Improved Curtis-Godson Approximation in a Non-Homogeneous Atmosphere[1]

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(Manuscript received 2 February 1972, in revised form 24 April 1972)

ABSTRACT

This paper presents a method of improving the Curtis-Godson approximation for computing transmission along a non-homogeneous path. The present approximation, as well as those by Curtis-Godson and by Goody, are applied to model atmospheres with typical distributions of three important absorbers, CO_2, H_2O and O_3. Error estimates for each are discussed.

1. Introduction

The transmission through a non-homogeneous atmosphere has been studied by Curtis (1952), Godson (1953), Goody (1964) and Armstrong (1968). Recently Yamamoto and Aida (1970; henceforth referred to as YA) have studied a restricted case, i.e., the transmission through a non-homogeneous atmosphere with an absorbing gas having a constant mixing ratio. In that study they introduced an auxiliary parameter relating to the optical path to increase the accuracy of evaluation. In the present work a method of improving the Curtis-Godson approximation is developed by introducing a similar parameter. From the standpoint of Goody's classification the present method belongs to a three-parameter approximation.

2. Present method

The Curtis-Godson approximation gives the exact result in two limits, i.e., the strong and weak limits of absorption. However, it yields an over-estimation of absorption in the intermediate range of optical paths. According to our interpretation, it seems that the equivalent half-width used in the Curtis-Godson approximation is too large to give the exact result for intermediate path lengths. Therefore, one way of improving the Curtis-Godson approximation is that of using a smaller equivalent half-width.

With regard to single-line absorption let us assume that the equivalent homogeneous atmosphere corresponding to a given non-homogeneous one between two levels u_1 and u_2 of absorbing gas, is characterized by the parameters, $\tilde{S}$, $\bar{\gamma}$ and ϵ, i.e.,

$$\tilde{S}=\int_{u_1}^{u_2} S\,du \Big/ \int_{u_1}^{u_2} du \equiv \tilde{S}_{CG}, \tag{1}$$

$$(\bar{\gamma})^{\epsilon}=\int_{u_1}^{u_2} S\gamma^{\epsilon}\,du \Big/ \int_{u_1}^{u_2} S\,du, \tag{2}$$

where S is the line intensity, γ the half-width, and the subscript CG is the abbreviation of Curtis-Godson. The parameter ϵ is introduced to make the equivalent half-width $\bar{\gamma}$ smaller than that of the Curtis-Godson approximation in an intermediate path length. Therefore, the value of ϵ should be smaller than one. The case of $\epsilon=1$ corresponds to the strong limit of absorption where $\bar{\gamma}$ defined by Eq. (2) coincides with that of the Curtis-Godson approximation. It is confirmed numerically that ϵ decreases with decrease of the amount of an absorbing gas in the path. In addition, the condition that absorption is independent of the line width at the weak limit of absorption is also involved in (2), if we make $\epsilon=0$ at this limit.

In a special case of an isothermal atmosphere with an absorber of constant mixing ratio, in which the line intensity S is constant and both γ and u are proportional to pressure, the exact form of transmission is known by YA. Therefore, more precise information on ϵ will be available by considering this case. From Eq. (2)

$$\bar{\gamma}_{CG}=\int_{u_1}^{u_2} S\gamma\,du \Big/ \int_{u_1}^{u_2} S\,du, \tag{3}$$

we have for the path between two pressure levels p_1 and p_2

$$\bar{\gamma}_{CG}-\bar{\gamma}=\frac{\gamma_2}{p_2}\left\{\tfrac{1}{2}(p_1+p_2)-\left[\frac{p_2^{1+\epsilon}-p_1^{1+\epsilon}}{(1+\epsilon)(p_2-p_1)}\right]^{1/\epsilon}\right\}, \tag{4}$$

[1] Partly supported by the National Oceanic and Atmospheric Administration under Contract 1-35123.

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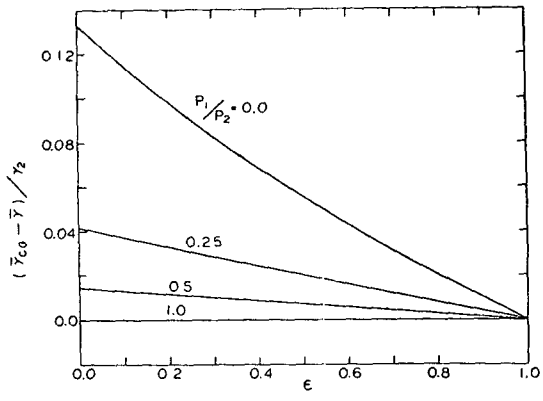

FIG. 1. Variation of $(\bar{\gamma}_{CG}-\bar{\gamma})/\gamma_2$ for $1\geqslant\epsilon\geqslant 0$ and $1\geqslant p_1/p_2\geqslant 0$.

where γ_2 is the half-width at p_2. Considering that the error of the Curtis-Godson approximation is due to overestimation of the half-width and that $\bar{\gamma}_{CG}-\bar{\gamma}\geqslant 0$ for $1\geqslant\epsilon\geqslant 0$ (see Fig. 1), obtained by numerical calculation from (4), the use of $\bar{\gamma}$ is expected to result in an improved approximation. Further, since we have the advantage of knowing the exact form of the integrated absorption in this case, the value of ϵ can be determined by equating the exact absorption formula for a non-homogeneous path (refer to YA) to the Ladenburg-Reiche formula for the corresponding homogeneous path; i.e.,

$$A_{\text{exact}}(\lambda)=A_{LR}(x), \tag{5}$$

where A denotes the integrated absorption for a single Lorentz line, and

$$A_{\text{exact}}(\lambda)=2\pi\lambda\gamma_2\left[F(-\lambda,\tfrac{1}{2},1,z)-\left(\frac{p_1}{p_2}\right)^2 F(1-\lambda,\tfrac{1}{2},1,z)\right], \tag{6}$$

$$\lambda=\frac{Sqp_2}{2\pi\gamma_2\rho g\xi}, \tag{7}$$

$$z=1-(p_1/p_2)^2,$$

where F is the hypergeometric function, q the mixing ratio of the absorbing gas, ρ the density of the absorbing gas at STP, g the acceleration of gravity, ξ the cosine of zenith angle,

$$A_{LR}(x)=2\pi\bar{\gamma}xe^{-x}[I_0(x)+I_1(x)], \tag{9}$$

$$x=\bar{S}\int_{u1}^{u2} du\Big/2\pi\bar{\gamma}, \tag{10}$$

LR is the abbreviation of Ladenburg-Reiche, and I_0 and I_1 are Bessel functions with an imaginary argument of zeroth and first order respectively. We shall here introduce another variable x_{CG} given by

$$x_{CG}=\bar{S}\int_{u1}^{u2} du\Big/2\pi\bar{\gamma}_{CG},$$

$$=\frac{2(p_2-p_1)}{(p_2+p_1)}\lambda \quad [\text{by Eq. (7)}],$$

$$=\bar{\gamma}x/\bar{\gamma}_{CG} \quad [\text{by Eq. (10)}]. \tag{11}$$

The introduction of this quantity is convenient, because it is linearly related to λ, while x is not so.

From Eqs. (2), (5), (10) and (11) the values of ϵ can be determined as a function of x_{CG}. The relation between ϵ and x_{CG} for the case of the path between $p_1=0$ and $p_2=p_0$ (surface pressure) is shown by the solid line in Fig. 2. Similar relations between ϵ and x_{CG} exist for any p_1 and p_2 values. Referring to the relation between ϵ and x_{CG} shown in Fig. 2, we can obtain the exact value of A in the present case by determining 1) x_{CG} corresponding to a given λ value [Eq. (11)], 2) ϵ from the solid line in Fig. 2, 3) $\bar{\gamma}$ [Eq. (2)], 4) x [Eq. (10)], and 5) using the Ladenburg-Reiche formula.

We shall next proceed to treat a general non-homogeneous atmosphere in which the exact solution for absorption is unknown. Although the exact relation between ϵ and x_{CG} is also unknown in this case, it will not be unreasonable to assume that a relation similar to that shown by the solid line in Fig. 2 holds as a first approximation. In addition, in order to reserve some freedom in determining this relation, we shall assume the general form

$$\epsilon(x_{CG})=\left(\frac{x_{CG}}{1+x_{CG}}\right)^n, \tag{12}$$

where n is a constant to be determined in connection with the distribution of the absorbing gas. If we determine the value of n so as to best match the solid line in Fig. 2, we have $n=1.6$ as shown by broken curves in Fig. 2. Although the introduction of Eq. (12) is based on the analysis for the path between $p_1=0$ and $p_2=p_0$, we intend to adopt it for any path between p_1 and p_2. This is because the error in evaluating A de-

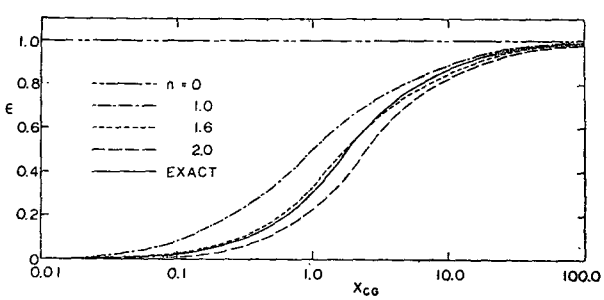

FIG. 2. Variation of the exact and approximate values of ϵ for n in Eq. (12). Solid line shows the exact value.

TABLE 1. Spectral parameters of absorbers. The symbols S_0 and γ_0 are the line intensity and halfwidth at the surface condition, respectively.

CO_2 (15μ)*

Line	γ_0 (cm^{-1})	S_0 (cm^{-2} atm^{-1})	E (cm^{-1})	ω
P(16)	0.0728	3.7015	77.091	0.64
P(46)	0.0670	0.2803	813.98	0.59
P(70)	0.0581	0.001833	1799.5	0.36

H_2O (6.3μ)**

Line	γ_0 (cm^{-1})	S_0 (cm^{-2} atm^{-1} $\times 10^4$)	E (cm^{-1})	ω
R(4)	0.0791	4.059	142.28	0.59
R(8)	0.0694	0.5614	586.48	0.43
R(11)	0.0581	0.0575	1114.56	0.39

O_3 (9.6μ)***

Line	γ_0 (cm^{-1})	S_0 (cm^{-2} atm^{-1})	E (cm^{-1})	ω
R(14)	0.0781	0.4940	189.02	0.5
R(29)	0.0781	0.3173	453.52	0.5
R(44)	0.0781	0.04554	908.18	0.5

* From Gray (1965, 1967, 1970) and Yamamoto *et al.* (1969).
** From Benedict and Calfee (1967) and Benedict and Kaplan (1959).
*** From Clough and Kneizys (1965) and Walshaw (1955).

creases with a decrease of the path length for any approximation; thus, the error in determining ϵ from (12) for smaller path lengths is less sensitive.

3. Application of the present method to a non-homogeneous atmosphere

We shall proceed to apply the present method to a non-homogeneous model atmosphere and examine the accuracy of approximation.

Taking a single Lorentz line, absorption by the total air column from the top of the atmosphere to the surface is calculated by the Curtis-Godson, Goody and present methods as well as by an entirely numerical method to compare the errors of the various approximations. The

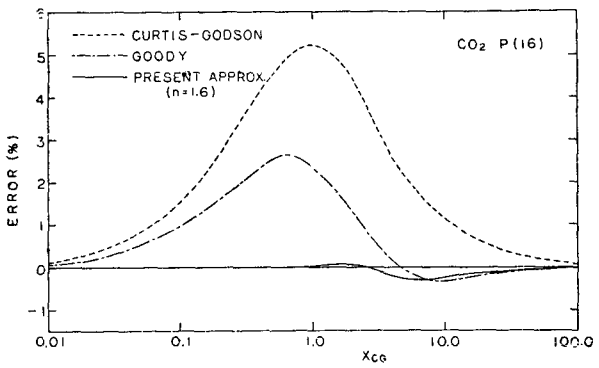

FIG. 3. Percentage errors, $100\times(A_{\text{approx.}}-A_{\text{exact}})/A_{\text{exact}}$, as a function of x_{CG} for the path between $p=0$ and $p=p_0$ for the CO_2 P(16) line.

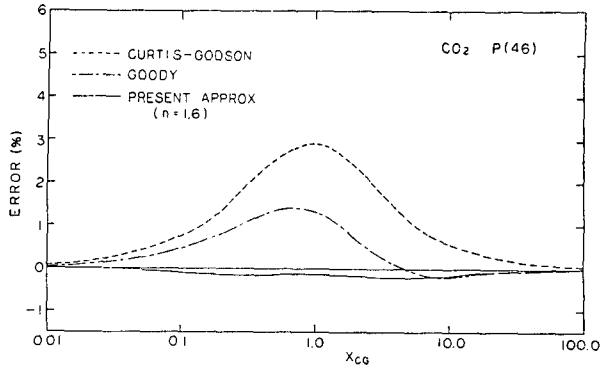

FIG. 4. Same as Fig. 3 except for the CO_2 P(46) line.

following temperature dependences of the line intensity and halfwidth are assumed:

$$S=S_0\left(\frac{T_0}{T}\right)^m \exp\left[-\frac{E}{k}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right], \tag{13}$$

$$\gamma=\gamma_0\frac{p}{p_0}\left(\frac{T_0}{T}\right)^\omega, \tag{14}$$

where T is the temperature, E the energy of the lower state, k the Boltzmann constant, m the numerical factor relating to the absorber, i.e., $m=1$, $\frac{3}{2}$ and $\frac{5}{2}$ for CO_2, H_2O and O_3, respectively, ω the numerical factor, and the suffix 0 denotes the surface condition. The spectral parameters of absorption lines selected for this calculation are listed in Table 1.

The approximation used in the present method involves the determination of:

1) $\bar{S}$ from Eq. (1)
2) $\bar{\gamma}_{CG}$ from Eq. (3)
3) x_{CG} from Eq. (11)
4) ϵ from Eq. (12), using values of n given in subsequent sections
5) $\bar{\gamma}$ from Eq. (2)
6) x from Eq. (10)
7) A from Eq. (9).

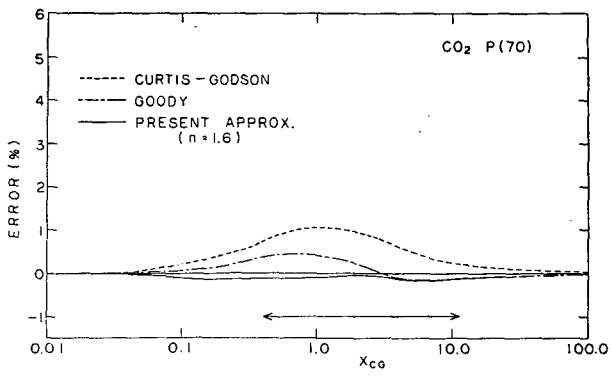

FIG. 5. Same as Fig. 3 except for the CO_2 P(70) line. The arrow-marked region of x_{CG} really occurs in the slant paths of Benporad's atmosphere.

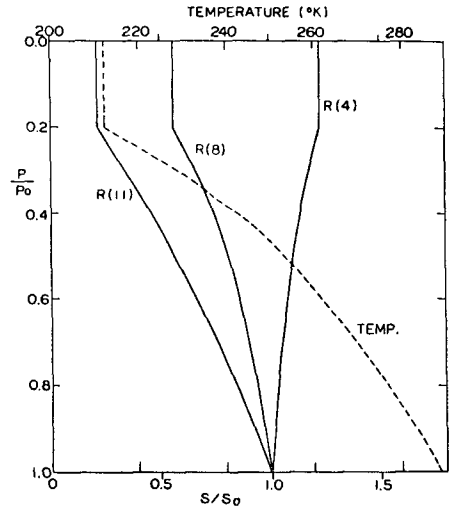

FIG. 6. Model atmosphere of the H_2O absorber. The broken line is the temperature profile, the solid line the temperature variation of line intensity normalized by the value at the ground level for each line.

a. CO_2 model atmosphere

The assumed temperature distribution, the CO_2 mixing ratio, and the adopted three lines are the same as used in YA. In this case the value of n for minimizing the errors is 1.6 and the errors are shown in Figs. 3–5. It is seen that the errors of the present approximation and YA are of the same order and are smaller than those of the Curtis-Godson and Goody approximations.

b. H_2O model atmosphere

The same temperature distribution as of the CO_2 case is assumed and the mixing ratio of H_2O [gm gm^{-1}] is assumed to be

$$q=0.00925(p/p_0)^2+0.00001, \tag{15}$$

which corresponds approximately to the average humidity profile in mid-latitudes during autumn. Three absorption lines, R(4), R(8) and R(11) of the R-branch in the 6.3 μ band, whose temperature dependence is as shown in Fig. 6, are adopted. Again errors are minimized for $n=1.6$, and are shown in Figs. 7–9.

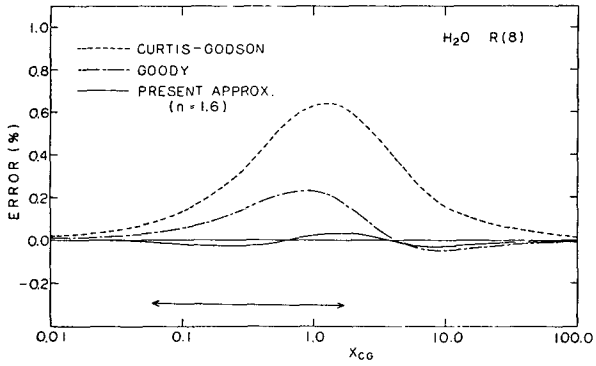

FIG. 8. Same as Fig. 3 except for the H_2O R(8) line.

c. O_3 model atmosphere

The mixing ratio of O_3 [gm gm^{-1}] is assumed to be

$$\left.\begin{aligned} &q=4.67\times10^{-8}, \quad \text{for} \quad 1.0\geqq p/p_0>0.5 \\ &q=0.2925(p/p_0)^2/[1.0+5.0000(p/p_0)^2], \\ &\qquad\qquad \text{for } 0.5\geqq p/p_0 \end{aligned}\right\}. \tag{16}$$

This distribution of q is shown in Fig. 10, in which average distributions of O_3 for 60N, 30N and the equator (Hering and Borden, 1964) are also shown for reference. The adopted profile does not coincide with observed ones for $0.002>p/p_0$, but this discrepancy may be disregarded in the present problem because of small absorption in that layer. The assumed temperature distribution is somewhat sophisticated; it corresponds to the O_3 distribution, and is given by

$$\left.\begin{aligned} &T=288.16(p/p_0)^{0.19026}, && \text{for } 1.0\geqq p/p_0>0.2 \\ &T=212.15, && \text{for } 0.2\geqq p/p_0>0.02 \\ &T=212.15[0.02/(p/p_0)]^{0.087811}, && \\ &\qquad\qquad && \text{for } 0.02\geqq p/p_0>0.002 \\ &T=259.65, && \text{for } 0.002\geqq p/p_0 \end{aligned}\right\}. \tag{17}$$

The adopted absorption lines are R(14), R(29) and R(44) in the R-branch of ν_3 fundamental band, and the

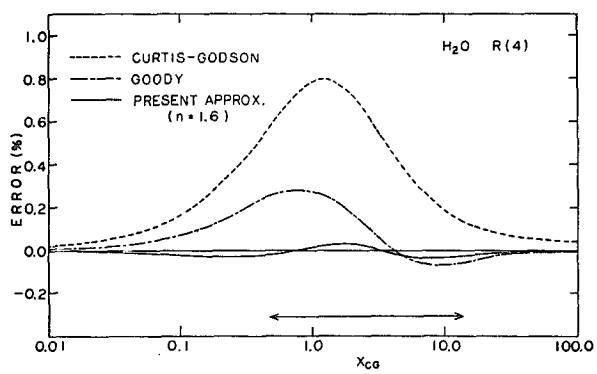

FIG. 7. Same as Fig. 3 except for the H_2O R(4) line.

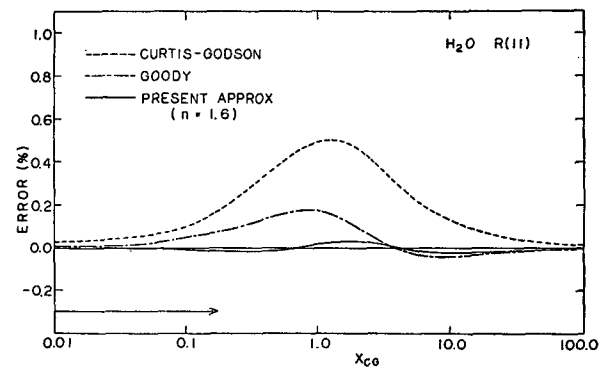

FIG. 9. Same as Fig. 3 except for the H_2O R(11) line.

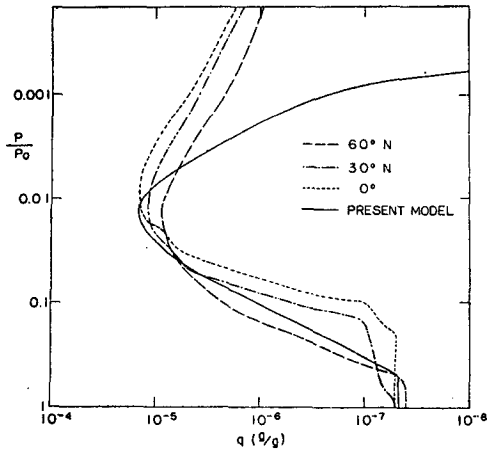

FIG. 10. Vertical distributions of O_3 mixing ratio. The solid line is the model defined by Eq. (16), the other lines the actual mean profiles at latitudes, 60N, 30N, and the equator.

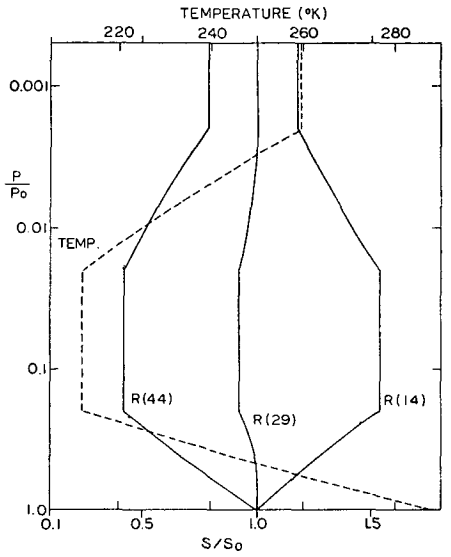

FIG. 11. Model atmosphere of O_3 absorber. Same as Fig. 6 except for the O_3 lines.

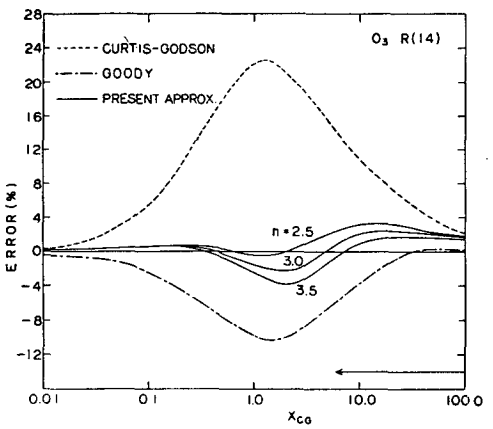

FIG. 12. Same as Fig. 3 except for the O_3 R(14) line.

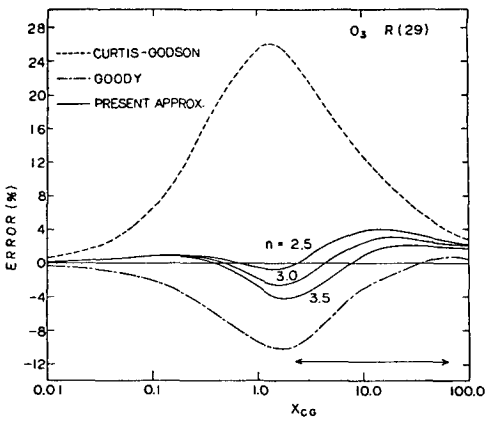

FIG. 13. Same as Fig. 3 except for the O_3 R(29) line.

temperature dependence of these lines is shown in Fig. 11. Figs. 12, 13 and 14 show the errors of the integrated absorption of these lines by the present approximation for $n=2.5$, 3 and 3.5, as well as by the Curtis-Godson and Goody approximations. These figures indicate that errors for all approximate methods are quite large in the case of O_3 due, perhaps, to the unique distribution of O_3. However, the present approximation is a remarkable improvement over the other two. The best value of n will be about 3.2, although the error estimation using this value is not carried out. If the present method is applied to a thin layer such as $0.2>p/p_0>0$, the value of n is reduced to 2.5; it will gradually approach 1.6 as the layers become even thinner.

4. Discussion

Although the present study is limited to single-line absorption, it is applicable to a spectral band interval with a minor modification. In order to do so, the band interval should be divided into small sub-intervals, each of which includes a single line. In addition to the single line, then, each sub-interval necessarily involves the wings of other lines outside of sub-interval and also

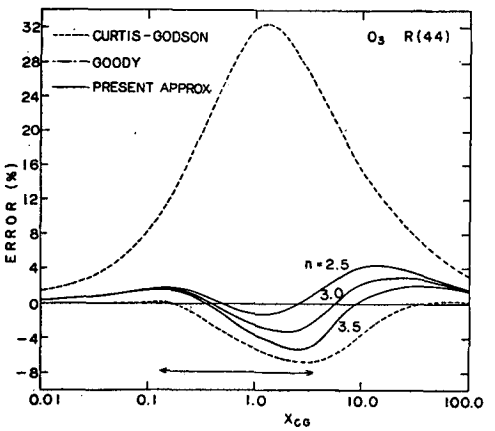

FIG. 14. Same as Fig. 3 except for the O_3 (R44) line.

contains various weak lines. Therefore, in applying the present method we need to determine an equivalent single-line intensity which involves the effect of the wings of other lines as well as that of weak lines. This evaluation can be made by calculating the transmission of the sub-interval for a given homogeneous path by two methods, one by using the given spectrum and the other by assuming an equivalent single line. In this process the half-width of the original strong line contained in the sub-interval can be assumed as that of the equivalent line.

The present method is also applicable to band models such as those of Elsasser and Goody, by treating the representative line intensity and half-width as if they applied to the case of a single line and by using respective transmission formulas.

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