{xxzz}=0. \tag{2.23}$$

3. An example

For computational convenience the unit of depth is redefined as the Ekman depth, i.e., $z=E^{\frac{1}{2}}\xi$. Furthermore, we specify that $E^{\frac{1}{2}}=1/40$. With this change (2.23) becomes

$$P_{8\xi}+P_{4\xi}+\lambda^{-2}P_{\xi\xi xx}=0, \tag{3.1}$$

where

$$\lambda^{-1}=(\sigma s\delta E)^{\frac{1}{2}}. \tag{3.2}$$

As in Stommel and Veronis (1957) periodic boundary conditions are used to eliminate the need for horizontal boundaries. The system is driven mechanically with a stress at $\xi=0$. The thermal boundary condition is that the perturbation temperature be zero at $\xi=0,-40$. This is consistent with the basic assumption that the total temperature is maintained at $T_0+\Delta T$ at $z'=0$ and T_0 at $z'=-D$.

We consider a problem with the boundary conditions

$$\left.\begin{array}{l}\dfrac{\partial v}{d\xi}=\cos x,\quad \dfrac{\partial u}{\partial \xi}=0,\quad \omega=T=0;\quad \text{at }\xi=0\\ u=v=\omega=T=0;\quad \text{at }\xi=-40\end{array}\right\}. \tag{3.3}$$

The solution to (3.1) is of the form

$$P(x,\xi)=P(\xi)\sin x, \tag{3.4}$$

where

$$P(\xi)=\sum_{i=1}^{6}\Pi_i\exp(\sigma_i\xi)+\Pi_7\xi+\Pi_8. \tag{3.5}$$

The σ_i's satisfy the condition that

$$\sigma_i^6+\sigma_i^2-\lambda^{-2}=0. \tag{3.6}$$

The σ_i's were determined numerically for various values of λ. A short list of them is given in Table 1. Note that σ_1 is real and $\sigma_2=\sigma_{2r}+i\sigma_{2i}$. The following relations are true:

$$\sigma_4=-\sigma_1,\quad \sigma_2=\sigma_3^*,\quad \sigma_5=-\sigma_2=\sigma_6^*, \tag{3.7}$$

where $(\ )^*$ denotes the complex conjugate.

Application of the boundary conditions determines the Π_i's:

$$\Pi_1=1/[\lambda^2\sigma_1^3(\sigma_1^2-\sigma_3^2)(\sigma_2^2-\sigma_1^2)]+\Pi_4\exp(-40\sigma_1), \tag{3.8}$$

$$\Pi_2=1/[\lambda^2\sigma_2^3(\sigma_2^2-\sigma_1^2)(\sigma_3^2-\sigma_2^2)], \tag{3.9}$$

$$\Pi_3=\Pi_2^*, \tag{3.10}$$

$$\begin{aligned}\Pi_4=\{&[\sigma_1^3(\sigma_2^3-\sigma_3^3)-\sigma_1^4(\sigma_3^2-\sigma_2^2)-\sigma_1\sigma_3^2(\sigma_2-\sigma_3)\sigma_2^2]\\ &\times\exp(-40\sigma_1)[1/(\lambda^2\sigma_1^3(\sigma_1^2-\sigma_3^2)(\sigma_2^2-\sigma_1^2))]\\ &-\Pi_7\sigma_2^2\sigma_3^2(\sigma_2-\sigma_3)\}/\{[\sigma_1(\sigma_1^3-\sigma_2^3)(\sigma_3^2-\sigma_2^2)\\ &-(\sigma_1^2-\sigma_2^2)(\sigma_3^3-\sigma_2^3)]-[\sigma_1^3(\sigma_2^3-\sigma_3^3)-\sigma_1^4\\ &\times(\sigma_3^2-\sigma_2^2)-\sigma_1\sigma_2^2\sigma_3^2(\sigma_2-\sigma_3)\exp(-80\sigma_1)]\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\Pi_5=&[2\sigma_1^4(\sigma_3^2-\sigma_1^2)\Pi_1\exp(-40\sigma_1)-\Pi_7(\sigma_3^3\sigma_1^2\\ &-\sigma_3^2\sigma_1^3)]/\{\sigma_2[(\sigma_2^3-\sigma_3^3)(\sigma_1^2-\sigma_3^2)\\ &-(\sigma_2^2-\sigma_3^2)(\sigma_1^3-\sigma_3^3)]\},\end{aligned} \tag{3.12}$$

TABLE 1. Some values of σ_1 and σ_2 for various values of λ.

λ	σ_1	σ_{2r}	σ_{2i}
0.033	3.1016	1.5591	2.6909
0.100	2.1377	1.0938	1.8658
0.333	1.3857	0.7762	1.2500
1.000	0.8260	0.6593	0.8808
3.000	0.3313	0.6896	0.7283
10.000	0.1000	0.7053	0.7089
30.000	0.0333	0.7069	0.7073
100.000	0.0100	0.7071	0.7071

$$\Pi_6 = \Pi_5^*, \quad (3.13)$$

$$\Pi_7 = -\{1/[\lambda^2\sigma_1^2(\sigma_1^2-\sigma_3^2)(\sigma_2^2-\sigma_1^2)] + \mathrm{Real}(2\sigma_2\Pi_2)\}, \quad (3.14)$$

$$\Pi_8 = 40\Pi_7 - [\lambda^2\sigma_1^2\Pi_4 + 2\mathrm{Real}(\lambda^2\sigma_2^2\Pi_6)] - \lambda^2\sigma_1^2\Pi_1 \times \exp(-40\sigma_1). \quad (3.15)$$

Utilizing the momentum equations, the fields are represented in terms of the pressure as

$$v = [P(\xi) - \lambda^2 P(\xi)_{6\xi}]\cos x, \quad (3.16)$$

$$u = \lambda^2 P(\xi)_{4\xi}\cos x, \quad (3.17)$$

$$w = E^{\frac{1}{2}}\lambda^2 P(\xi)_{3\xi}\sin x, \quad (3.18)$$

$$T = (1/E^{\frac{1}{2}})P(\xi)_{\xi}\sin x. \quad (3.19)$$

These are plotted in Figs. 1–4 for various values of λ, the value of λ being indicated on each curve. The symbol H designates the homogeneous case.

4. Discussion of results

We consider (2.23) and note that for x scales of O(1) there exist certain natural scales for z which arise out of the possible balances between the three terms. These are as follows:

1) THE EKMAN LAYER. A balance between the first and second terms occurs if $(\sigma s\delta E) \ll O(1)$. Then the vertical scale is $O(E^{\frac{1}{2}})$. The scaled vertical coordinate ξ in the Ekman layer is defined as

$$\xi = E^{-\frac{1}{2}}z. \quad (4.1)$$

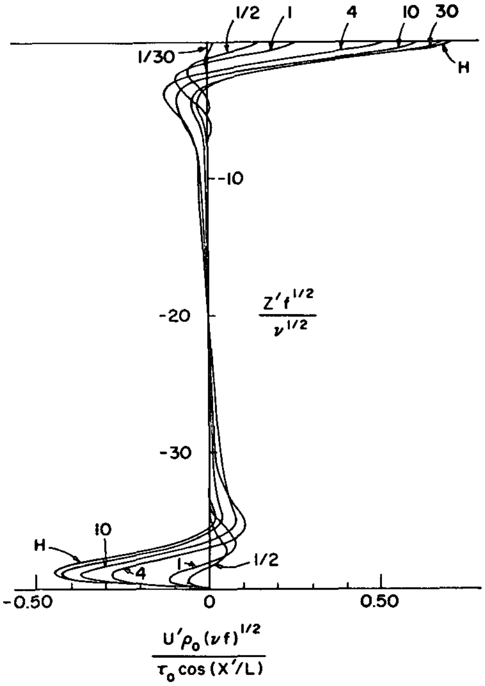

FIG. 2. u velocity as a function of stratification.

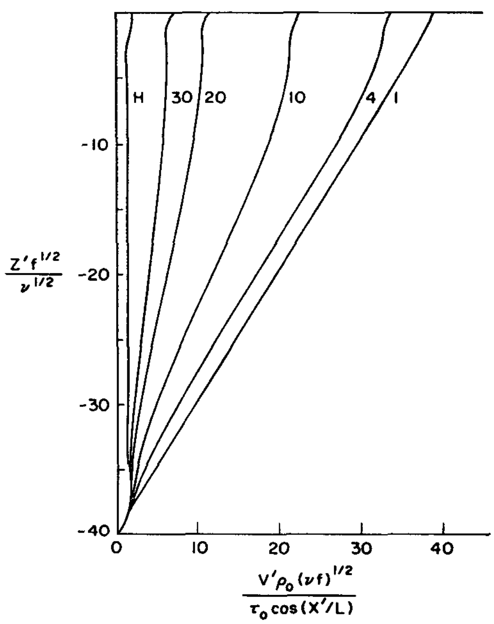

FIG. 1. v velocity as a function of stratification.

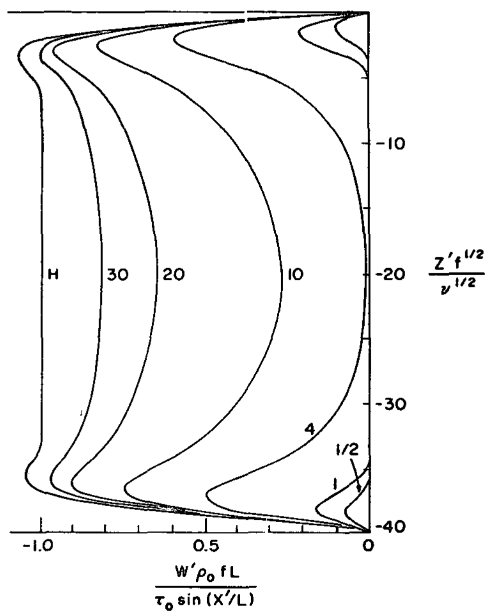

FIG. 3. w ve ocity as [a function of stratification.

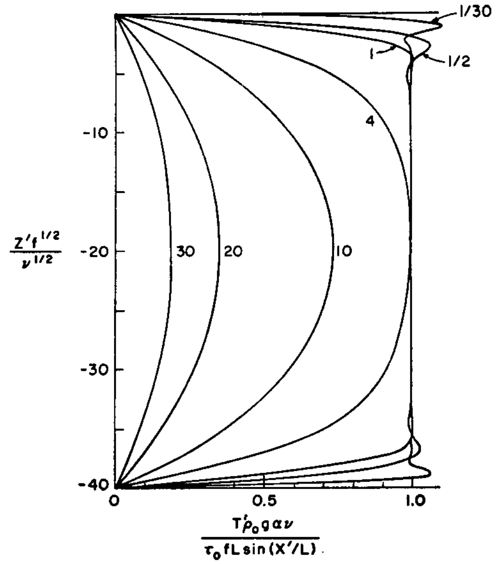

FIG. 4. Temperature as a function of stratification.

2) THE LINEYKIN SOLUTION. A balance between the second and third terms occurs if $(\sigma s\delta E)\ll O(1)$. Then the vertical scale is $O[(\sigma s\delta)^{-\frac{1}{2}}]$ and the appropriate scaled variable is

$$\mu=(\sigma s\delta)^{\frac{1}{2}}z. \qquad (4.2)$$

The Lineykin solution (Lineykin, 1955) only exists as a boundary layer for $(\sigma s\delta)^{\frac{1}{2}}>O(1)$. When $(\sigma s\delta)\leqslant O(1)$ this solution exists throughout the interior of the fluid.

3) THE THERMAL $E^{\frac{1}{3}}$ LAYER. A balance between the first and third terms occurs if $(\sigma s\delta E)\gg O(1)$. Then the vertical scale is $O[(E^2/\sigma s\delta)^{1/6}]$ and the scaled variable is

$$\eta=\left(\frac{E^2}{\sigma s\delta}\right)^{-1/6}z. \qquad (4.3)$$

4) THE COUETTE SOLUTION. The solution, for which all three terms in the pressure equation are identically zero, is

$$p(x,z)=A(x)z+B(x). \qquad (4.4)$$

The solution to (3.1) consists of the Couette flow and six exponential terms. Three of the exponentials grow and three decay away from $\xi=0$. For convenience the decaying ones are called the upper solution, and the growing ones, which contribute mainly at $\xi=-40$, are called the lower solution.

For $(\sigma s\delta E)\ll O(1)$ the real exponentials correspond to the Lineykin solution, and the complex exponentials are the Ekman solution. Consider a stratification for which both the Lineykin and Ekman solutions exist but the separation between the upper and lower surfaces is large enough so that the upper and lower Lineykin solutions decay to zero before they reach each other. Integration of the v momentum equation for the top solution shows, for each of these exponentials, that if a is a depth greater than the decay depth, the surface stress is

$$\int_{-a}^{0} ud\xi=\int_{-a}^{0} v_{\xi\xi}d\xi=v_\xi(0)-v_\xi(-a)=v_\xi(0), \qquad (4.5)$$

balanced by a body force. In the homogeneous case, all of the surface stress is balanced by a body force in the Ekman layer. Stratification, however, introduces a real exponential which can also balance a stress through a body force. The inclusion of stratification has introduced two additional solutions among which the surface stress can be distributed—the Lineykin and Couette solutions. In the other limit of $(\sigma s\delta E)\gg O(1)$, the real and complex exponentials no longer correspond to the Lineykin and Ekman solutions, but the idea of the stress being balanced by the real exponential, the complex exponentials, and the Couette flow still holds.

The distribution of surface stress among the real exponentials, the complex exponentials, and the Couette solution at $\xi=0$ for various values of λ is shown in Table 2. In the limit of $\lambda\ll O(1)$ all of the stress is absorbed by the Couette flow. For $\lambda\gg O(1)$, it is not clear which of the solutions balances the stress. This is resolved in the next section.

TABLE 2. The distribution of stress at $z=0$ for various values of λ.

λ	Real exponentials	Complex exponentials	Couette solution
0.100	−0.016	0.016	1.00
0.250	−0.055	0.055	1.00
0.500	−0.149	0.149	1.00
1.000	−0.417	0.417	1.00
2.000	−0.856	0.856	1.00
4.000	−0.989	0.989	1.00
10.000	−1.000	1.000	1.00

a. The limit $(\sigma s\delta E)\ll O(1)$

The complex exponentials in this limit correspond to the Ekman solution. The real exponentials correspond to the Lineykin solution. For an $O(1)$ applied stress in the Ekman layer u, $v=O(E^{\frac{1}{2}})$; $\omega=O(E)$; $T=O(\sigma s\delta E)$. For the Lineykin solution we define the variables

$$\left.\begin{aligned}\mu&=(\sigma s\delta)^{\frac{1}{2}}z\\ \hat{v},\hat{p}&=[1/(\sigma s\delta)^{\frac{1}{2}}](v,p)\\ \hat{T}&=T\\ \hat{u}&=E(\sigma s\delta)^{\frac{1}{2}}u\\ \hat{\omega}&=Ew\end{aligned}\right\}. \qquad (4.6)$$

The equations these satisfy are:

$$\hat{v}=\hat{p}_x, \qquad (4.7)$$

$$\hat{u}=\hat{v}_{\mu\mu}, \qquad (4.8)$$

$$0=-\hat{p}_\mu+T, \qquad (4.9)$$

$$\hat{w}=\hat{T}_{\mu\mu}, \qquad (4.10)$$

$$0=\hat{u}_x+\hat{\omega}_\mu. \qquad (4.11)$$

The Couette flow variables are denoted by an overbar, and v, $\bar{p}$, $\bar{T}$, $z=\mathrm{O}(1)$; $\bar{\omega}=\bar{u}=0$.

Because the order of magnitude of the temperature in the Lineykin and Couette solutions is much greater than it is in the Ekman solution, these solutions satisfy the temperature boundary conditions by themselves. Consequently, $T=0$ implies that $\bar{T}=-\hat{T}$. Through the thermal wind relationship this implies that $\hat{v}_\mu+\bar{v}_z=0$. All the stress in this limit is taken up by the Ekman layer.

b. The limit $(\sigma s\delta E)\gg O(1)$

In this limit the exponentials correspond to the thermal $\mathrm{E}^{\frac{1}{3}}$ layer. This layer is the thermal equivalent to the Stewartson $\mathrm{E}^{\frac{1}{3}}$ layer (Veronis 1967a,b). It was also examined by Stommel and Veronis (1957). While the dynamics of this layer are nonrotating, there is a rotational effect in that the $\tilde{v}$ can be calculated once $\tilde{u}$ is known. Thus, we have

$$0=-\tilde{p}_x+\tilde{u}_{\eta\eta}, \qquad (4.12)$$

$$\tilde{u}=\tilde{v}_{\eta\eta}, \qquad (4.13)$$

$$0=-\tilde{p}_\eta+\tilde{T}, \qquad (4.14)$$

$$0=\tilde{u}_x+\tilde{\omega}_\eta, \qquad (4.15)$$

$$\tilde{\omega}=\tilde{T}_{\eta\eta}, \qquad (4.16)$$

where the variables are defined by

$$\left.\begin{aligned} \eta&=(\sigma s\delta/\mathrm{E}^2)^{1/6}z\\ \tilde{p}&=(\mathrm{E}^2/\sigma s\delta)^{1/6}p\\ \tilde{u}&=[1/(\sigma s\delta)^{\frac{1}{2}}]u\\ \tilde{\omega}&=[\mathrm{E}/(\sigma s\delta)^2]^{\frac{1}{3}}w\\ \tilde{v}&=1/[(\sigma s\delta)^{5/6}\mathrm{E}^{\frac{1}{3}}]v\\ \tilde{T}&=T \end{aligned}\right\}. \qquad (4.17)$$

The temperature in this layer is assumed to be the same order as the interior temperature. This allows the thermal boundary conditions at the top and bottom surfaces to be satisfied. The boundary layer contribution to the surface stress is $\mathrm{O}[1/(\sigma s\delta\mathrm{E})^{\frac{2}{3}}]$, i.e., $\tilde{v}_\eta\ll\mathrm{O}(1)$.

Because there is a thermal $\mathrm{E}^{\frac{1}{3}}$ layer, the question arises as to why there is not a thermal $\mathrm{E}^{\frac{1}{4}}$ layer in this problem. The equations for this layer are the same as those for the thermal $\mathrm{E}^{\frac{1}{3}}$ layer except the first momentum equation which becomes

$$\hat{p}_x=0. \qquad (4.18)$$

This implies that $\hat{T}$ is independent of x. However, for this simple solution, all the variables are functions of x. Consequently, this layer does not occur.

c. Convergence to the homogeneous limit

Consider the situation in which the separation between the upper and lower surfaces is large enough so that the upper and lower Lineykin solutions essentially decay to zero before they reach each other. As was shown in the first part of Section 4, the fraction of the applied stress taken up by each exponential solution at $\xi=0$ is a direct indication of the net transport to the right of the stress associated with that exponential. Consequently, for those solutions which decay rapidly enough, there is no net transport to the right of the surface stress. The integrated transport in the Ekman layer is balanced by an equal, but opposite, integrated transport in the Lineykin layer.

Suppose the stratification is sufficiently weak that the upper Lineykin solution begins to contribute at the lower surface and vice versa. The value of the stress at $\xi=0$ is then the sum of the upper and lower contributions. This sum is always -1, but it is no longer simply related to the net transport in the upper Lineykin solution. However, the upper and lower Lineykin solutions have opposite but equal transports. Consequently, as their decay depth gets larger and larger, their transports begin to cancel. For extremely weak stratifications, neither solution decays appreciably in the distance between the upper and lower surface. As a result, the net Lineykin transport in this region is zero; transports occur only in the lower and upper Ekman layers. This is the homogeneous limit for the u velocity. Fig. 5 illustrates the manner in which the Ekman, Lineykin and Couette solutions combine to give the v velocity for $\lambda=100$. It appears that the Lineykin exponentials are well represented for weak stratifications by the first two terms in their Taylor expansions,

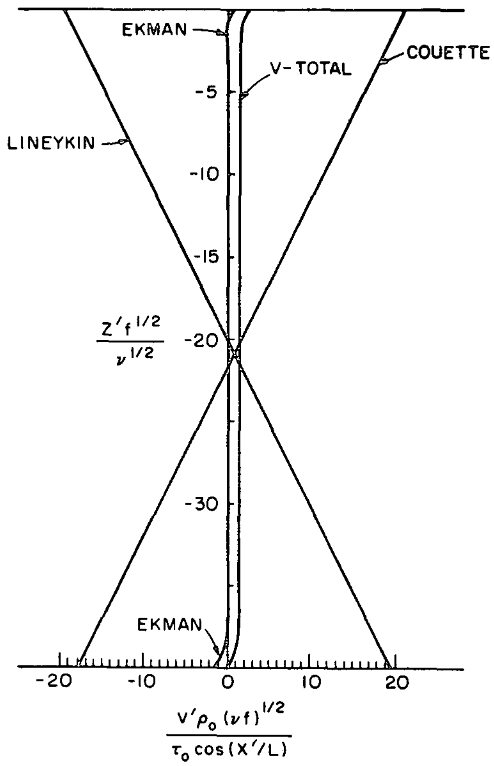

FIG. 5. Decomposition of the v velocity for $\lambda=100$.

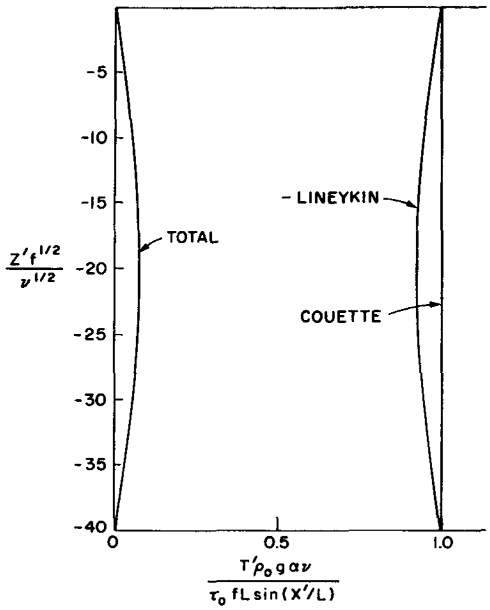

FIG. 6. Decomposition of the temperature for $\lambda = 50$.

i.e., $1+\mu$. Fig. 6 is a similar illustration for the temperature field when $\lambda = 50$.

d. Comparison with previous work

In the studies of Barcilon and Pedlosky (1967a,b), a stratification ($\sigma s \delta$; $\delta = 1$) of order $E^{1/2}$ represented an important transition in the dynamics of the fluid. This occurs because the order of the vertical velocity at the base of an Ekman layer driven by velocity boundary conditions is $E^{1/2}$. For stratifications less than this value, Ekman layers play a dominant role in the dynamics. For stratifications greater than this, the stratification inhibits the interior vertical velocity to be less than $O(E^{1/2})$; consequently, the Ekman layers are thought to be nondivergent, and viscous-diffusive processes control the dynamics of the fluid.

However, if, as in the example studied here, we specify a stress boundary condition, then the w at the base of the Ekman layer is $O(E)$. Consequently $E^{1/2}$ no longer appears as a crucial stratification. Now divergent Ekman layers can exist, and are influential in the dynamics, up to $(\sigma s \delta E) \approx O(1)$. At this point the Ekman and Lineykin layers combine to give the thermal $E^{1/3}$ layer. For stratifications greater than this the dynamics of the fluid are essentially non-rotating

The $O(E)$ flux out of the Ekman layer is absorbed by the Lineykin solution. The Lineykin solution only exists as a boundary layer for a small range in stratification. For weaker stratifications the boundary layer merges with the interior and is instrumental in bringing about the convergence to the homogeneous limit.

The Lineykin layer is physically analogous to the hydrostatic layer (Barcilon and Pedlosky, 1967b). The formal difference occurs only in that for the hydrostatic layer the diffusion of momentum and heat occurs in the vertical direction. In the examples which were considered, the small aspect ratio and periodic boundary conditions made it possible to neglect the effects of the hydrostatic layer and concentrate primarily on the effects of the Lineykin solution.

5. Some aspects of thermal forcing

We again consider (3.1) and now apply the boundary conditions:

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial \xi} = a\cos x, \quad T = b\sin x, \quad \dfrac{\partial u}{\partial \xi} = \omega = 0; \quad \text{at } \xi = 0 \\ u = v = \omega = T = 0; \quad \text{at } \xi = -40 \end{array}\right\}. \tag{5.1}$$

The solution is of the form

$$p(x,\xi) = [\sum_{i=1}^{6} \Pi_i \exp \sigma_i \xi + \Pi_7 \xi + \Pi_8] \sin x, \tag{5.2}$$

where it can be shown that $\Pi_7 \equiv a$.

For $(\sigma s \delta E) < O(1)$ the solution again consists of Ekman layers, a Lineykin solution, and a Couette flow. As before, the Lineykin and Couette solutions have to satisfy the thermal boundary condition by themselves, i.e.,

$$\hat{T} + \bar{T} = b, \quad \text{at } \xi = 0. \tag{5.3}$$

Since $\bar{T} \equiv \Pi_7 \equiv a$, we have $\hat{T} = b - a$. Through the thermal wind relationship

$$\hat{v}_\xi = \hat{T} = b - a. \tag{5.4}$$

The stress condition at $\xi = 0$ is

$$\hat{v}_\xi + \bar{v}_\xi + v_{\text{E.L.}\,\xi} = a, \tag{5.5}$$

or

$$v_{\text{E.L.}\,\xi} = a - b, \tag{5.6}$$

where $(\ )_{\text{E.L.}}$ stands for an Ekman layer variable. Because of the thermal forcing, the net transport in the Ekman layer is now proportional to $a - b$. For example, if $b = 2a$ then the net transport in the Ekman layer is, in fact, to the *left* of the "apparent" applied stress a! This results, of course, because of the thermal wind shear associated with the thermal forcing.

In the limit $(\sigma s \delta E) \gg O(1)$ the only effect of the thermal forcing would be to modify the solution near $\xi = 0$. The interior solution would still be the Couette flow

$$\bar{v} = a(\xi + 40), \tag{5.7}$$

$$\bar{T} = a. \tag{5.8}$$

This solution doesn't converge to the homogeneous limit unless some provision is made to let the applied temperature perturbation go to zero as the stratification decreases.

6. Geophysical implications

We recall that

$$\sigma s\delta=\left(\frac{\nu}{\kappa}\right)\left(\frac{g\alpha\Delta T}{f^2L}\right)\left(\frac{D}{L}\right).$$

From this it is apparent that decreasing the horizontal length scale L is equivalent to increasing the basic mean stratification. For typical mid-oceanic values of the parameters, $(\sigma s\delta)$ is of the order of 10^{-3}. Consequently, the effects of stratification are quite small. However, near boundaries, where features with small horizontal length scales might be important, the effects of stratification can be quite large. Such a situation was studied in Leetmaa (1969), where these ideas were applied to a model of coastal upwelling.

Acknowledgments. I would like to thank Prof. Henry Stommel for suggesting this problem. This research was supported by the National Science Foundation under Grant GA-1613.

REFERENCES

Barcilon, V., and J. Pedlosky, 1967a: Linear theory of rotating stratified fluid motions. *J. Fluid Mech.*, **29**, 1–16.

——, and ——, 1967b: A unified linear theory of homogeneous and stratified rotating fluids. *J. Fluid Mech.*, **29**, 609–621.

Leetmaa, A., 1969: On the theory of coastal upwelling. Ph.D. thesis, Dept. of Meteorology, Massachusetts Institute of Technology.

Lineykin, P. S., 1955: On the determination of the thickness of the baroclinic layer in the ocean. *Dokl. Akad. Nauk SSSR*, **101**, 461–464.

Stommel, H., and G. Veronis, 1957: Steady convective motion in a horizontal layer of fluid heated uniformly from above and cooled non-uniformly from below. *Tellus*, **9**, 401–407.

Veronis, G., 1967a: Analogous behavior of rotating and stratified fluids. *Tellus*, **19**, 620–633.

——, 1967b: Analogous behavior of homogeneous, rotating fluids and stratified, non-rotating fluids. *Tellus*, **19**, 326–335.