NOTES AND CORRESPONDENCE

On the Stability of Low-Latitude Quasi-Geostrophic Flow in a Conditionally Unstable Atmosphere[1]

CHIH-PEI CHANG

Dept. of Atmospheric Sciences, University of Washington, Seattle

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1 Introduction

There is mounting evidence that the latent heat released in deep cumulus convection is an important energy source for planetary waves in the tropical troposphere (Riehl, 1959; Chang *et al.*, 1970; Nitta, 1970). This has led to the belief that the concept of Conditional Instability of the Second Kind (CISK)[2] which has been instrumental in understanding the dynamics of hurricanes (Charney and Eliassen, 1964) may also be applicable to small-amplitude tropical disturbances.

Recently, Yamasaki (1969) studied this mechanism theoretically and showed that the observed preferred horizontal scale on the order of several thousand kilometers as reported by many investigators (Riehl, 1954; Yanai *et al.*, 1968; Wallace and Chang, 1969; Chang *et al.*, 1970) for low-latitude disturbances can be explained by the CISK mechanism in a two-dimensional quasi-geostrophic model (e.g., a normal modes stability analysis indicates that the maximum growth rate for unstable modes occurs for zonal wavelengths in the range of several thousand kilometers). However, since his model is limited to two dimensions, it remains to see if his results are relevant to tropical disturbances which are bounded laterally.

The purpose of this note is to investigate the CISK mechanism in a three-dimensional quasi-geostrophic model applied to low latitudes, and, in particular, to examine the validity of Yamasaki's results. The variation of mean zonal flow will be excluded.

2. Basic equations

The nondimensionalized, quasi-geostrophic, potential vorticity equation in a uniform zonal flow can be written as

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left[\Phi_{xx}+\Phi_{yy}+e^{z}\frac{d}{dz}(e^{-z}\epsilon\Phi_z)\right]+\Phi_x\beta = e^{z}\frac{d}{dz}(e^{-z}\epsilon Q), \quad (1)$$

where Φ is the perturbed geopotential, $\bar{u}$ the mean zonal velocity (constant), Q the diabatic heating rate being nondimensionalized by $(c_p c^3)/(RL)$ (where c_p is the specific heat at constant pressure, R the gas constant, c the characteristic horizontal velocity, and L the characteristic horizontal scale), β is the "Rossby parameter", and x, y, z are the coordinates directed eastward, northward and upward, respectively; in this case $z=-\ln(p/p_s)$, where p is the local pressure and p_s a standard reference pressure. The nondimensional parameter ϵ is scaled to the order of unity by

$$\epsilon=\left(\frac{f_0^2L^2}{gH^2}\right)\Big/\left(\frac{\partial\ \ln\theta_s}{\partial z^*}\right),$$

where f_0 is the Coriolis parameter, g the gravitational constant, H the scale height, θ_s the potential temperature of the basic state, and $z^*\equiv Hz$. The static stability $\partial(\ln\theta_s)/\partial z^*$ will be treated as constant.

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[2] The cooperative interaction between deep cumulus convection and large-scale disturbances, wherein the latent heat released in organized convection provides the horizontal density gradients for driving the large-scale disturbances, and the large-scale motion field provides the low-level convergence for maintaining convection in a conditionally unstable atmosphere. This interaction leads to a self-amplification of the large-scale disturbance.

Eq. (1) can be derived from the general equations of motion, continuity, and hydrostatic and thermodynamic energy (including diabatic heating) in the normal quasi-geostrophic scale-analysis approach (e.g., see Pedlosky, 1964).

Solutions of (1) can be assumed to be westward propagating waves with zonal wavenumber k and frequency ω, i.e.,

$$\begin{Bmatrix}\Phi\\ Q\end{Bmatrix}=\begin{Bmatrix}\phi(y,z)\\ q(y,z)\end{Bmatrix}e^{i(kx+\omega t)+z/2}. \tag{2}$$

Substituting (2) into (1) leads to

$$i\hat{\omega}\left[-k^2\phi+\frac{\partial^2\phi}{\partial y^2}+\epsilon\left(\frac{\partial^2}{\partial z^2}-\frac{1}{4}\right)\phi\right]+ik\beta\phi=\epsilon\left(\frac{\partial}{\partial z}-\frac{1}{2}\right)q, \tag{3}$$

where $\hat{\omega}=\omega+k\bar{u}$ is the complex Doppler-shifted frequency. Eq. (3) can be separated into meridional and vertical dependence in terms of a separation constant $-h_n$. Thus, by expanding the variables in the form

$$\phi=\sum_n \phi_n(z)Y_n(y),$$

$$q=\sum_n q_n(z)Y_n(y),$$

Eq. (3) becomes

$$\frac{d^2Y_n}{dz^2}+\left(\frac{k\beta}{\hat{\omega}}-k^2-h_n\right)Y_n=0, \tag{4}$$

$$\frac{d^2\phi_n}{dy^2}+\left(\frac{h_n}{\epsilon}-\frac{1}{4}\right)\phi_n=-i\left(\frac{\partial}{\partial z}-\frac{1}{2}\right)\frac{q_n}{\hat{\omega}}. \tag{5}$$

Adopting the annulus approximation, the boundary conditions for (4) are $Y_n=0$ at $y=\pm l$. This, in turn, gives

$$h_n=\frac{k\beta}{\hat{\omega}}-k^2-\lambda_n^2, \tag{6}$$

where $\lambda_n=n\pi/(2l)$ is a measurement of meridional wavenumber. In the following sections the subscript n will be dropped by setting $n=1$.

3. The CISK model

As in the usual CISK parameterization (Charney and Eliassen, 1964; and others), the diabatic heating function $q(z)$ is related to the vertical velocity w_{TE} at the top of the Ekman layer by

$$q(z)=m\eta(z)w_{TE}, \tag{7}$$

where $\eta(z)$ is the vertical profile of heating and m a proportionality constant. Thus, the heat released from condensation will be proportional to the boundary layer convergence. The top of the Ekman layer will be set at 900 mb.

Note that Eq. (7) also implies negative condensation heating when downward motion is taking place at the top of the Ekman layer. Charney and Eliassen have excluded this effect by specifying that $q(z)=0$ when $w_{TE}<0$. However, for the sake of facilitating the analytical work here, the assumption of negative heating will be used, as in Yamasaki (1969).

Charney and Eliassen (1949) have shown that for quasi-geostrophic flow, w_{TE} is proportional to the vorticity at the top of the Ekman layer. As a result we can write

$$w_{TE}=\alpha\nabla^2\Phi_{TE}=-\alpha(k^2+\lambda^2)\Phi_{TE}, \tag{8}$$

where α is a proportionality constant determined by the depth of the Ekman layer and the ageostrophic deviation angle at the surface.

To model the tropical atmosphere, we set $p_s=900$ mb so that $z=0$ at the top of Ekman layer. We let $H=7$ km and specify the heating profile $\eta(z)$ as

$$\eta(z)=\begin{cases}\sin\dfrac{\pi z}{2}, & z\leqslant 2\\ 0, & z>2\end{cases} \tag{9}$$

The actual heating profile is $e^{z/2}\sin(\pi z/2)$ (shown in Fig. 1) which gives maximum heating near the 300-mb level, as suggested from the spectral study by Chang *et al.* (1970).

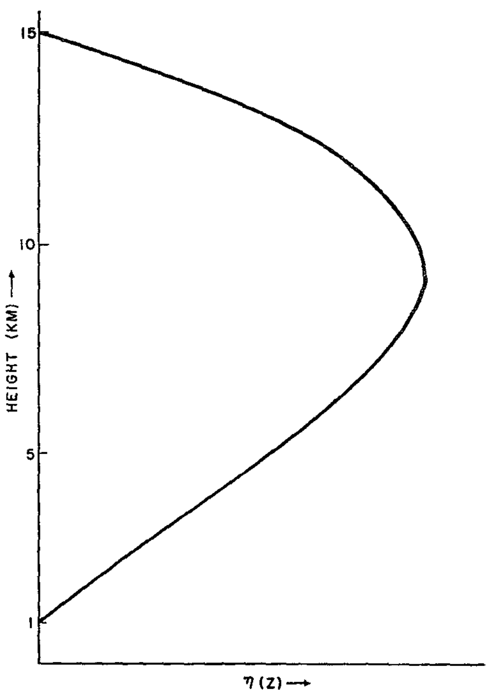

Fig. 1. Vertical profile of the heating function.

Boundary conditions applicable to (5) are as follows:

Lower boundary condition

(Derived from nondimensional thermodynamic energy equation)

$$i\hat{\omega}\left(\frac{\partial}{\partial z}+\frac{1}{2}\right)\phi+\frac{w_{TE}}{\epsilon}=q=0 \text{ at top of Ekman layer} \quad (10)$$

Interface conditions

Geopotential field (ϕ) continuous at $z=2$ (11)

Temperature field ($d\phi/dz$) continuous at $z=2$ (12)

The solutions to the system (5), (7), (8), (9) are then

$$\phi_n=A\sin\gamma z+B\cos\gamma z+\frac{i\pi/\hat{\omega}}{(2\gamma^2-\pi^2/2)}m\alpha(k^2+\lambda^2)\phi_{TE}\cos\frac{\pi z}{2}-\frac{i/\hat{\omega}}{(2\gamma^2-\pi^2/2)}m\alpha(k^2+\lambda^2)\phi_{TE}\sin\frac{\pi z}{2}, \quad \text{for } z\leqslant 2, \quad (13a)$$

and

$$\phi_n=C\sin\gamma z, \quad \text{for } z>2, \quad (13b)$$

where $\gamma=[(h/\epsilon)-\frac{1}{4}]^{\frac{1}{2}}$ and A, B, C are constants to be determined by the boundary conditions.

Applying (8), (10), (11), (12) to (13) and rearranging terms, we obtain

$$\hat{\omega}\left(2\gamma^2-\frac{\pi^2}{2}\right)-i\pi m\alpha(k^2+\lambda^2)=\frac{1}{\gamma}\left[\frac{\pi m\alpha(k^2+\lambda^2)}{2}-\frac{\alpha(k^2+\lambda^2)(2\gamma^2-\pi^2/2)}{\epsilon}\right]+\frac{i\hat{\omega}(2\gamma^2-\pi^2/2)}{2\gamma}+\frac{\pi(1/2+i\gamma)\alpha(k^2+\lambda^2)}{\gamma\exp(-2i\gamma)}. \quad (14)$$

Since the last term of (14) is small compared to other terms, it will be neglected in the following discussions.

4. Discussion

For the case of $\beta=0$, the equivalent depth and hence the vertical structure term γ are independent of $\hat{\omega}$. From (6) and the fact that $\epsilon\approx 1$, we have

$$\gamma^2=\frac{h}{\epsilon}-\tfrac{1}{4}\approx-(k^2+\lambda^2), \quad (15)$$

and from (14), letting $\epsilon=1$,

$$\hat{\omega}=\frac{\pi m\alpha(k^2+\lambda^2)\left(i+\frac{1}{2\gamma}\right)-\frac{\alpha(k^2+\lambda^2)}{\gamma}(2\gamma^2-\pi^2/2)}{(2\gamma^2-\pi^2/2)\left(1-\frac{1}{2\gamma}\right)}. \quad (16)$$

This is a purely imaginary $\hat{\omega}$ which gives the growth or damping rate of the waves.

Substituting (15) into (16) gives

$$\hat{\omega}=\frac{i\pi m\alpha(k^2+\lambda^2)}{\left[\frac{\pi^2}{2}+2(k^2+\lambda^2)\right]}+\frac{i\alpha(k^2+\lambda^2)^{\frac{1}{2}}}{\left[1-\frac{1}{2(k^2+\lambda^2)^{\frac{1}{2}}}\right]}. \quad (17)$$

In the right-hand side of (17), the first term is the heating term, which tends to cause amplification of the waves, while the second term is the friction term which tends to cause decay of the waves. If the width of the channel flow is constant and k increases (zonal wavelength decreases), the heating term will tend to increase the growth rate while the friction term will tend to decrease the growth rate.

Fig. 2 shows the growth rate curves for $\alpha=0.03$ and $m=12$, corresponding to a kinematic eddy viscosity coefficient of 10 m^2 sec^{-1} [a value suggested by Brunt (1939)], a latitude of 15°, and a maximum heating rate of 10C per day near the 300-mb level. Thin line I is the curve corresponding to $\lambda=6.2$ (which is 1500 km lateral width for the channel flow) and no β effect. The growth rate increases from small scale to large scale, first rapidly, then slowly, and then seems to be asymptotic to a constant. Therefore, there is no "most unstable" wave scale. But if we simplify the model by assuming no meridional variation (set $\lambda=0$ and the lateral width becomes infinity), a growth rate (shown as thin line II in Fig. 2) can be obtained with a maximum ~4000 km. This falls in the range of Yamasaki's findings. Varying the parameters in the range reasonable for the tropical atmosphere gives similar characteristic differences of the growth rate curve between three-dimensional and two-dimensional channel flow.

This difference can readily be seen from (7). If $\lambda=0$, as we increase the zonal wavelength (decrease k), the increase of growth rate due to the heating term will be slower than in the case of a non-zero λ, corresponding to a channel width on the order of a few thousand kilometers. But the effect of varying λ on the damping due to the friction term is relatively smaller. Thus, the growth rate curve can bend back at the longer wavelength end.

If we include the β effect, then γ will be dependent on $\hat{\omega}$ and the order of the frequency equation will be increased. This may be the reason that Yamasaki (1969) found multiple roots when he included the β effect in his

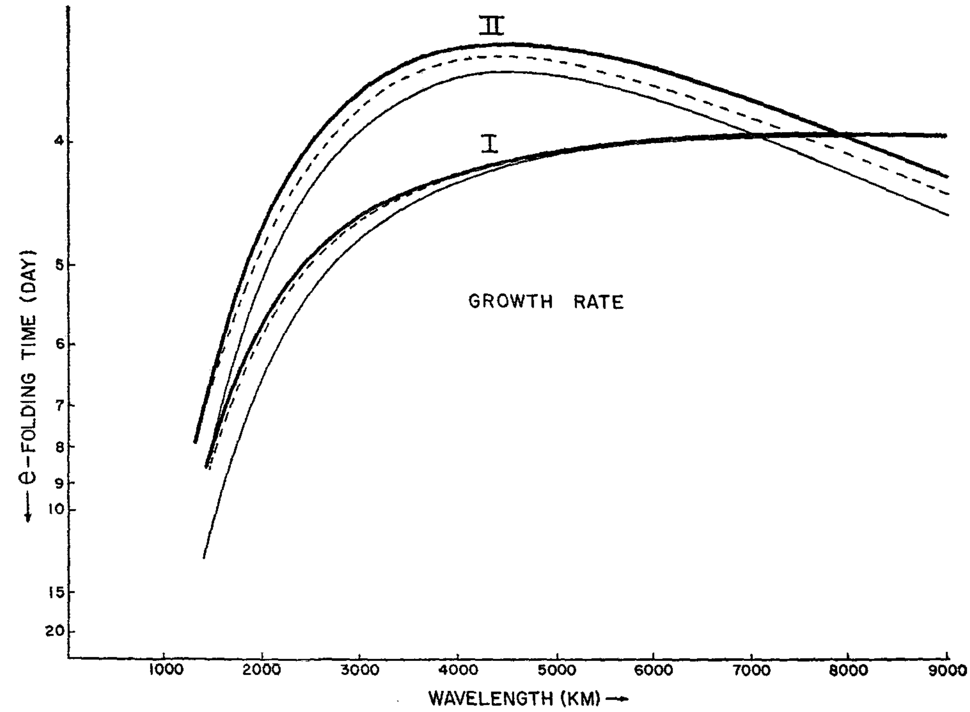

FIG. 2. Growth rate as a function of wavelength. Group I curves apply to laterally bounded waves and group II to two-dimensional waves. Thin lines are growth rate curves without β effect, thick lines with β effect, and dashed lines with β effect and including the vorticity advection for parameterizing the vertical velocity at the top of Ekman layer.

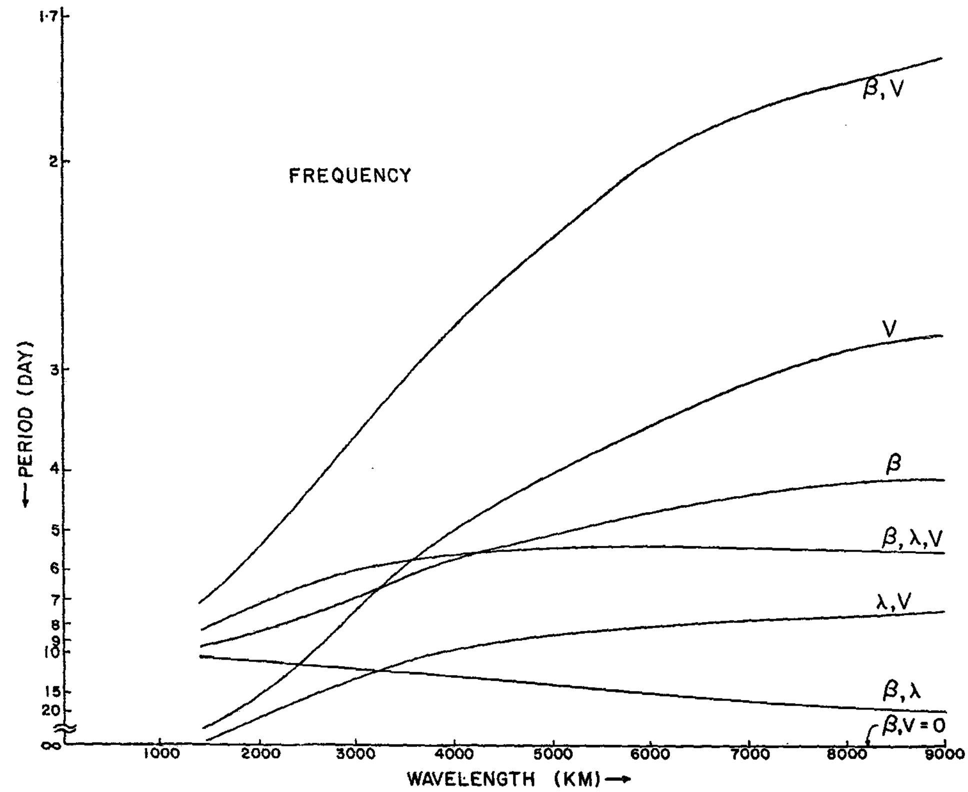

FIG. 3. Frequency as a function of wavelength. Symbols on the curves are defined as follows: β, β effect is included; λ, lateral boundaries are imposed; V, vorticity advection is included in the parameterization of the vertical velocity at the top of Ekman layer.

numerical model. Thick lines I and II in Fig. 2 give the growth rate for waves possessing reasonable frequencies[3] for the three-dimensional and two-dimensional cases, respectively. Here we see that the β effect slightly destabilizes the laterally bounded waves at the shorter wavelength end, while it destabilizes the unbounded waves in the whole spectrum.

A noticeable difference between Yamasaki's work and this model is the treatment of the vertical velocity at the top of Ekman layer. In Yamasaki's model, w_{TE} is not only produced by boundary layer convergence as parameterized here, but also as a result of vorticity advection. The dashed lines in Fig. 2 illustrate the inclusion of this effect, and suggest no significant change in the growth rate.

The real part of the Doppler-shifted frequency $\hat{\omega}$, i.e., the frequency of waves relative to mean zonal flow, is shown in Fig. 3. As expected, it shows that the β effect increases the westward phase speed. The exclusion of lateral boundaries ($\lambda=0$) also seems to give an unsatisfactory explanation of the frequency for a substantial part of the wave spectrum.

5. Concluding remarks

From the foregoing discussion, it is suggested that the shape of the growth rate curve found by Yamasaki for unstable wave modes in a two-dimensional, quasi-geostrophic flow driven by the CISK mechanism may be attributed to the over-simplification of his model.

The failure to obtain a maximum growth rate on the scale of observed tropical waves in the simple model described above may be a consequence of the use of geostrophic boundary layer solution in the parameterization of the CISK mechanism. However, the recent analysis of Holton *et al.* (1971) suggests that the geostrophic boundary layer solution may still be a good approximation at latitudes as low as 10° for disturbances with periods of 5 days.

Perhaps a more important defect in the model is the use of quasi-geostrophic equations to represent the dynamics of low-latitude flow. Matsuno (1966) has shown that as k becomes smaller the quasi-geostrophic approximation becomes less valid in the equatorial region. Thus, if non-geostrophic effects are included in the model, the growth rate might possibly decrease in the longer wavelength end from what has been shown in Fig. 2, leaving a maximum growth rate at some intermediate wavelength.

[3] The frequency equation becomes transcendental when $\beta\neq0$ and is solved by a numerical method.

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