Limit Cycles and Unstable Baroclinic Waves[1]

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ABSTRACT

An analytical theory is presented for long-period pulsations of a finite-amplitude baroclinic wave. It is shown that for small dissipation a limit cycle is possible *whether or not* the steady wave regime is stable to infinitesimal disturbances. Moreover, the limit cycle is shown to be stable. A second limit cycle is shown to exist only when the steady wave regime is stable but in that case the *second* limit cycle is unstable.

1. Introduction

This paper continues a study of the finite-amplitude dynamics of an unstable wave in a two-layer baroclinic current. In a recent paper[2] (Pedlosky, 1971) the effect of a small amount of viscous dissipation was considered. It was shown that the presence of even a little friction produced profound effects on the finite-amplitude dynamics. One of the most important of these was the eventual loss of "memory" by the wave in the finite-amplitude regime. This is manifested in two ways and depends on the degree of viscous dissipation. Either the wave amplitude tends asymptotically to a steady value independent of the initial conditions, or, if the dissipation is small enough, the wave *amplitude* continues to pulsate periodically. Moreover, the amplitude oscillation which appears is a limit cycle. That is, the amplitude pulsation will change its amplitude and period continuously with time until the limit cycle oscillation is achieved, after which all elements of the amplitude oscillation such as the period and amplitude (of the amplitude oscillation) remain fixed. From experiments performed with an analog computer it appeared, as reported in L. C., that two distinct limit cycles are possible. In the first case (which we will now refer to as the $\mathcal{E}>0$ limit cycle)[3] the amplitude of the baroclinic wave (as seen by an observer moving with the speed of the wave phase) oscillates between maximum and minimum values which are equal in magnitude but opposite in sign. In the second case (the $\mathcal{E}<0$ limit cycle) the amplitude of the wave oscillates between maximum and minimum values which are unequal but identical in sign. Thus, an observer travelling with the phase speed of the wave (which in this case is just the mean zonal velocity of the undisturbed flow) can expect, *a priori*, to witness one of three types of asymptotic behavior.

The wave pattern may remain fixed. This state is qualitatively similar to the steady wave regime reported in the annulus experiments of Hide (1953). Otherwise, the observer will witness one of the two types of *amplitude vacillation*, qualitatively similar to the experimentally observed amplitude vacillations reported by Pfeffer and Chiang (1967). In the first mode of vacillation (the $\mathcal{E}>0$ limit cycle) the baroclinic wave will actually periodically disappear as the wave amplitude passes through zero. The zonal flow then possesses its maximum vertical shear. The mean shear reaches its minimum value when the wave amplitude attains its maximum value. In the second mode (the $\mathcal{E}<0$ limit cycle) the wave is present throughout the entire vacillation. Furthermore, in the second mode of vacillation an observer originally travelling with a crest continues to do so, while in the first mode a crest, for example, becomes a trough as the wave amplitude changes sign.

The purpose of this paper is to present an analytical theory of the limit cycle vacillations. In L. C. only the $\mathcal{E}>0$ limit cycle was actually observed in the computer experiments although it was possible to plausibly infer the existence of the $\mathcal{E}<0$ limit cycle. It is natural to conjecture that this implies that only the $\mathcal{E}>0$ limit cycle is stable in the sense that solutions at some time "near" the $\mathcal{E}>0$ limit cycle will asymptotically converge to it but that solutions "near" the $\mathcal{E}<0$ limit cycle will diverge from it tending eventually either to the steady wave regime or the stable $\mathcal{E}>0$ limit cycle. This conjecture is strengthened by our observation in L. C. that the $\mathcal{E}<0$ limit cycle appeared to be possible only when the steady wave state is stable, a situation which provides an asymptotically attainable state for the solution trajectory in phase space that starts near the $\mathcal{E}<0$ limit cycle.

The present study takes as its starting point Eqs. (2.1a) and (2.1b) of Section 2 which describe the interaction of the wave and the mean flow, and the continuous alteration of the mean flow by the nonlinear

[1] This research was supported by the National Science Foundation under Grant GA-11951.

[2] Hereafter refered to as L. C.

[3] Please see Eq. (3.7).

self-interaction of the wave. For small values of the viscosity the form of the amplitude vacillation is shown to be *almost* completely determined from inviscid theory, but the complete description and determination of the vacillation is obtained only by the consideration of certain integral constraints which are viscous in character. Nonetheless, it is remarkable but true that the constraints are themselves independent of the magnitude of the viscosity as long as it is small but nonzero so that the complete structure of the vacillation is viscosity-independent. The theory shows that the $\mathcal{E}>0$ limit cycle (the one with the periodically disappearing wave amplitude) is possible and stable for all linearly unstable waves, but that the $\mathcal{E}<0$ limit cycle is possible only when the steady wave regime is stable (and this depends only on the horizontal aspect ratio of the wave) and when the $\mathcal{E}<0$ limit cycle is possible it is nonetheless unstable.

Thus, from the theory presented below we can infer that when the steady regime is possible amplitude vacillation is also possible, at least for small enough viscosity, and it is of the $\mathcal{E}>0$ stable limit cycle form. However, a large disturbance of the steady wave state is then required to push the system into amplitude vacillation. When the steady wave state is unstable the $\mathcal{E}>0$ limit cycle is the only possible asymptotic state.

It is important to recall that the present theory suffers the defect that it is valid only in the parameter range for which, according to linear theory, the wave is only slightly unstable. Furthermore the viscous spin-up time[4] must be long compared to the e-folding time for unstable growth given by linear theory. On the other hand, the present theory is deductive and does not require the use of either ad hoc truncated series or extensive numerical analysis, which for me at any rate often obscure the essentials of the theory.

2. The amplitude equation

In L. C. the consideration of the dynamics of a slightly unstable wave in a baroclinic zonal flow led to the following equations for the wave amplitude $A(T)$ and the correction to the zonal flow $\Phi^{(2)}(y,T)$, *viz*:

$$\frac{d^2A}{dT^2}+\frac{3}{2}\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{dA}{dT}-\frac{\Delta}{|\Delta|}\frac{k^2(U_1-U_2)^2}{4a^2}A+\frac{Ak^2(U_1-U_2)}{2a^2}m\pi\int_0^1\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}\sin 2m\pi y\,dy=0, \quad (2.1a)$$

$$\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}-a^2\Phi_1^{(2)}\right]+\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{\partial^2\Phi_1^{(2)}}{\partial y^2}=\frac{a^2m\pi}{2(U_1-U_2)}\left[\frac{d|A|^2}{dT}+\frac{2r}{|\Delta|^{\frac{1}{2}}}|A|^2\right]\sin 2m\pi y. \quad (2.1b)$$

[4] See Section 5.

The reader is referred to L. C. for a detailed discussion of the analysis which results in (2.1a) and (2.1b). For our present purposes it is only necessary to recall that A is the nondimensional amplitude of a quasi-geostrophic wave which perturbs a uniform zonal flow in a two-layer model. The flow is confined to an infinitely long channel on the f plane of width L, while the thickness on each layer of fluid in the absence of any motion in a coordinate frame rotating with angular velocity Ω is $D/2$. The fluid in each layer has constant density, that in the upper fluid being ρ_1, the lower ρ_2 ($\rho_2>\rho_1$). Initially, the nondimensional zonal flow (in the x direction) in the upper layer is U_1 and U_2 in the lower. In each layer the lowest order nondimensional quasi-geostrophic wave field has the form

$$\left.\begin{aligned}\psi&=\mathrm{Re}\,Ae^{ik(x-ct)}\sin(m\pi y)\\ c&=(U_1+U_2)/2\end{aligned}\right\}. \quad (2.2)$$

The parameter which measures the stability of the basic flow is

$$F=[4\Omega^2L^2]\Big/\left[g\frac{(\rho_2-\rho_1)}{\rho_2}\frac{D}{2}\right], \quad (2.3)$$

and is chosen to exceed its critical value F_c by a small amount Δ. In L. C. the magnitude of the frictional effects (provided by Ekman layers on the bounding horizontal surfaces) were taken to be small. The effect of viscosity is measured by the parameter

$$r=\frac{2(\nu\Omega)^{\frac{1}{2}}}{D}\frac{L}{U}, \quad (2.4)$$

where ν is the kinematic viscosity of the fluid and U a characteristic horizontal velocity. The parameter r was chosen to be $O(\Delta^{\frac{1}{2}})$ and a straightforward multiple time-scale analysis of the resulting nonlinear stability problem to (2.1a, b). For

$$\frac{r}{|\Delta|^{\frac{1}{2}}}\leqslant O(1),\quad F_C=\frac{a^2}{2}\equiv(k^2+m^2\pi^2)/2.$$

The wave amplitude A is a function of the "slow time" variable

$$T=|\Delta|^{\frac{1}{2}}t. \quad (2.5)$$

The proper boundary condition for $\Phi_1^{(2)}$ was shown in L. C. to be

$$\frac{\partial^2\Phi_1^{(2)}}{\partial y\partial T}=0,\quad y=0,\,1. \quad (2.6)$$

We note in passing that the linear part of (2.1a) [i.e., the first three terms on the left-hand side of (2.1a)] yields the exponential growth predicted by linear stability theory, while the final term in (2.1a) represents the alteration of the wave amplitude evolution due to the interaction of the wave with the changing mean flow. In the same way (2.1b) relates the change of the mean zonal flow to the amplitude of the baroclinic wave. Indeed, (2.1b) is nothing more than the x-independent form of the potential vorticity equation. The first term in (2.1b) is the slow rate of change of the mean potential vorticity and this is produced by the frictional dissipation of potential vorticity via the Ekman-layer pumping [the second term in (2.1b)] and the mean transport of potential vorticity by the wave [the right-hand side of (2.1b)].

As discussed in L. C. the only apparently feasible way of making progress in the solution of (2.1a, b) was to relax (2.6) and accept spatially periodic solutions for $\Phi_1^{(2)}$ of the form

$$\Phi_1^{(2)}=B(T)\sin(2m\pi y), \tag{2.7}$$

$$B(T)=\frac{m\pi a^2}{2(U_1-U_2)(4m^2\pi^2+a^2)}\Big[|A|^2-|A(0)|^2e^{-\sigma\eta T}+\beta\sigma\eta\int_0^T|A(T')|^2e^{-\sigma\eta(T-T')}dT'\Big], \tag{2.8}$$

where

$$\sigma^2=\frac{\Delta}{|\Delta|}\frac{k^2(U_1-U_2)^2}{4a^2} \tag{2.9}$$

is chosen to be positive, while

$$\left.\begin{aligned}\eta&=\left(\frac{r}{\sigma|\Delta|^{\frac12}}\right)\left(\frac{4m^2\pi^2}{4m^2\pi^2+a^2}\right)\\ \beta&=(k^2+3m^2\pi^2)/(2m^2\pi^2)\end{aligned}\right\}. \tag{2.10}$$

If we let

$$\left.\begin{aligned}R&=[Akm^2\pi^2]/[\sigma(2a^2+8m^2\pi^2)^{\frac12}]\\ \theta&=\sigma T\end{aligned}\right\}, \tag{2.11}$$

we are led directly to the governing equations for the development of the wave-amplitude, *viz*:

$$\frac{d^2R}{d\theta^2}+\alpha\eta\frac{dR}{d\theta}-R+R[R^2-D]=0, \tag{2.12a}$$

$$\frac{dD}{d\theta}+\eta D+\beta\eta R^2=0, \tag{2.12b}$$

where

$$\left.\begin{aligned}D&=R^2(0)e^{-\eta\theta}-\beta\eta\int_0^\theta R^2(\theta')e^{-\eta(\theta-\theta')}d\theta'\\ \alpha&=\tfrac{3}{8}(k^2+5m^2\pi^2)/(m^2\pi^2)\end{aligned}\right\}. \tag{2.13}$$

We have also restricted our attention to cases where R is real. Note that the evolution of the amplitude depends on two parameters only; i.e., on η which is a measure of the degree of viscous dissipation and $k^2/(m^2\pi^2)$ which measures the aspect ratio of the wave. The parameters α and β are related by

$$\alpha=\tfrac{3}{4}(\beta+1). \tag{2.14}$$

Steady solutions of (2.12a, b) (which represent steady wave regimes), i.e.,

$$\left.\begin{aligned}R^{(s)}&=\pm(1+\beta)^{-\frac12}\\ D^{(s)}&=-\beta/(1+\beta)\end{aligned}\right\} \tag{2.15}$$

are always possible, but they are unstable if

$$\eta^2<\frac{2(\beta-\alpha)}{\alpha(\alpha+1)(\beta+1)}=\frac{1}{2}\frac{(\alpha-3)}{\alpha^2(\alpha+1)}. \tag{2.16}$$

Instability thus requires $\beta>\alpha$. In the limit of small η the steady wave solutions will be unstable whenever $\alpha/\beta<1$.

The computer calculations presented in L. C. implied that when the equilibrium points which represent (2.15) in the three-dimensional $R,(dR/d\theta),D$ phase space are unstable, the trajectory of the solution of (2.12a, b) enters a limit cycle. Such a situation is shown in Fig. 1. For small η, but with $\alpha/\beta>1$, the equilibrium points are stable but nonetheless a similar limit cycle behavior is possible if the initial conditions are sufficiently distant from the equilibrium points. Furthermore, when $\alpha/\beta>1$ a second limit cycle appears which seems to be a closed trajectory on a bounding surface about the equilibrium point such that when $\alpha/\beta>1$ trajectories within this surface wind inward and end as steady solutions, while initially more distant trajectories wind outward and end on a limit cycle similar to the one that appears when $\alpha/\beta<1$. For $\eta=O(1)$ no limit cycles were found and the steady solutions were stable.

In short, the calculations in L. C. indicated the possibility of two limit cycles. For small η one of these existed for all values of α/β. The main qualitative features of this limit cycle are 1) it seems to be a stable limit cycle and 2) during the limit cycle (as can be seen in Fig. 1) the amplitude oscillates in sign. The second limit cycle was found on the analog only when the steady solutions were stable ($\alpha/\beta>1$) and it characteristically had an amplitude R of a single sign (Fig. 5 of L. C.).

3. The limit cycles

In the analog experiments the limit cycles appear only when η is small. We will therefore present an asymptotic theory for the limit cycle for $\eta\ll1$. Eqs. (2.12a, b) may be written as

$$\begin{aligned}\frac{d}{d\theta}\left[\frac12\left(\frac{dR}{d\theta}\right)^2-\frac{R^2}{2}+\frac{R^4}{4}-\frac{R^2D}{2}\right]&=-\frac{R^2}{2}\frac{dD}{d\theta}-\alpha\eta\left(\frac{dR}{d\theta}\right)^2\\ &=\eta\left[\frac{R^2}{2}D+\beta\frac{R^4}{2}-\alpha\left(\frac{dR}{d\theta}\right)^2\right],\end{aligned} \tag{3.1}$$

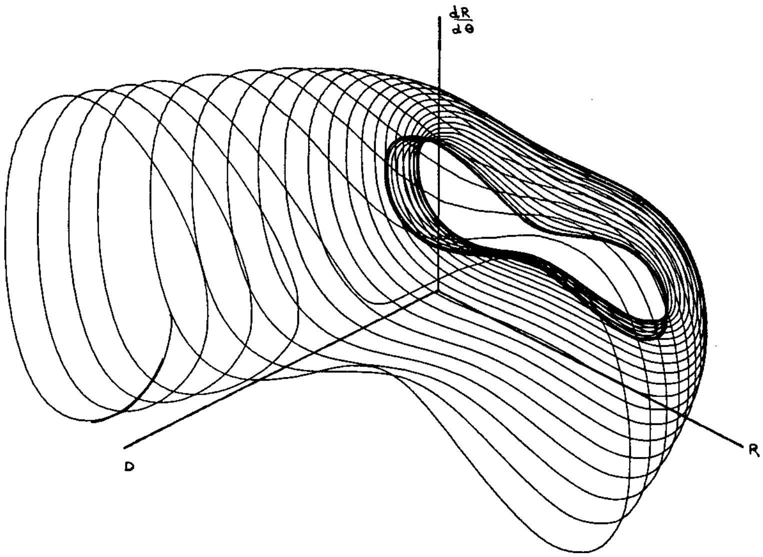

FIG. 1. A perspective view in the R, $dR/d\theta$, D phase space showing the gradual drift of the solution trajectory into a limit cycle. The limit cycle is distinguished by the heavy black line.

where we have used the relation

$$\frac{dD}{d\theta}=-\eta[D+\beta R^2]. \tag{3.2}$$

If there are closed trajectories (limit cycles) in the $R, dR/d\theta, D$ phase space, then integrals of the form

$$\frac{1}{T_p}\int_0^{T_p}\frac{dH}{d\theta}d\theta$$

must vanish if T_p is the time required to traverse the closed trajectory. If we integrate (3.1) and (3.2) around such a hypothesized trajectory we obtain, for $\eta\neq 0$,

$$\overline{R^2D}+\beta\overline{R^4}=2\alpha\overline{\left(\frac{dR}{d\theta}\right)^2}, \tag{3.3}$$

$$\overline{D}=-\beta\overline{R^2}, \tag{3.4}$$

where an overbar defines the integral

$$\overline{(\ )}=\frac{1}{T_p}\int_0^{T_p}(\ )d\theta. \tag{3.5}$$

The constraints (3.3) and (3.4) hold for all η. For small η, R and D can be expanded in an asymptotic series

$$\left.\begin{aligned}R&=R_0+\eta R_1+\cdots\\ D&=D_0+\eta D_1+\cdots\end{aligned}\right\}. \tag{3.6}$$

In general, for small η, R and D will depend on two time variables, θ and $\eta\theta$. The first represents the time behavior during one (open or closed) oscillation in the phase space while the dependence on $\eta\theta$ represents the drift of the trajectory itself in phase space. We will return to this point later. For the limit cycle, however, the trajectory in phase space is fixed; consequently, when in the limit cycle the solution depends only on θ. Therefore, for small η the first approximation to (3.1) and (3.2) yields

$$\frac{1}{2}\left(\frac{dR_0}{d\theta}\right)^2-\frac{R_0^2}{2}+\frac{R_0^4}{4}-\frac{R_0^2D_0}{2}=\mathcal{E}, \tag{3.7}$$

$$\frac{dD_0}{d\theta}=0, \tag{3.8}$$

Hence, D_0 and $\mathcal{E}$ are constants. The problem for R_0 is identical to the problem of the amplitude oscillation of an *inviscid* baroclinic wave (Pedlosky, 1970) with one crucial difference. If the oscillation were truly nondissipative, D_0 and $\mathcal{E}$ would be determined by the initial conditions [i.e., the values of $R(0)$ and $dR/d\theta(0)$]. In the case at hand the initial conditions have long been forgotten and $\mathcal{E}$ and D_0 are to be considered as unknowns. They are determined from the viscous constraints (3.3) and (3.4). Since D_0 is constant for $\eta\ll 1$, we use (3.4) to obtain

$$D_0=-\beta\overline{R_0^2}, \tag{3.9}$$

$$\overline{R_0^4}-(\overline{R_0^2})^2=2\frac{\alpha}{\beta}\overline{\left(\frac{dR_0}{d\theta}\right)^2}. \tag{3.10}$$

Since R_0 is a function of D_0 and $\mathcal{E}$, (3.9) and (3.10) can be considered as two equations for D_0 and $\mathcal{E}$ whose solutions will completely determine the nature of the limit cycle oscillations (if any) deduced from (3.7).

It is convenient at this stage to consider the possibilities $\mathcal{E} \gtrless 0$ separately. If $\mathcal{E}>0$ it is a simple matter to show that the solution of (3.7) is the elliptic cosine, i.e., that

$$R_0(\theta)=R_m \operatorname{cn}\left(\frac{R_m}{\sqrt{2}n}\theta,\kappa\right), \tag{3.11}$$

where the constant, maximum amplitude of the oscillation is

$$R_m=\{(1+D_0)+[(1+D_0)^2+4\mathcal{E}]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{3.12}$$

and where the modulus κ of the elliptic function is given by

$$\kappa^2=\frac{(1+D_0)+[(1+D_0)^2+4\mathcal{E}]^{\frac{1}{2}}}{2[(1+D_0)^2+4\mathcal{E}]^{\frac{1}{2}}}. \tag{3.13}$$

The period of the pulsation is

$$T_p=\sqrt{2}4K(\kappa)/R_m, \tag{3.14}$$

where $K(\kappa)$ is the complete elliptic integral

$$K(\kappa)=\int_0^1 \frac{dx}{[(1-x^2)(1-\kappa^2x^2)]^{\frac{1}{2}}}. \tag{3.15}$$

The oscillation given by (3.11) yields the same type of phase space trajectory as the limit cycle shown in Fig. 1. As we shall see later the oscillation with $\mathcal{E}>0$ can be identified with the *stable* limit cycle.

On the other hand, if $\mathcal{E}<0$ the oscillation has a completely different form, *viz*:

$$R_0=R_{\max} dn\left(\frac{R_{\max}\theta}{\sqrt{2}},\kappa\right), \tag{3.16}$$

where again the maximum amplitude is given by

$$R_{\max}=\{(1+D_0)+[(1+D_0)^2+4\mathcal{E}]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{3.17}$$

but where, however, the *form* of the oscillation is quite different. The function dn is defined by

$$dn^{-1}(x,\kappa)\equiv\int_x^1 \frac{dx'}{[(1-x'^2)(x'^2-1+\kappa^2)]^{\frac{1}{2}}}, \tag{3.18}$$

and has the property that it is of a single sign during an oscillation. The amplitude, for $\mathcal{E}<0$, has a *minimum* value given by

$$R_{\min}=\{(1+D_0)-[(1+D_0)^2+4\mathcal{E}]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{3.19}$$

$$\kappa^2=(R_{\max}{}^2-R_{\min}{}^2)/R_{\max}{}^2. \tag{3.20}$$

The oscillation given by (3.16) has the same form as the unstable limit cycle found in L. C.; we will see later that such an identification is a valid one.

In reality, of course, the form of the solution for both $\mathcal{E} \gtrless 0$ depends on the value of κ and R_m. Indeed, rather than $\mathcal{E}$ and D_0, it is more convenient to consider R_m and κ (or for $\mathcal{E}<0$, $R_{\max}$ and κ) as the two elements of the oscillation required to complete the solution which is to be determined from (3.9) and (3.10). Furthermore, it is not clear at this stage whether the oscillations given by (3.11) and (3.16) are possible, or, more precisely, for what values of $k/(m\pi)$ (or equivalently α/β) the solutions (3.11) and (3.16) will satisfy (3.9) and (3.10).

To answer these questions we substitute the oscillatory solutions into (3.9) and (3.10). In each case certain definite integrals involving integrands which are powers of elliptic functions are required. In all cases the integrals are tabulated and can be found in Byrd and Friedman (1954). The details are straightforward. For the case $\mathcal{E}>0$ the resulting transcendental equation for κ determined directly from (3.10) is of the form

$$\frac{\alpha}{\beta}=\left\{(2-3\kappa^2)(1-\kappa^2)+2(2\kappa^2-1)\frac{E(\kappa)}{K(\kappa)} -3\left[\frac{E(\kappa)}{K(\kappa)}-(1-\kappa^2)\right]^2\right\}\Bigg/ \left[(2\kappa^2-1)\frac{E(\kappa)}{K(\kappa)}+(1-\kappa^2)\right], \tag{3.21}$$

where $E(\kappa)$ is the complete elliptic integral of the second kind, i.e.,

$$E(\kappa)=\int_0^{\pi/2}(1-\kappa^2\sin^2\theta)^{\frac{1}{2}}d\theta. \tag{3.22}$$

Both $E(\kappa)$ and $K(\kappa)$ are tabulated functions of the modulus κ, which ranges in value from zero to one. Hence, it is an easy matter to solve (3.21). However, rather than solving for κ as a function of α/β, we determine what value of α/β corresponds to a given value of κ in the range $0\leqslant\kappa\leqslant1$. Now in terms of $k^2/(m^2\pi^2)$, we have

$$\frac{\alpha}{\beta}=\tfrac{3}{4}[(k/m\pi)^2+5]/[(k/m\pi)^2+3]. \tag{3.23}$$

Hence, physically meaningful values of α/β lie in the range

$$0.75\leqslant\alpha/\beta\leqslant1.15. \tag{3.24}$$

Thus, acceptable values of κ are only those which yield values of α/β in this range. Table 1 gives the result of a simple numerical evaluation of (3.21). Thus, for $\mathcal{E}>0$ physically acceptable solutions are valid for *all* values of α/β in the range given by (3.24). It is interesting to note that the corresponding values of κ all lie in the range $0.928<\kappa^2$. In particular, this includes the values $1<\alpha/\beta\leqslant1.25$ for which the equilibrium solutions (2.15) are *stable*. We inferred in L. C. from computer calculations that for small η a limit cycle could be present even if the steady wave solutions are

TABLE 1. Modulus of the solution as a function of α/β.

κ^2	α/β
0.928	0.7472217
0.929	0.750844
0.93	0.754507
0.94	0.7935515
0.95	0.837873
0.96	0.88927856
0.97	0.95086432
0.98	1.02884461
0.99	1.14020397
0.992	1.17085253
0.994	1.20712501
0.995	1.22838801
0.99589	1.24988581
0.9959	1.25014457

stable to infinitesimal disturbances; the above results confirm those speculations. From (3.9) and the relation

$$R_m^2(1-\tfrac{1}{2}\kappa^2)=1+D_0, \tag{3.25}$$

we have

$$R_m^2=\left\{1-\frac{1}{2\kappa^2}+\frac{\beta}{\kappa^2}\left[\frac{E(\kappa)}{K(\kappa)}-1+\kappa^2\right]\right\}^{-1}, \tag{3.26}$$

which, with (3.21), yields the amplitude of the oscillation as a function of wavenumber (i.e., α/β). The relationship is shown in Fig. 2. Note that no solution is possible outside the range $0.75\leqslant\alpha/\beta\leqslant 1.25$. Further, the amplitude of the limit cycle pulsation increases as α/β increases [or as $(k/m\pi)^2$ decreases]. This, heuristically, can be explained as follows. As α/β increases, β decreases, and hence $R^{(s)}$ as given by (2.15) increases. The limit cycle, as was shown in L. C., always encompasses the equilibrium point in the R, $dR/d\theta$ plane, and hence as $R^{(s)}$ increases we expect the amplitude of the corresponding limit cycle to increase. The interesting thing to note is that the amplitude of the $\mathcal{E}>0$ limit cycle is largest for those values of α/β for which the steady wave solutions are stable to *infinitesimal* disturbances. Again this conclusion is plausible for we found in L. C. that when the equilibrium solutions were

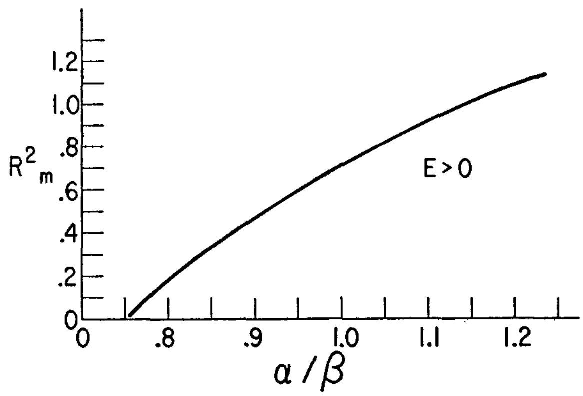

FIG. 2. Maximum amplitude in the $\mathcal{E}>0$ limit cycle as a function of α/β.

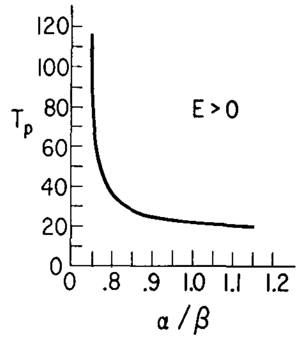

FIG. 3. The period of the $\mathcal{E}>0$ limit cycle.

stable only trajectories which were initially distant from the equilibrium points in the solution phase space end on the $\mathcal{E}>0$ limit cycle. The period of the limit cycle pulsation is shown in Fig. 3. Note that the shorter periods of pulsation corresponds to larger pulsation amplitudes.

For $\mathcal{E}<0$, the analysis is very similar. The integral constraint (3.10) now yields, as the equation for κ^2,

$$\frac{\alpha}{\beta}=\frac{\{[2(2-\kappa^2)E(\kappa)/K(\kappa)]-1+\kappa^2-3[E(\kappa)/K(\kappa)]^2\}}{(2-\kappa^2)[E(\kappa)/K(\kappa)]-2(1-\kappa^2)}. \tag{3.27}$$

The corresponding α/β, κ relation is shown in Table 2. Thus, almost all values of κ are possible but they correspond to the *restricted* range for α/β

$$1\leqslant\alpha/\beta\leqslant 1.25.$$

The $\mathcal{E}<0$ limit cycle is possible only for those values of $k/(m\pi)$ for which the steady wave solutions are *stable*. This was expected since we have tentatively identified the $\mathcal{E}<0$ limit cycle with the unstable limit cycle inferred from the analog calculations and the unstable limit cycle appears only when the equilibrium points are stable. Hence the asymptotic theory reproduces the inferred parametric dependence for the existence of each of the two possible limit cycles. The maximum and minimum amplitude for the $\mathcal{E}>0$ limit cycle as functions of α/β are shown in Fig. 4. Note that the unstable limit cycle shrinks to a point when $\alpha/\beta=1$. For that value of α/β (for which $\beta=\alpha=3$) the calculations show that $R_{\max}^2=R_{\min}^2=0.25$. From (2.15) we note that

$$(R^{(s)})^2\phi=1/(1+\beta),$$

which, for $\beta=3$, yields $(R^{(s)})^2=0.25$. Hence, the $\mathcal{E}<0$ limit cycle collapses onto the point representing the equilibrium solution as that point undergoes the transition from an unstable to stable point.

In the next section we verify our supposition that the $\mathcal{E}>0$ limit cycle is stable and the $\mathcal{E}<0$ limit cycle is unstable.

4. The stability of the limit cycles

We mentioned in passing in Section 3 that when η is small R and D are functions of the two time variables θ and $\eta\theta$. The dependence on the first variable describes the oscillation in phase space on a closed or nearly closed trajectory while the second variable characterizes the drift of the trajectory either to or from a limit cycle or into the steady solution. In determining the existence and character of the limit cycles we could consider solutions which were *steady* as far as the second variable is concerned. Obviously, to discuss the stability of the limit cycles a more general formulation is required which allows for the dependence of the solutions on the second time scale. Thus, for $\eta \ll 1$ we consider R and D to have the form

$$\left.\begin{aligned} R&=R(\theta_*,\tau)\\ D&=D(\theta_*,\tau)\end{aligned}\right\}, \tag{4.1}$$

where

$$\tau=\eta\theta$$

is the new long time variable.[5] It is essential in the following analysis that a new time variable θ_* be chosen to replace θ. The variable θ_* is of the same order in η as θ. We must choose θ_* in such a way that the period of oscillation of R as a function of θ_* be independent of τ. This is a familiar requirement in such multiple time-scale problems in order to avoid spurious secularities in the solution. The reader is referred to Cole (1968) for a more detailed discussion of this point. A relation between θ_* and θ is postulated to have the form

$$\frac{\partial\theta_*}{\partial\theta}=f(\tau), \tag{4.2}$$

where $f(\tau)$ is the function to be determined by the requirement discussed above.

The amplitude equations (2.12a, b) then become

$$\left.\begin{aligned} &f^2\frac{\partial^2 R}{\partial\theta_*^2}-R+R[R^2-D]\\ &\quad=-\eta\left[\frac{df}{d\tau}\frac{\partial R}{\partial\theta_*}-2f\frac{\partial^2 R}{\partial\tau\partial\theta_*}-\alpha f\frac{\partial R}{\partial\theta_*}\right]\\ &\qquad\qquad-\eta^2\left[\frac{\partial^2 R}{\partial\tau^2}+\alpha\frac{dR}{d\tau}\right]\\ &f\frac{\partial D}{\partial\theta_*}=-\eta\left[\frac{\partial D}{\partial\tau}+D+\beta R^2\right]\end{aligned}\right\}. \tag{4.4}$$

As in Section 3 an asymptotic representation for R and D will be sought in the form

$$\left.\begin{aligned} R&=R_0(\theta_*,\tau)+\eta R_1(\theta_*,\tau)+\cdots\\ D&=D_0(\theta_*,\tau)+\eta D_1(\theta_*,\tau)+\cdots\end{aligned}\right\}. \tag{4.5}$$

Inserting (4.5) in (4.3) and (4.4), and equating like powers of η, yields the following problem sequence. The O(1) problem is

$$\frac{\partial}{\partial\theta_*}\left[\frac{f^2}{2}\left(\frac{\partial R_0}{\partial\theta_*}\right)^2-\frac{R_0^2}{2}+\frac{R_0^4}{4}-D_0\frac{R_0^2}{2}\right]=0, \tag{4.6}$$

$$\frac{\partial}{\partial\theta_*}D_0=0. \tag{4.7}$$

The O(η) problem yields

$$-f\frac{\partial D_1}{\partial\theta_*}=\frac{\partial D_0}{\partial\tau}+D_0+\beta R_0^2, \tag{4.8}$$

$$f^2\frac{\partial^2 R_1}{\partial\theta_*^2}-R_1+3R_0^2R_1-R_0D_1-R_1D_0$$
$$=-\frac{df}{d\tau}\frac{\partial R_0}{\partial\theta_*}-2f\frac{\partial^2 R_0}{\partial\tau\partial\theta_*}-\alpha f\frac{\partial R_0}{\partial\theta_*}. \tag{4.9}$$

[5] The reader will recall that the amplitude equations (2.12a, b) were derived from a two-time scale analysis. In that context τ should perhaps be called a "super long" time scale.

TABLE 2. Modulus of the solution with $\mathcal{E}<0$ as a function of α/β.

κ^2	α/β
0.15	1.00081041
0.25	1.00129164
0.35	1.00291142
0.45	1.0055481
0.55	1.00985649
0.65	1.01688288
0.75	1.02900924
0.85	1.0527437
0.95	1.12017668
0.99	1.2364896

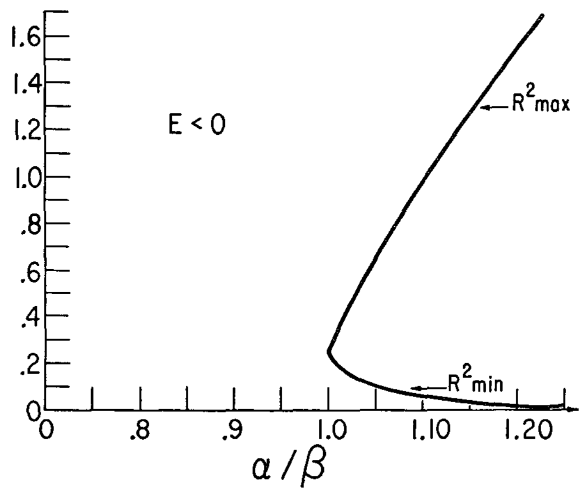

FIG. 4. The maximum and minimum amplitudes of the $\mathcal{E}<0$ limit cycle.

Thus, (4.7) informs us that D_0 is a function of τ only, i.e., it is a slowly varying function, while the integral of (4.6) yields

$$\frac{f^2}{2}\left(\frac{\partial R_0}{\partial\theta_*}\right)^2-\frac{R_0^2}{2}+\frac{R_0^4}{4}-D_0\frac{R_0^2}{2}=\mathcal{E}(\tau), \qquad (4.10)$$

where $\mathcal{E}(\tau)$ is to be determined. The solutions of (4.10) will yield oscillatory solutions identical in form to (3.11) and (3.16). These solutions will be periodic in θ_* but not in θ, unless, of course, f is a constant in which case we are in the limit cycle. Let T_p be the unknown period of the oscillation in θ_*. The following feature is then of great importance. Since D_0 is independent of θ_* and R_0 is a periodic function of θ_*, we note from (4.8) that D_1 will increase linearly with θ_*, thus eventually rendering our asymptotic expansion (4.5) invalid for time $\eta\theta_*\equiv\tau$ of O(1). We obviously must retain the validity of our expansion at least this long to investigate the behavior of the drift to or from the limit cycle. *We can obviate this difficulty if we insist that the integral over the period T_p, i.e., the right-hand side of (4.8), vanishes.* This yields the *secularity condition* (where again an overbar defines an integral mean over an oscillation period)

$$\left[\frac{dD_0}{d\tau}+D_0=-\beta\overline{R_0^2}\right]. \qquad (4.11)$$

Note that the steady form of (4.11) is identical to (3.9). Combining (4.11) with (4.8) yields

$$f\frac{\partial D_1}{\partial\theta_*}=\beta(\overline{R_0^2}-R_0^2). \qquad (4.12)$$

Similarly by letting

$$R_1=G(\theta_*,\tau)R_0(\theta_*,\tau)$$

and substituting into (4.9), we find that

$$f^2\left(\frac{\partial R_0}{\partial\theta_*}\right)^2\frac{\partial G}{\partial\theta_*}=\int_0^{\theta_*}\left\{D_1\frac{\partial}{\partial\theta_*'}\frac{R_0^2}{2}-\frac{\partial}{\partial\tau}\left[f\left(\frac{\partial R_0}{\partial\theta_*'}\right)^2\right]-\alpha f\left(\frac{\partial R_0}{\partial\theta_*'}\right)^2\right\}d\theta_*'. \qquad (4.13)$$

If, over a period of the oscillation, R_1 is not to increase secularly then G must also be a periodic function of θ^* with period T_p, which in turn requires that $(\partial G/\partial\theta_*)(T_p)=0$ or

$$\int_0^{T_p}\left\{D_1\frac{\partial R_0^2/2}{\partial\theta_*}-\frac{\partial}{\partial\tau}\left[f\left(\frac{\partial R_0}{\partial\theta_*}\right)^2\right]-\alpha f\left(\frac{\partial R_0}{\partial\theta_*}\right)^2\right]d\theta_*=0. \qquad (4.15)$$

If we integrate the first term in (4.15) by parts and use (4.12), we obtain

$$f\frac{\partial}{\partial\tau}f\overline{\left(\frac{\partial R_0}{\partial\theta_*}\right)^2}+\alpha f^2\overline{\left(\frac{\partial R_0}{\partial\theta_*}\right)^2}-\frac{\beta}{2}[\overline{R_0^4}-(\overline{R_0^2})^2]=0. \qquad (4.16)$$

Now, if R_0 is not a function of τ, i.e., if the solutions are functions of only θ_*, as in the limit cycles, $\theta=\theta_*$ and (4.16) becomes identical to (3.10). Hence, the steady solutions of (4.11) and (4.16) determine the limit cycles. That task is done. The solutions of (4.11) and (4.16) which are time-dependent, i.e., dependent on τ, will describe the slow drift of the solution in the phase space. More precisely, (4.11) and (4.16) may be considered as ordinary differential equations for $\mathcal{E}(\tau)$ and $D_0(\tau)$.

We turn first to the solution of (4.10). For $\mathcal{E}(\tau)>0$

$$R_0(\theta_*,\tau)=R_m(\tau)\ \mathrm{cn}\{\omega(\tau)\theta_*,\kappa(\tau)\}, \qquad (4.17)$$

where

$$\omega(\tau)=R_m(\tau)/[\sqrt{2}\kappa(\tau)f(\tau)], \qquad (4.18)$$

where $R_m(\tau)$ and $\kappa(\tau)$ are again determined by (3.12) and (3.13), but since $\mathcal{E}$ and D_0 are functions of τ so are R_m and κ. We must insist that the period of R_0 as a function of θ_* be independent of τ. The period in θ_* must be a constant and with no loss of generality we can choose that constant to be unity. Any other choice of constant would simply enter as a constant multiple of f and not affect the behavior of the amplitude as a function of θ. This requires that

$$f(\tau)=\frac{d\theta_*}{d\theta}=R_m(\tau)/\{4\sqrt{2}K[\kappa(\tau)]\kappa(\tau)\}, \qquad (4.19)$$

so that

$$R_0(\theta_*,\tau)=R_m(\tau)\ \mathrm{cn}[4K(\tau)\theta_*,\kappa(\tau)]. \qquad (4.20)$$

Substitution of (4.20) and (4.19) into (4.11) and (4.16) yields two ordinary differential equations for R_m and κ whose determination is equivalent to (but more useful than) the determination of $\mathcal{E}$ and D_0. After much reduction we obtain the coupled nonlinear ordinary differential equations

$$\frac{n^3}{KR_m^3}\frac{d}{d\tau}\left\{\frac{KR_m^3}{\kappa^3}\left[(2\kappa^2-1)\frac{E}{K}+1-\kappa^2\right]\right\}+\alpha\left\{(2\kappa^2-1)\frac{E}{K}+1-\kappa^2\right\}$$
$$=\beta\left[(2-3\kappa^2)(1-\kappa^2)+2(2\kappa^2-1)\frac{E}{K}-3\left(\frac{E}{K}-1+\kappa^2\right)^2\right], \qquad (4.21)$$

$$\frac{d}{d\tau}\left[R_m^2(1-\tfrac{1}{2}\kappa^2)\right]$$

$$+R_m^2\left[(1-\tfrac{1}{2}\kappa^2)+\frac{\beta}{\kappa^2}\left(\frac{E}{K}-1+\kappa^2\right)\right]=1, \qquad (4.22)$$

the steady solutions of which yield (3.21) and (3.26). A general solution of (4.21) and (4.22) for R_m and κ is out of the question. We need only note the relationship between the functions K and E to κ to recognize the impossibility of that task. On the other hand it is relatively easy to perturb the steady solutions of (4.21) and (4.22) to determine the stability of the limit cycle. That is, if we let

$$\left.\begin{aligned}R_m-R_{ms}&=V(\tau)R_{ms}\\ \kappa-\kappa_s&=\nu(\tau)\kappa_s\end{aligned}\right\}, \qquad (4.23)$$

where R_{ms} and κ_s are the values of R_m and κ in the limit cycle, we obtain two linear ordinary differential equations for $V(\tau)$ and $\nu(\tau)$ if the perturbations are small. After a great deal of manipulation, the equations for ν and V become

$$a_1\frac{dV}{d\tau}+a_2\frac{d\nu}{d\tau}+a_3\nu=0, \qquad (4.24)$$

$$b_1\frac{dV}{d\tau}+b_2\frac{d\nu}{d\tau}+b_3\nu+b_4V=0, \qquad (4.25)$$

where

$$a_1=3\left[\frac{(2\kappa_s^2-1)E(\kappa_s)}{K(\kappa_s)}+1-\kappa_s^2\right]$$

$$a_2=3\left[\frac{E(\kappa_s)}{K(\kappa_s)}-1\right]$$

$$a_3=\alpha\left[\left(2\frac{E(\kappa_s)}{K(\kappa_s)}-1\right)(4\kappa_s^2-1)+\frac{(1-2\kappa_s)^2}{(1-\kappa_s^2)}\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2\right]$$

$$-\beta\left[4(\kappa_s^2-1)+\frac{E(\kappa_s)}{K(\kappa_s)}(14-8\kappa_s^2)+\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2\right.$$

$$\left.\times\frac{(14\kappa_s^2-6)}{(1-\kappa_s^2)}+6\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^3\frac{1}{(1-\kappa_s^2)}\right]$$

$$b_1=1-\tfrac{1}{2}\kappa_s^2$$

$$b_2=\tfrac{1}{2}\kappa_s^2$$

$$b_3=\frac{1}{2\kappa_s^2}+\frac{\beta}{2\kappa_s^2}-\frac{\beta}{2\kappa_s^2}\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2\frac{1}{(1-\kappa_s^2)}$$

$$b_4=1/R_{ms}^2.$$

The stability of the limit cycle can be determined by seeking solutions of (4.24) and (4.25) in the form $e^{\lambda\tau}$. A quadratic equation for λ is then obtained whose solutions λ_1 and λ_2 are given by

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}=-\frac{(a_1b_3-a_3b_1-a_2b_4)}{2(a_1b_2-a_2b_1)}$$

$$(\pm)\frac{\left[(a_1b_3-a_2b_4-a_3b_1)^2+4a_3b_4(a_1b_2-a_2b_1)\right]^{\frac{1}{2}}}{2(a_1b_2-a_2b_1)}. \qquad (4.26)$$

If both λ_1 and λ_2 are negative the limit cycle is stable. Now the a_j and b_j are functions of κ_s and R_{ms} and hence functions of α/β. Table 3 contains the results of the calculation of λ_1 and λ_2. Thus, the $\mathcal{E}>0$ limit cycle is indeed stable as we expected, and this is true over the entire range α/β for which the limit cycle exists.

For $\mathcal{E}<0$ a similar analysis once again leads to a pair of linear differential equations governing the stability of the solution which now has the form of (3.16). The stability equations are again (4.24) and (4.25) where, however, for $\mathcal{E}<0$, we have

$$a_1=3\left[\frac{(2-\kappa_s^2)E(\kappa_s)}{K(\kappa_s)}+2(\kappa_s^2-1)\right]$$

$$a_2=3\kappa_s^2\left(\frac{1-E(\kappa_s)}{K(\kappa_s)}\right)$$

$$a_3=\alpha\left[4\frac{E(\kappa_s)}{K(\kappa_s)}(1-\kappa_s^2)\right.$$

$$\left.+\frac{(\kappa_s^2-2)}{(1-\kappa_s^2)}\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2+(5\kappa_s^2-2)\right]$$

$$-\beta\left[\frac{E(\kappa_s)}{K(\kappa_s)}(14-8\kappa_s^2)\right.$$

$$+\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2\frac{(14\kappa_s^2-16)}{1-\kappa_s^2}$$

$$\left.+4(\kappa_s^2-1)+6\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^3\Big/(1-\kappa_s^2)\right]$$

$$b_1=(1-\kappa_s^2/2)$$

$$b_2=-\kappa_s^2/2$$

$$b_3=-\frac{\kappa_s^2}{2}+\frac{\beta}{2}\left[2\frac{E(\kappa_s)}{K(\kappa_s)}\right.$$

$$\left.-1-\left(\frac{E(\kappa_s)}{K(\kappa_s)}\right)^2\frac{1}{(1-\kappa_s^2)}\right]$$

$$b_4=1/R_{ms}^2.$$

TABLE 3. Decay rates for the perturbations of the $\mathcal{E}>0$ limit cycle.

α/β	λ_1	λ_2
0.8378	−7.826	−71.99
0.8893	−5.131	−52.82
0.9508	−3.696	−44.49
1.0288	−2.85	−42.213
1.1402	−2.2161	−29.95

Again, the stability (or instability) of the limit cycle is determined from the solutions of (4.26). For $\mathcal{E}>0$, we find that the limit cycle is *unstable* as can be seen by the values of λ_2 in Table 4. Thus, the inferences drawn from the computer calculations are verified.

5. Conclusions

When the effect of friction is small, that is, when the spin down time for the system, $D/(\nu\Omega)^{\frac{1}{2}}$, is much longer than the characteristic *e*-folding time for unstable baroclinic development, two final asymptotic states may be achieved by a finite-amplitude baroclinic wave. Either the wave may eventually settle into a state with a fixed amplitude, or, if the dissipation is small enough, the wave amplitude may perpetually oscillate in a limit cycle. For small values of the friction parameter the solution will *always* tend to a limit cycle if the steady wave solution is unstable to infinitesimal disturbances. This is the $\mathcal{E}>0$ limit cycle calculated in Section 3. If the steady wave solution is stable to infinitesimal disturbances the solution will tend to the *steady* solution only if it is initially close to the steady wave solution; otherwise, it again will tend to the $\mathcal{E}>0$ limit cycle. A second limit cycle is theoretically possible[6] if $\alpha/\beta>1$ ($\mathcal{E}<0$, but as we have seen in Section 4 it is unstable and never realized. On the other hand the $\mathcal{E}>0$ limit cycle is stable and solutions initially (or at some time) close to it will converge to it. The results of these calculations indicate to me that, with small dissipation, long-term pulsations of baroclinic waves are a natural asymptotic state in the flow, even more so than a steady wave state. The steady wave state, although theoretically possible, always seems to be unstable to a *finite*-amplitude disturbance which will place it on a $\mathcal{E}>0$ limit cycle trajectory. The results of L. C. indicate that

TABLE 4. Decay and growth rates for the $\mathcal{E}<0$ limit cycle.

α/β	λ_1	λ_2
1.0081	−3.976	245.8
1.0029	−3.9322	55.98
1.0386	−3.172	4.59
1.23645	−0.418	39.69

[6] In terms of the horizontal aspect ratio of the wave we recall that this requires $k^2/(m^2\pi^2)>3$.

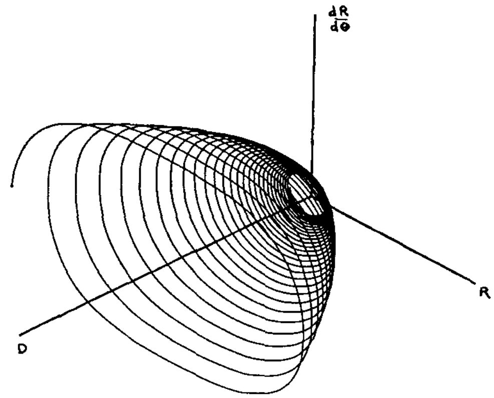

FIG. 5. A perspective view of the R, $dR/d\theta$, D phase space showing the gradual decay of the finite amplitude oscillation when the baroclinic flow is linearly stable.

only for $\eta=O(1)$, which is outside the scope of the theory presented here, will the steady wave regime be the "natural" one, i.e., only when the *e*-folding time for baroclinic development is as long as a spin-down time.

While I believe these results may well be of general validity, it is important to recall the simplicity and idealization of the basic model. Significant changes may well occur if, for example, the variation of the Coriolis parameter is included,[7] if more than one wave is included in the initial conditions, or if spatial modulation of R is allowed.

If the mean flow is linearly stable [$\sigma^2<0$ in (2.9)] the *only* equilibrium point in the solution phase space is the stable point at $R=dR/d\theta=D=0$. Computer calculations show that all trajectories spiral that point and no limit cycles are possible (Fig. 5).

Finally, because of the presence of dissipation in the wave, it is easy to show that although the average amplitude in the $\mathcal{E}>0$ limit cycle is zero the average potential energy conversion from the mean flow is not zero. Using Eqs. (A8) from Pedlosky (1970) and (4.8) from L. C., the transfer of energy from the mean flow to the wave (which is proportional to the eddy flux), integrated over a wavelength, is

$$E_{MW}=|\Delta|^{\frac{1}{2}}\left[\frac{1}{2}\frac{d}{d\theta}|A|^2+\frac{r}{|\Delta|^{\frac{1}{2}}}|A|^2\right]. \qquad (5.1)$$

Over one period of the limit cycle the average energy transfer is

$$\overline{E_{MW}}=r\overline{|A|^2}>0. \qquad (5.2)$$

Acknowledgments. The computer drawings shown in Figs. 1 and 5 were done in L. F. McGoldrick's laboratory

[7] Indeed Holopainen (1961) has shown for the linear problem that the presence of the beta effect has the singular effect of shifting the region of stability if only a minute amount of Ekman-layer friction is introduced.

at the University of Chicago. It was Prof. McGoldrick who suggested that the phase space trajectories could be presented in a three-dimensional perspective and he kindly gave his time to produce the drawings. His laboratory is partially supported by ONR Contract NOOO14-67-A-0285-0002.

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