Size Distribution of Raindrops Generated by their Breakup and Coalescence[1]

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ABSTRACT

Some computations of the evolution of raindrop-size distributions under the operation of the processes of drop breakup and coalescence are presented. It is found that the size distribution approaches a stationary state. The stationary size distributions for different rainfall rates are roughly parallel to each other when $\log f(r)$ is plotted against r, $f(r)$ being the concentration density of drops of radius r. A comparison with observations shows that the computed distributions are flatter than the observed ones. It is suggested that other processes besides the breakup and coalescence of raindrops shape their distribution with size.

1. Introduction

The modification of the size distribution of raindrops by coalescence and evaporation was investigated by Mason and Ramanadham (1954), Rigby *et al.* (1954), Sivaraman and Sivaramakrishnan (1962), and Hardy (1963) among others. When large unstable drops are present, the process of their breakup also modifies the size distribution. This paper is concerned with the evolution of drop-size spectra under the operation of the processes of drop disintegration and coalescence.

One application of these studies lies in their possible bearing on the explanation of observed raindrop size distributions. The main processes involved in the development of raindrops by the "warm" rain process are the condensation of water vapor, collisions and coalescence between drops, and drop breakup. In spite of the differing physical characteristics of clouds giving rise to different rates of condensation of water vapor, averaged raindrop size spectra for a given rainfall rate are found to be remarkably similar. This suggests that the processes of coalescence between raindrops and their breakup might be the major factors controlling the size distribution of raindrops. At low rainfall rates, large unstable drops are scarce, and drop breakup may not be significant, leaving coalescence between raindrops as the major factor; this was suggested by Hitschfeld (1955). Calculations by Srivastava (1967) showed that the exponential size distribution of raindrops (Marshall and Palmer, 1948) is quasi-stable with respect to coalescence in the sense that while the exponential distribution changes rather slowly, narrow distributions tend rather rapidly toward the exponential shape. At large rainfall rates, however, the concentration of unstable drops is comparatively large; their breakup, therefore, may also be a significant factor governing the shape of the dropsize distribution curve. Indeed, Komabayasi (1965) and Blanchard and Spencer (1970) suggested that observed raindrop size spectra may be the result of the processes of coalescence and drop breakup. Coalescence tends to produce progressively larger drops which then disintegrate to produce smaller drops. Eventually a balance may be reached between coalescence and breakup, leading to a stationary distribution; the purpose of this study is to compute this stationary distribution.

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2. Governing equation

The equation governing the evolution of the drop-size distribution under the processes of coalescence and breakup may be written as

$$\frac{\partial f(x,t)}{\partial t}=\frac{1}{2}\int_0^x f(y,t)f(x-y,t)K(y,x-y)dy - f(x,t)\int_0^\infty f(y,t)K(x,y)dy - f(x,t)P(x)+\int_x^\infty f(y,t)Q(y,x)P(y)dy, \quad (1)$$

where $f(x,t)$ is the concentration density of drops of mass x at time t, that is, $f(x,t)\Delta x$ is the number per unit volume of drops of mass x to $x+\Delta x$. Sometimes the concentration density function $f(x,t)$ will be simply written as $f(x)$, dropping explicit mention of the time dependence. Besides the mass x, the concentration density may be expressed in terms of any function of x; thus, if we use the radius r, then $f(r)\Delta r$ is the number

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per unit volume of drops of radius r to $r+\Delta r$. Obviously,

$$f(r)=f(x)\frac{dx}{dr}=\frac{3xf(x)}{r}. \qquad (2)$$

In terms of the natural logarithm of r, which will be used below as a size variable, we have

$$f(\ln r)=\frac{f(r)}{d\ln r/dr}=rf(r). \qquad (3)$$

In addition to the concentration density function, the mass density function xf will be used; this gives the mass of drops per unit volume per unit size range expressed in terms of the argument of f.

The first two terms on the right-hand side of Eq. (1) represent the rate of change of the drop concentration by coalescence, and the third and fourth terms its rate of change by drop breakup. The function $K(x,y)$ is the so-called collection kernel and is such that $f(x)f(y)K(x,y)\Delta x\Delta y\Delta t$ is the probability of a collision between drops of mass x to $x+\Delta x$ and y to $y+\Delta y$ in time Δt. The collection kernel is a symmetric function of its arguments. For geometric sweepout by virtue of relative fall velocity (which will be assumed here), we have

$$K(x,y)=\pi(r_1+r_2)^2|V_1-V_2|, \qquad (4)$$

where r_1, r_2 and V_1, V_2 are the radii and terminal velocities corresponding to the drops of mass x and y. Since we shall be concerned with drops greater than 0.1 mm radius, (4) is probably an adequate representation of the kernel. Without the last two terms, Eq. (1) is the so-called stochastic collection equation; the interpretation of this equation may be found in various places (e.g., Berry, 1967). The term $f(x,t)P(x)$ represents the loss of drops of mass x by disintegration, $P(x)\Delta t$ being the probability of a drop of mass x disintegrating in time Δt. The last term on the right-hand side of (1) represents the gain in the concentration of drops of mass x by the disintegration of larger drops, $Q(y,x)\Delta x$ being the number of drops of mass x to $x+\Delta x$ formed by the breakup of one drop of mass y. Obviously, we must have

$$Q(y,x)=0, \quad y<x. \qquad (5)$$

In addition, from mass conservation considerations, we have

$$y=\int_0^y xQ(y,x)dx. \qquad (6)$$

By multiplying both sides of (1) by x and integrating with respect to x over its entire range, and making use of the symmetry of $K(x,y)$ and Eq. (6), it may be shown that the total mass under the distribution $f(x,t)$ does not change with time.

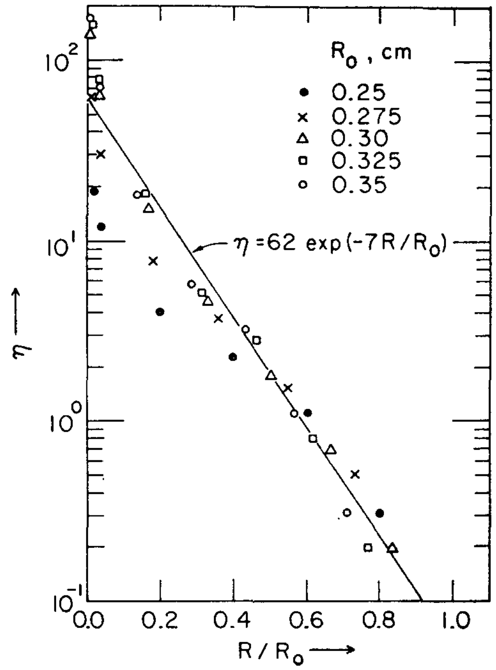

FIG. 1. Cumulative size distribution of fragments resulting from the disintegration of a drop of radius R_0. The ordinate is the number of fragments exceeding the indicated normalized radius, R/R_0. This figure is based on data from Komabayasi *et al.* (1964).

3. Drop breakup

The functions $P(x)$ and $Q(y,x)$ can be determined from experiments. The breakup of large drops was experimentally studied by Blanchard (1950), Fournier d'Albe and Hidayetulla (1955), and Komabayasi *et al.* (1964) among others. For the function $P(x)$ associated with the probability of breakup of a drop of mass x (radius r), the last named authors suggested

$$P(x)=2.94\times10^{-7}\exp(34r) \quad [\mathrm{sec}^{-1}], \qquad (7)$$

giving rise to a size distribution of the fragments resulting from the disintegration of a drop of radius r_0 of the form

$$Q(r_0,r)=10^{-1}r_0^3\exp(-15.6r). \qquad (8)$$

From $Q(r_0,r)$ the function $Q(y,x)$ can be found using a transformation similar to those in (2) and (3). Eq. (8) was suggested by Komabayasi *et al.* as a crude approximation to their observations. In particular, it does not satisfy condition (6) and is therefore unacceptable for theoretical or numerical work, since violation of (6) implies violation of water conservation. We have therefore reanalyzed the data reported by Komabayasi *et al.* and redetermined the function $Q(y,x)$.

Fig. 1 is a plot of the data of Komabayasi *et al.* and shows the size distribution of the fragments resulting from drops of various sizes. The abscissa is the normalized drop size R/R_0, R_0 and R being the radii of the parent and fragment drops, respectively. The ordinate η is the number of drops of normalized size greater than

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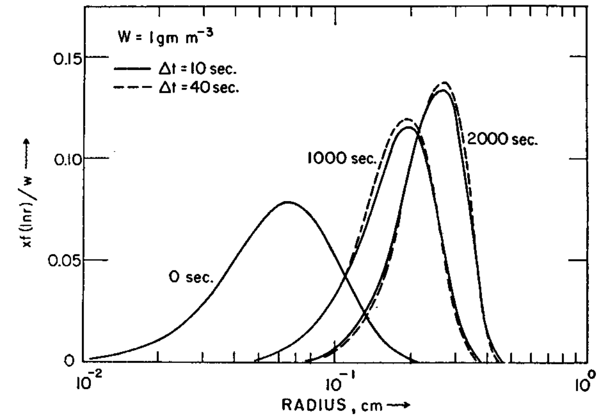

FIG. 2. Mass density function normalized by the water content of the distribution vs the radius. This figure shows development of a distribution by coalescence only. The computations were made with two values of the time increment, $\Delta t=10$ sec (full curves) and $\Delta t=40$ sec (dashed curves).

R/R_0 resulting from the breakup. It is seen that the data for the various R_0 may be approximately represented by

$$\eta=a\exp(-bR/R_0), \tag{9}$$

where a and b are constants. Obviously the curve in Fig. 1 must be truncated at $R=R_0$, so that

$$\eta=0, \quad R/R_0>1. \tag{10}$$

From (9), the size distribution is given by

$$Q(R_0,R)=\frac{ab}{R_0}\exp\left(-b\frac{R}{R_0}\right). \tag{11}$$

This may also be written in terms of the masses y and x of the drops of radii R_0 and R as

$$Q(y,x)=Q(R_0,R)\frac{dR}{dx}=\frac{ab}{3x}\left(\frac{R}{R_0}\right)\exp\left(-b\frac{R}{R_0}\right). \tag{12}$$

The data in Fig. 1 are fairly well represented by $b=7$. The constant a is now determined by the condition of mass conservation:

$$R_0^3=\frac{ab}{R_0}\int_0^{R_0}R^3\exp\left(-b\frac{R}{R_0}\right)dR, \tag{13}$$

which gives

$$a=\frac{b^3}{[6-\exp(-b)(b^3+3b^2+6b+6)]}. \tag{14}$$

With $b=7$, Eq. (14) gives $a=62.3$.

It should be remembered that (11) and (12) are only approximations to the data; however, they possess the virtue of satisfying (6).

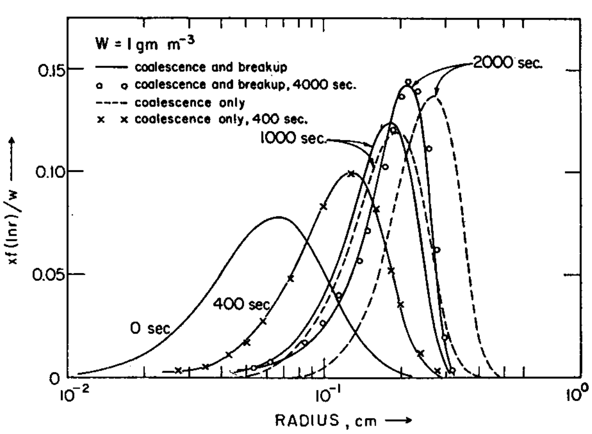

FIG. 3. Development of drop-size distribution by coalescence and breakup (full curves) and by coalescence only (dashed curves).

4. Method of calculation

The stationary distribution under drop breakup and coalescence satisfies Eq. (1) with its left-hand side set equal to zero. Since it is difficult to solve such a nonlinear equation, the stationary solution was obtained by the so-called marching method. In this method we start with an assumed initial distribution and solve the complete equation (1) to find the time evolution of the distribution. After some time the distribution, $f(x,t)$, becomes independent, or approximately independent, of time; this gives the required stationary solution. For a given water content the stationary distribution is independent of the assumed initial distribution; however, the time required to reach the stationary distribution depends upon the initial distribution.

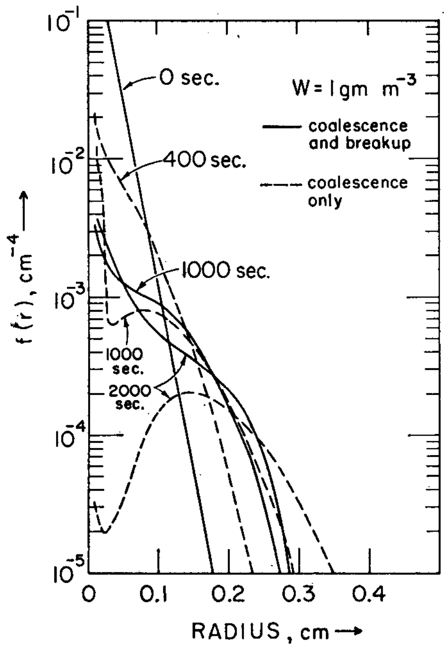

FIG. 4. Replot of Fig. 3 as concentration density $f(r)$ vs radius r.

The method used for solving (1) was that described by Berry (1967) with some minor modifications. The radius range from 0.1–5 mm was considered and a

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logarithmic radius scale was adopted such that the drop mass doubled at every fourth grid point of the radius scale. The integrals on the right-hand side were evaluated using up to a five-point Newton-Cotes integration formula. In evaluating the first integral on the right-hand side, the values of $f(x-y, t)$ which correspond to radii not on the selected radius scale were obtained using a four-point Lagrangian interpolation on the logarithm of $f(x,t)$. The solution was advanced in time steps of Δt using the simple scheme

$$f(x, t+\Delta t)=f(x,t)+A\Delta t, \qquad (15)$$

where A is the value of the right-hand side of (1). The time integration was not refined any further since the accuracy of the final stationary distribution is not critically dependent on that of the time integration.

The fall velocity data reported by Gunn and Kinzer (1949) and its extrapolation was used to compute the kernel $K(x,y)$. To avoid development of excessively large drops by coalescence, the fall velocity of drops greater than 3.2 mm radius was set equal to that of the drop of radius 3.2 mm.

The accuracy of the computations was checked in a number of ways. First of all, Eq. (1) was solved considering coalescence only and using a collection kernel equal to the sum of the volumes of the drops. The numerical solution was found to agree closely with the analytical solution for this case given by Golovin (1963). The degree to which the water content is conserved is another test of the accuracy of the numerical solution. The water content did not change by more than 5% in any of the calculations. In a calculation starting with a rather steep initial exponential distribution having a water content of 1 gm m^{-3}, the water content changed by less than 2% at the end of 4000 sec. This attests to the accuracy of the computation of the right-hand side of (1). To test the accuracy of the time integration, the calculation was done with time steps of 10 and 40 sec, ignoring drop breakup (see Fig. 2). The concentrations in the two cases did not differ by more than 3% over the range of sizes which contributed significantly to the water content.

5. Results and discussion

Fig. 3 is a plot of the development of the same initial distribution as in Fig. 2, but considering both coalescence and breakup (full curves). For comparison, the curves for no drop breakup are also included (dashed curves). While the curve with no drop disintegration advances progressively toward larger sizes, the curve with breakup approaches a stationary distribution in about 1500 sec (not shown), with the curves for 2000 and 4000 sec being virtually identical. In the literature raindrop size distributions are usually plotted on scales of $\log f(r)$ vs r; for comparison a replot of Fig. 3 on comparable scales is given in Fig. 4. It is interesting to note, with breakup, that the concentration of small drops increases between 1000 and 2000 sec.

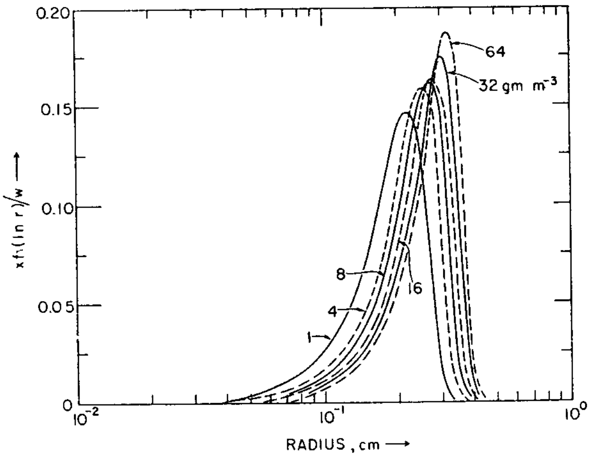

Fig. 5. Stationary distributions of the indicated water contents resulting from drop disintegration and coalescence.

Calculations similar to those on which Fig. 3 is based were repeated for water contents W equal to 4, 8, 16, 32 and 64 gm m^{-3}. The initial distributions used were similar to that for $W=1$ gm m^{-3}, the concentrations being stepped by a constant factor to give the required water content. As the water content was increased, a stationary distribution was achieved faster; with $W=32$ gm m^{-3}, the stationary distribution was attained in ~300 sec. Figs. 5 and 6 are plots of the stationary distributions on scales similar to that of Figs. 3 and 4, respectively. The rainfall rate for each distribution is shown in Fig. 6. For comparison, the Marshall-Palmer

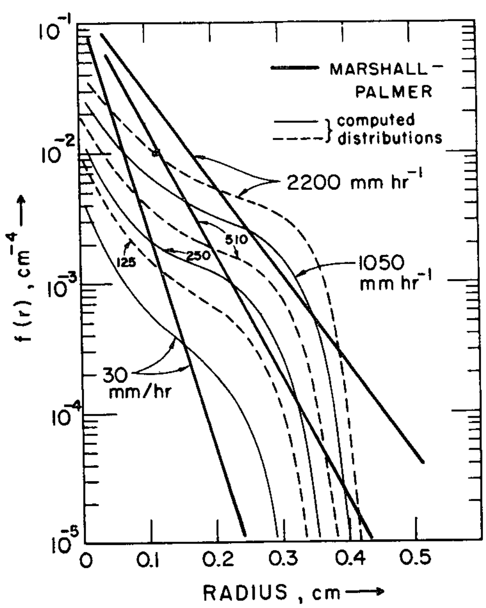

Fig. 6. Replot of Fig. 5 as concentration density vs radius. Rainfall rates are indicated and the Marshall-Palmer distributions for three rainfall rates are shown for comparison.

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distributions for three rainfall rates are also shown. It is seen that in the middle size range the stationary distributions are flatter than the Marshall-Palmer distribution. Moreover, in contrast to the Marshall-Palmer distribution, the stationary distributions for various rainfall rates are roughly parallel to each other. It may be pointed out that observed distributions for high rainfall rates display both these characteristics; this may be seen by reference to Fig. 9 in the paper by Blanchard and Spencer (1970). Sekhon and Srivastava (1970) found this to be the case for drop-size distributions deduced from Doppler radar observations of a thunderstorm.

A comparison of stationary and observed distributions is made in Fig. 7. For the sake of clarity only the distributions for 30, 125 and 510 mm hr^{-1} are shown. The observed distributions for 25, 100 and 500 mm hr^{-1} are taken from Fig. 9 of Blanchard and Spencer. The distribution observed by Blanchard in his rain column experiment at a rainfall rate of 410 mm hr^{-1} is also shown. It is seen that the distribution observed by Blanchard is similar in *form* to the computed distribution but has comparatively more large and small drops and less medium sized drops. This may be because the fall distance in the observations was not enough to allow a balance between drop breakup and coalescence to be achieved.

It is also seen from the figure that the observed distributions are distinctly steeper and are limited to smaller sizes than the computed distributions. This suggests that other factors besides drop breakup and coalescence are responsible for shaping the size distribution of raindrops. Of course, observed distributions may not have achieved equilibrium. Moreover, we have assumed only breakup of individual drops, while observations reported in the literature show that drops may break up on collision. This effect was not considered in formulating Eq. (1) because of lack of data on the probability of such breakup, and the size distribution of the fragments resulting therefrom. Qualitatively, it appears that the inclusion of this effect would tend to bring the computed distributions closer to the observed ones. Another shortcoming in the computations is that condensation and drops of radius <0.1 mm were not considered. By providing a continuing supply of small drops, these effects would tend to produce a steeper distribution.

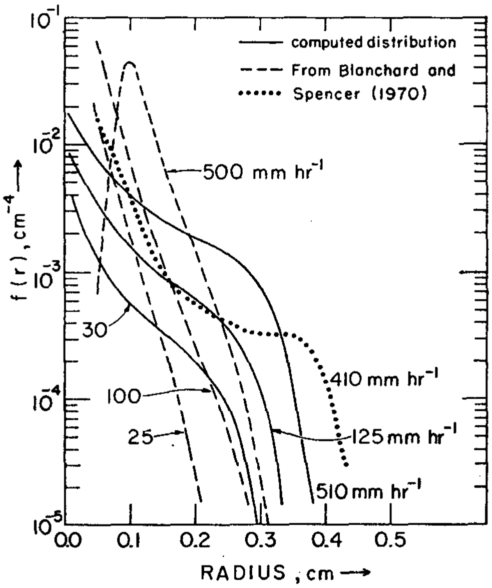

FIG. 7. Comparison of computed stationary distributions (full cruves); distributions observed in Southern Rhodesia and Miami [as quoted by Blanchard and Spencer (1970)] (dashed); and a distribution observed in the experiments by Blanchard and Spencer (dotted).

6. Summary

The computations have shown that under the action of drop breakup and coalescence a stationary distribution results. Similar to observations, the stationary distributions are roughly parallel to each other and are relatively flat in the medium size range. However, observed distributions are steeper. This may be because the processes of drop disintegration on collision, condensation of water vapor, and drops of radius <0.1 mm were not considered in the calculations.

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REFERENCES

Berry, E. X, 1967: Cloud droplet growth by collection. *J. Atmos. Sci.*, **24**, 688–701.

Blanchard, D. C., 1950: The behavior of water drops at terminal velocity in air. *Trans. Amer. Geophys. Union*, **31**, 836–842.

——, and A. T. Spencer, 1970: Experiments on the generation of raindrop-size distributions by drop breakup. *J. Atmos. Sci.*, **27**, 101–108.

Fournier d'Albe, E. M., and M. S. Hidayetulla, 1955: The break-up of large water drops falling at terminal velocity in free air. *Quart. J. Roy. Meteor. Soc.*, **81**, 610–613.

Golovin, A. M., 1963: The solution of the coagulation equation for cloud droplets in a rising air current. *Bull. Acad. Sci. URSS, Ser. Geofiz.*, No. 5, 783–791.

Gunn, R., and G. D. Kinzer, 1949: The terminal velocity of fall for water droplets in stagnant air. *J. Meteor.*, **6**, 243–248.

Hardy, K. R., 1963: The development of raindrop-size distribution and implications related to the physics of precipitation. *J. Atmos. Sci.*, **20**, 299–312.

Hitschfeld, W., 1955: Size distribution generated by a random process. *Artificial Stimulation of Rain*, New York, Pergamon Press, 224–228.

Komabayasi, M., 1965: Probability of disintegration of water drops as a factor determining size distribution of raindrops. *Proc. Intern. Conf. Cloud Physics*, Tokyo and Sapporo, 260–264.

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——, T. Gonda and K. Isono, 1964: Lifetime of water drops before breaking and size distribution of fragment droplets. *J. Meteor. Soc. Japan*, **42**, 330–340.

Marshall, J. S., and W. McK. Palmer, 1948: The distribution of raindrops with size. *J. Meteor.*, **5**, 165–166.

Mason, B. J., and R. Ramanadham, 1954: Modification of the size distribution of falling raindrops by coalescence. *Quart. J. Roy. Meteor. Soc.*, **80**, 388–394.

Rigby, E. C., J. S. Marshall and W. Hitschfeld, 1954: The development of the size distribution of raindrops during their fall. *J. Meteor.*, **11**, 363–372.

Sekhon, R. S., and R. C. Srivastava, 1970: Doppler radar observation of dropsize distribution in a thunderstorm. *Preprints of Papers 14th Radar Meteorology Conference*, Tucson, Amer. Meteor. Soc., 171–175.

Sivaraman, K. R., and M. V. Sivaramakrishnan, 1962: Modification of the size distribution of raindrops with distance fallen. *Indian J. Meteor. Geophys.*, **13**, Spec. No. 17–20.

Srivastava, R. C., 1967: On the role of coalescence between raindrops in shaping their size distribution. *J. Atmos. Sci.*, **24**, 287–292.

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