On the Depth of the Equatorial Planetary Boundary Layer[1]

CHIH-PEI CHANG

Dept. of Meteorology, Naval Postgraduate School, Monterey, Calif. 93940

(Manuscript received 24 August 1972, in revised form 20 December 1972)

ABSTRACT

The solutions of the linear equatorial boundary layer are evaluated and found to be dependent on both the depth of the modelled boundary layer and the mode of symmetry about the equator. The inclusion of the frictional effect at levels higher than the usually assumed 1-km boundary layer depth is crucial to the concentration of convergence around the critical latitude. It is therefore suggested that the top of the boundary layer in tropical wave models should be properly selected in order to represent the deep convergence around the critical latitude.

1. Introduction

It has been previously noted (Greenspan, 1968) that for oscillatory motions in a contained rotating viscous fluid, the depth of the boundary layer near a *critical latitude* (where the frequency of oscillation equals the local Coriolis frequency) is inversely proportional to the square root of the difference between the oscillation frequency and the Coriolis frequency. As a consequence of this fact the vertically integrated mass transport in the boundary layer possesses a singularity at the critical latitude. Because tropical waves are usually observed at latitudes where the Coriolis frequencies are comparable to the wave frequencies, Holton *et al.* (1971) and Yamasaki (1971a) have suggested that this property may be important for modelling synoptic waves in the tropics. In particular, the critical latitude singularity may be important for the so-called CISK studies in which the heating due to cooperative cumulus convection is parameterized as being proportional to the vertical motion at the top of the boundary layer.

Attempts to replace Ekman-type parameterizations of boundary layer convergence in CISK models by the critical latitude concept have been carried out by Yamasaki (1971b), Hayashi (1971b) and Hoyer (1972). However, they all treated the boundary layer as a very shallow layer with the top at or around 900 mb, a common assumption among the Ekman-layer CISK models. In a related study, Hayashi (1971a) concluded that surface drag terms applied at the ground can be used as a good approximation to the friction layer. However, there are several problems associated with those treatments. Theoretically, the critical latitude singularity results from the upward extension of the boundary layer to very high altitudes. Although it is true that, as will be shown later, the convergence can sometimes reach a local maximum near the critical latitude at a constant-height level, the increase of the vertically integrated convergence is more generally a consequence of increasing depth toward the critical latitude. In addition to this theoretical consideration, observational evidence by Williams (1970), Wallace (1971) and Reed and Recker (1971) has shown that mass convergence in the tropical waves is not mainly confined to the very low levels below 900 mb as is usually assumed by the inclusion of an Ekman-type boundary layer in tropical models. Their observations indicate that 1) only a fraction, probably 10–30%, of the synoptic-scale convergence takes place below 900 mb, and 2) the ratio of synoptic-scale convergence in the layer below 900 mb to total upward mass flux [i.e., the parameter η defined by Ooyama (1969)] does not seem to be constant. It is probable that a certain amount of the convergence above 900 mb is due to entrainment of environmental air into cumulonimbus towers. But the deepening of boundary layer convergence around the critical latitude, where huge amounts of synoptic-scale convective activity is commonly observed, could also explain the observed convergence profile. Therefore, a CISK model allowing the possibility of deepened boundary layer convergence near the critical latitude is likely to be more realistic than using the assumption that all condensation heating is proportional to the convergence in a boundary layer with its top fixed at 900 mb.

In view of the foregoing discussion, the purpose of this paper is to examine the properties of equatorial boundary layer solutions with respect to the frequency and depth dependence of convergence and vertically integrated mass transport. A linearized model for a

[1] Contribution No. 279, Department of Atmospheric Sciences, University of Washington, Seattle.

forced barotropic wave in a resting Boussinesq atmosphere on the equatorial β-plane will be sufficient for our purpose.

2. The solution

Following the non-dimensionalization of Holton *et al.* (1971), we use a length scale $L\equiv(c/\beta)^{\frac{1}{2}}$, a time scale $\tau\equiv(c\beta)^{-\frac{1}{2}}$ and a depth scale H, where $c\equiv(gH)^{\frac{1}{2}}$ is a horizontal velocity scale, $\beta\equiv 2\Omega/a$ with Ω the angular velocity of the earth, and a the radius of the earth; and g is the gravitational constant and H the scale height. In this form the equations of motion and continuity equation are:

$$\frac{\partial u^*}{\partial t}=-\frac{\partial p^*}{\partial x}+yv^*+K\frac{\partial^2 u^*}{\partial z^2}, \tag{1}$$

$$\frac{\partial v^*}{\partial t}=-\frac{\partial p^*}{\partial y}-yu^*+K\frac{\partial^2 v^*}{\partial z^2}, \tag{2}$$

$$\frac{\partial w^*}{\partial z}=-\frac{\partial u^*}{\partial x}-\frac{\partial v^*}{\partial y}, \tag{3}$$

where

$u^*(x,y,z)$	perturbation zonal velocity
$v^*(x,y,z)$	perturbation meridional velocity
$w^*(x,y,z)$	perturbation vertical velocity
$p^*(x,y)$	perturbation pressure
K	vertical eddy viscosity coefficient, non-dimensionalized by $H^2(c\beta)^{\frac{1}{2}}$, assumed constant
x, y, z	zonal, meridional and vertical direction, respectively

We assume that solutions exist of the form

$$\begin{Bmatrix} u^* \\ v^* \\ w^* \\ p^* \end{Bmatrix}=\begin{Bmatrix} u \\ v \\ w \\ p \end{Bmatrix} e^{i(\nu t+\lambda x)}, \tag{4}$$

where ν is the wave frequency and λ the zonal wavenumber. Substituting (4) into (1)–(3) leads to

$$i\nu u=yv-i\lambda p+K\frac{\partial^2 u}{\partial z^2}, \tag{5}$$

$$i\nu v=-yu-\frac{\partial p}{\partial y}+K\frac{\partial^2 v}{\partial z^2}, \tag{6}$$

$$\frac{\partial w}{\partial z}=-i\lambda u-\frac{\partial v}{\partial y}. \tag{7}$$

It is convenient to apply the classical method of Ekman (1905) to solve the equations of motion. We let

$$\left.\begin{aligned} W&\equiv u+iv \\ W^*&\equiv u-iv \end{aligned}\right\}, \tag{8}$$

and by adding and subtracting Eqs. (5) and (6), we obtain

$$\left.\begin{aligned} i(\nu+y)W&=-i\lambda p-i\frac{\partial p}{\partial y}+K\frac{\partial^2 W}{\partial z^2} \\ i(\nu-y)W^*&=-i\lambda p+i\frac{\partial p}{\partial y}+K\frac{\partial^2 W^*}{\partial z^2} \end{aligned}\right\}. \tag{9}$$

Now we will consider the top boundary conditions. If we let the frictional terms vanish at the top of boundary layer where $z=z_T$ and W_T and W_T^* are the solutions at this level, the system (9) becomes

$$\left.\begin{aligned} -i\lambda p-i\frac{\partial p}{\partial y}&=i(\nu+y)W_T \\ -i\lambda p+i\frac{\partial p}{\partial y}&=i(\nu-y)W_T^* \end{aligned}\right\}. \tag{10}$$

Substituting (10) into (9) leads to

$$\left.\begin{aligned} \frac{\partial^2 W}{\partial z^2}-\gamma^2(W-W_T)&=0 \\ \frac{\partial^2 W^*}{\partial z^2}-\alpha^2(W^*-W_T^*)&=0 \end{aligned}\right\}, \tag{11}$$

where

$$\gamma=\left[\frac{i(\nu+y)}{K}\right]^{\frac{1}{2}},$$

$$\alpha=\left[\frac{i(\nu-y)}{K}\right]^{\frac{1}{2}}.$$

The solution of (11) subject to the lower boundary conditions that all velocity components vanish at $z=0$ is

$$\left.\begin{aligned} W(z)&=W_T\left[\frac{1-e^{-\gamma z}+e^{-2\gamma z_T}(e^{\gamma z}-1)}{1-e^{-2\gamma z_T}}\right] \\ W(z)^*&=W_T^*\left[\frac{1-e^{-\alpha z}+e^{-2\alpha z_T}(e^{\alpha z}-1)}{1-e^{-2\alpha z_T}}\right] \end{aligned}\right\}. \tag{12}$$

Combining (8) and (12), we obtain the solution

$$\left.\begin{aligned} u&=\frac{1}{2}(W+W^*) \\ v&=-\frac{i}{2}(W-W^*) \end{aligned}\right\}. \tag{13}$$

When $z_T\rightarrow\infty$, solution (12) becomes the well-known geostrophic Ekman spiral if we let $\nu=0$ and replace y

by a constant Coriolis parameter f_0 (which is unity by our nondimensionalization). It is clear from (12) that if we integrate the continuity equation (3) from $z=0$ to $z=z_T\rightarrow\infty$, singularity will occur and the vertical velocity at the top of the boundary layer will become infinite when $\nu+y=0$ or $\nu-y=0$.

If the depth of the boundary layer is finite, i.e., $z_T<\infty$, the vertical velocity w_T at the top z_T can again be found by integrating (3) from $z=0$ to $z=z_T$. To investigate the behavior of w_T at the critical latitude $y=\nu$, we can evaluate the integral of the height-dependent part of W^* in (12):

$$I=\frac{\int_0^{z_T}[1-e^{-\alpha z}+e^{-2\alpha z_T}(e^{\alpha z}-1)]dz}{1-e^{-2\alpha z_T}},$$

$$=\frac{\alpha z_T+e^{-\alpha z_T}-1+e^{-2\alpha z_T}(e^{\alpha z_T}-1-\alpha z_T)}{\alpha-\alpha e^{-2\alpha z_T}}.$$

At the critical latitude $y=\nu$ and $\alpha=0$, we apply L'Hopital's rule twice to obtain

$$\lim_{\alpha\rightarrow 0} I=\frac{z_T}{2}.$$

A similar argument holds for the integral of W in (12) for $y=-\nu$. We conclude that the vertical velocity at the top z_T is finite if the boundary layer depth is finite, and that as z_T becomes larger, w_T will be closer to being proportional to z_T.

3. Simple examples

Since the depth of the boundary layer of the real tropical atmosphere must be finite, there will be no singularity of vertical velocity at the critical latitude. But we would like to know how the convergence is distributed in this boundary layer especially near the critical latitude where $y=\pm\nu$. To accomplish this, we have to specify the "free" atmospheric flow at $z=z_T$ in the boundary layer solution (12) and (13). For the purpose of simplicity, the two simplest non-trivial solutions for barotropic channel flow are used: an asymmetric mode (pressure is asymmetric with respect to the equator)

$$\left.\begin{aligned} u&\equiv u_T=l\sin ly\\ v&\equiv v_T=i\lambda\cos ly\end{aligned}\right\}\text{ at } z=z_T, \tag{14}$$

and a symmetric mode (pressure is symmetric with respect to the equator)

$$\left.\begin{aligned} u&\equiv u_T=-l\cos ly\\ v&\equiv v_T=i\lambda\sin ly\end{aligned}\right\}\text{ at } z=z_T, \tag{15}$$

where $l=\pi/D$, D being the width of the channel. The frequency of the barotropic Rossby wave is $\nu=\lambda/(\lambda^2+l^2)$.

In the following discussion, we let the width of the channel flow at the top be 50° for the asymmetric mode and 25° for the symmetric mode, so that all v_T's vanish at a poleward boundary of 25°. We also let the zonal wavelength be 4000 km and the eddy viscosity coefficient be 10 m² sec⁻¹. Because of symmetry, only results for the Northern Hemisphere will be shown in the diagrams.

In Fig. 1 the amplitude and phase of w_T for the asymmetric mode are plotted as functions of latitude, for four different boundary layer depths: $z_T=1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. For the $z_T=1$, $\frac{1}{2}$ and $\frac{1}{4}$ layers a maximum is found in the amplitude of w_T at the critical latitude, with sharper concentrations of w_T for deeper boundary layers. When the layer depth is shrunk to $z_T=\frac{1}{8}$, this maximum shifts poleward away from the critical latitude.

The patterns of phase change of these vertical motions are generally similar around the critical latitude. North of the critical latitude, upward motion is very close to coinciding with the phase line of positive vorticity (trough line), except near the critical latitude where it is ahead of the trough. South of the critical latitude toward the equator, vertical motion changes from leading the trough to lagging the trough. This phase change is largest for the deepest case ($z_T=1$), for which w_T changes from $\sim\frac{3}{8}$ cycle ahead of the trough to $\sim\frac{3}{8}$ behind the trough across a distance of $<3°$ in latitude; and is smaller for the shallower layers, with $\sim 1/18$ cycle separation in the entire channel for the $z_T=\frac{1}{8}$ case. This small phase change in w_T for the $z_T=\frac{1}{8}$ layer and its amplitude profile with maximum near the center of the channel actually resembles the Ekman relation of vertical velocity and vorticity (Charney and Eliassen, 1949).

The shape of the amplitude distribution of w_T for the symmetric mode is more dependent on the depth of the boundary layer. Fig. 2 presents the amplitude and phase distributions for the symmetric mode for the $z_T=1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ layers. The $z_T=1$ layer possesses a maximum of w_T at the critical latitude, but for the $z_T=\frac{1}{4}$ layer w_T is roughly a minimum around this region, with maxima north of it and at the equator. The north maximum is shifted further poleward to the center of the channel when the layer depth is decreased to $z_T=\frac{1}{8}$, which again resembles the geostrophic Ekman relation. For the intermediate depth $z_T=\frac{1}{2}$ layer, w_T is maximum at both a latitude slightly north of the critical latitude and the equator, while a minimum occurs on the equator side of the critical latitude so that the amplitude of w_T changes quite rapidly in that neighborhood.

The phase changes of w_T for the symmetric mode are generally opposite to those for the asymmetric mode. They change from lagging the trough line north of the critical latitude to leading the trough line south of the critical latitude. This change is again more abrupt for the deeper layers. It spans backward more than one-half cycle in a very short distance for the cases where

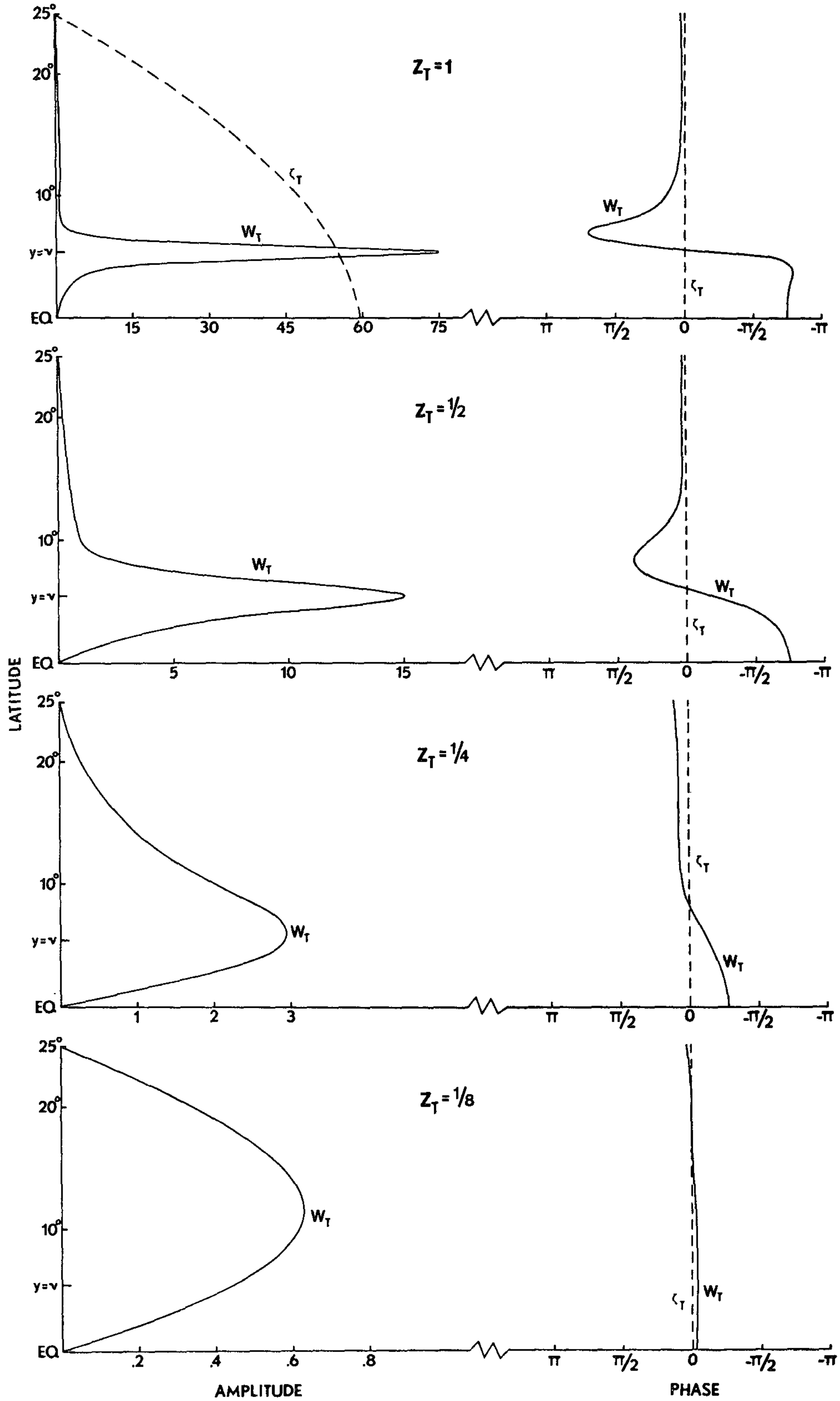

FIG. 1. Latitudinal distribution of amplitude and phase of vertical motion at top of boundary layer (w_T) for the asymmetric mode with $z_T=1$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. Dashed lines are vorticity at top of boundary layer (ζ_T). (Note change of scale in amplitude.)

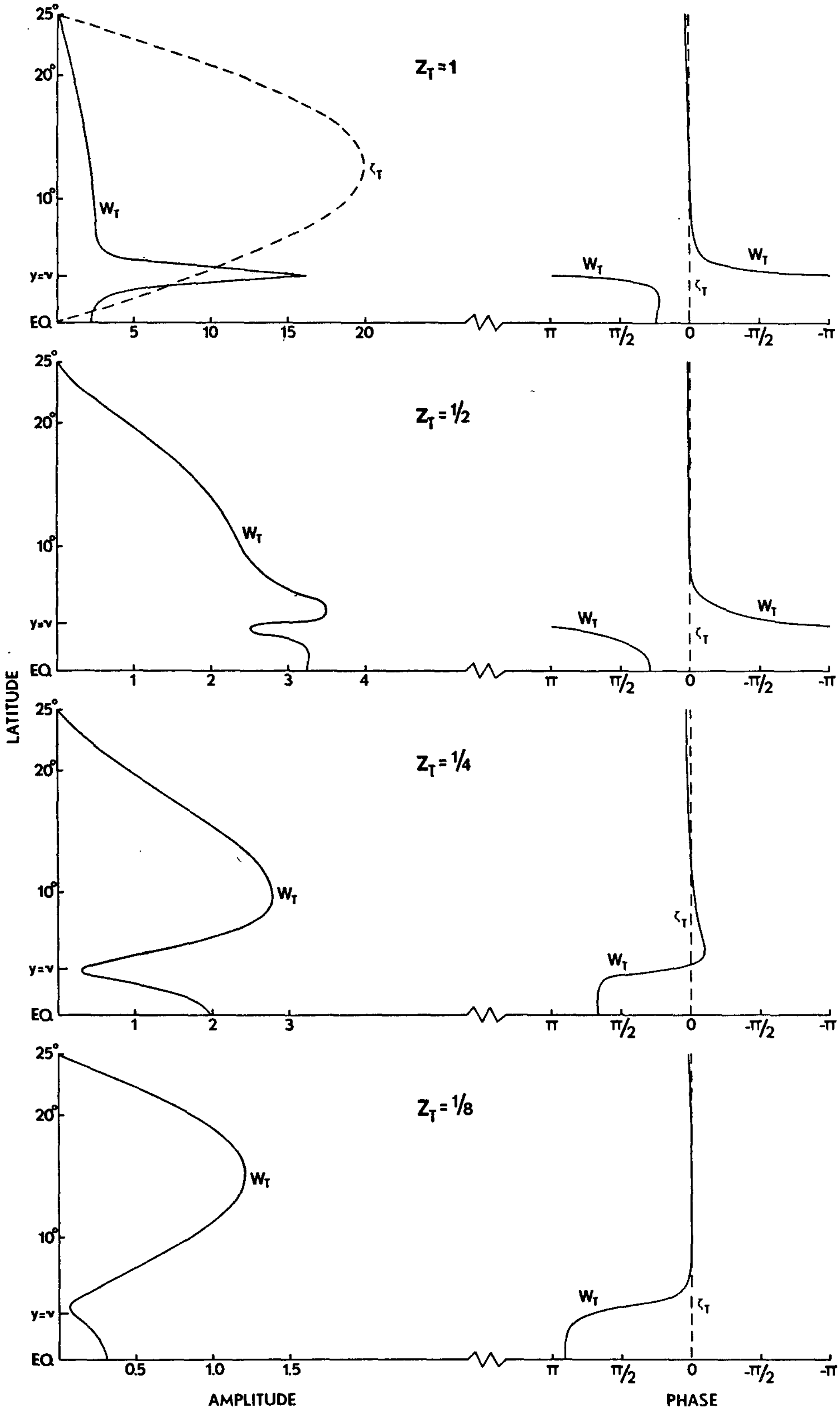

FIG. 2. Same as Fig. 1, except for the symmetric mode.

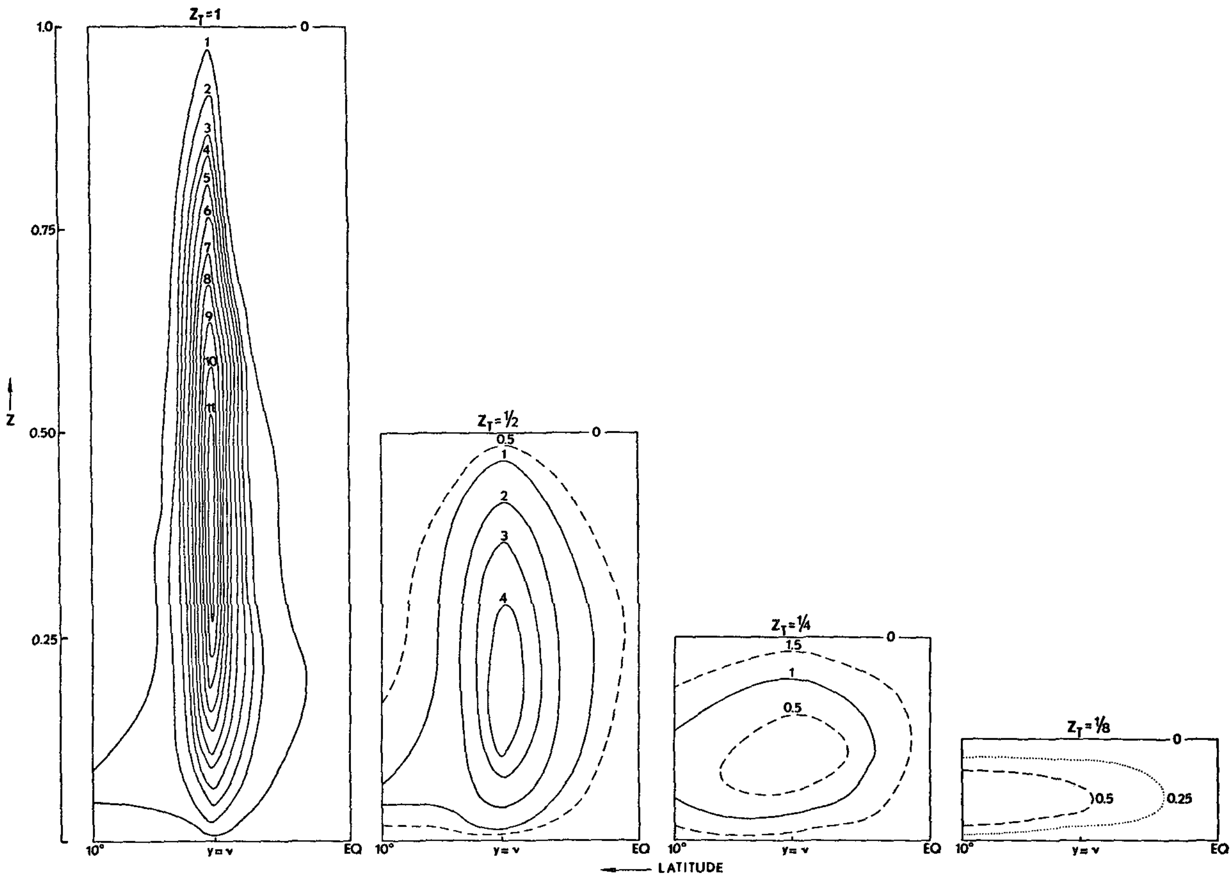

FIG. 3. Cross section of the amplitude of convergence for the asymmetric mode with $z_T=1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

w_T possesses a maximum around the critical latitude ($z_T=1$ and $z_T=\frac{1}{2}$). For the shallower layers, this change is accomplished by crossing the original trough line forward toward the equator.

The difference in depth dependence of the vertical motions produced by mass convergence in the boundary layer between the symmetric and asymmetric modes can be seen by comparing the vertical cross section of the amplitude of convergence for the two modes, as shown in Figs. 3 and 4. For the asymmetric mode (Fig. 3) the $z_T=\frac{1}{8}$ layer possesses convergence maximum poleward of the critical latitude at all levels. As the depth is increased to $z_T=\frac{1}{4}$, a tendency of concentration of convergence toward the critical latitude begins to show up, especially at upper levels. When z_T is increased to $\frac{1}{2}$ and higher, the maximum of convergence takes place at the critical latitude at all levels. Since the convergence at each latitude is nearly in phase in the vertical,[2] the above structures are consistent with the w_T distribution shown in Fig. 1. For the symmetric mode (Fig. 4), the convergence pattern is more complicated. The $z_T=\frac{1}{4}$ and $z_T=\frac{1}{8}$ layers both have two maxima of convergence: one to the north of the critical latitude and one at the equator; thus, the minimum occurs near the critical latitude. As z_T increases to $\frac{1}{2}$, another maximum of convergence is found at higher levels of the critical latitude. This maximum becomes dominant for the $z_T=1$ layer, as it intensifies and extends upward with the increase of the boundary layer depth. Apparently, this is the source for the maximum vertical velocity at the critical latitude for a deeper boundary layer with symmetric mode waves. We also note that even when the boundary layer is deep enough to possess a dominant critical latitude maximum, convergence at levels close to the ground is still a minimum around this latitude.

In addition to the above, Figs. 3 and 4 also show that northward of the critical latitude, frictional convergence is almost always confined to the lowest layer below $\frac{1}{4}$ of the scale height, which agrees qualitatively with the mid-latitude boundary layer theory. Therefore, an increase of boundary layer depth will only produce more vertical motion near and equatorward of the critical latitude.

[2] For both modes of symmetry, only in the deepest layer ($z_T=1$) does the convergence at the highest levels tend to be out of phase with the lower levels. This implies that vertical motion may reach a maximum at levels below $z=z_T$ for the very deep layers.

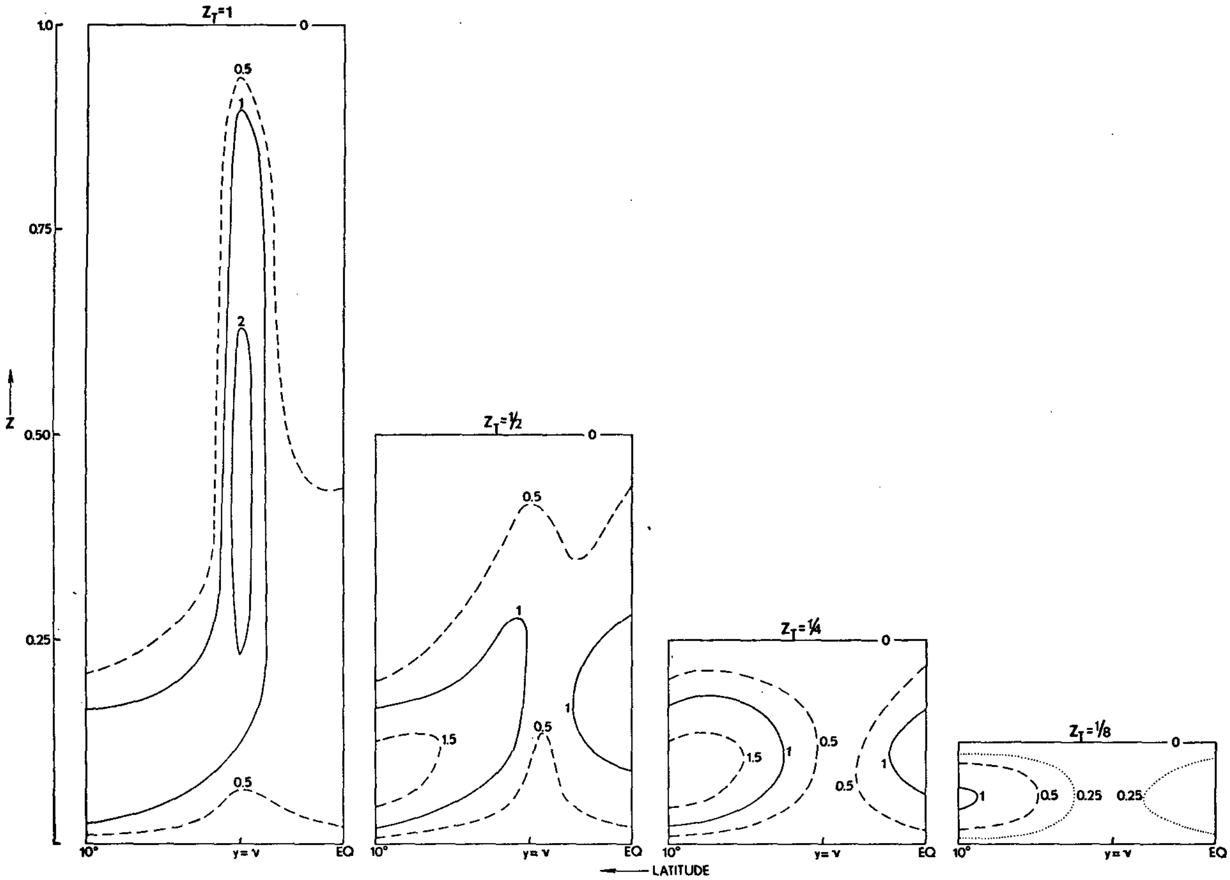

FIG. 4. Same as Fig. 3, except for the symmetric mode.

4. Concluding remarks

From the foregoing discussion, it appears that when incorporating the equatorial boundary layer process in a synoptic-scale CISK model, the selection of the boundary layer depth may effect the resultant convergence pattern and vertical motion significantly. The conclusion by Hayashi (1971a)[3] that the vertical velocity distribution produced by surface drag terms alone would resemble the complete critical latitude phenomena is thus not necessarily valid. The reason that Hayashi (1971a, b) and Hoyer (1972) found no critical latitude maximum of vertical velocity at the top of the boundary layer for the symmetric mode is apparently due to the fact that they limited the boundary layer in their models to the lowest 100–200 mb layer. In a complete tropical circulation model, a boundary layer much deeper than the usually adopted 100 mb depth may be needed to adequately represent the deep convergence around the critical latitude.

[3] Hayashi (1971a) used a lower boundary condition that the stress is equal to a drag force at the surface for (11). This will only change the convergence pattern near the ground (most noticeably the minimum near the critical latitude for the symmetric mode will disappear rapidly and become a maximum as z_T increases to $>1/2$), while the general features of depth dependence of the w_T profile remain roughly unchanged for both modes.

Acknowledgments. I wish to thank Prof. James R. Holton for his advice, Dr. Richard Anthes for pointing out an error in the boundary layer solution, and Prof. R. T. Williams for reading the final manuscript. This work was supported by the National Science Foundation under Grant GA-23488 to the University of Washington and by the Environmental Prediction Research Facility under Projects 55319 and 55514 at the Naval Postgraduate School.

REFERENCES

Charney, J. G., and A. Eliassen, 1949: A numerical method for predicting the perturbations of the middle-latitude westerlies. *Tellus*, **1**, 38–54.

Ekman, V. W., 1905: On the influence of the earth's rotation on ocean currents. *Arkiv. Math. Astron. Fysik.*, **2**, No. 11.

Greenspan, H. P., 1968: *The Theory of Rotating Fluids.* Cambridge University Press, 327 pp.

Hayashi, Y., 1971a: Frictional convergence due to large-scale equatorial waves in a finite-depth Ekman layer. *J. Meteor. Soc. Japan*, **49**, 450–456.

——, 1971b: Large-scale equatorial waves destabilized by convective heating in the presence of surface friction. *J. Meteor. Soc. Japan*, **49**, 458–465.

Holton, J. R., J. M. Wallace and J. A. Young, 1971: On boundary layer dynamics and the ITCZ. *J. Atmos. Sci.*, **28**, 275–280.

Hoyer, R. R., 1972: Planetary boundary layer effect upon equatorial tropospheric waves (submitted to *J. Atmos. Sci.*).

Ooyama, K., 1969: Numerical simulation of the life cycle of tropical cyclones. *J. Atmos. Sci.*, **26**, 3–40.

Reed, R. J., and E. E. Recker, 1971: Structure and properties of synoptic scale wave disturbances in the equatorial western Pacific. *J. Atmos. Sci.*, **28**, 1117–1133.

Wallace, J. M., 1971: Spectral studies of tropospheric wave disturbances in the tropical western Pacific. *Rev. Geophys. Space Phys.*, **9**, 557–612.

Williams, K. T., 1970: Characteristics of the wind, thermal, and moisture fields surrounding the satellite-observed mesoscale trade wind cloud clusters in the western north Pacific. *Preprints Symp. Tropical Meteorology*, Honolulu, Amer. Meteor. Soc., D IV-1 to D IV-8.

Yamasaki, M., 1971a: Frictional convergence in Rossby waves in low latitudes. *Syono Memorial Volume*, Meteor. Soc. of Japan, 691–698.

——, 1971b: A further study of wave disturbances in the conditionally unstable model tropics. *J. Meteor. Soc. of Japan*, **49**, 391–415.