

A Model for the Dynamics of the Inversion Above a Convective Boundary Layer

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ABSTRACT

The differential equations governing the strength Δ (a potential temperature difference) and the height h of inversions associated with dry penetrative convection are considered. No assumptions on the magnitude of the downward heat flux at the inversion base are needed to obtain an algebraic equation that relates h and Δ to the heating history of the boundary layer and to the initial conditions. After the nocturnal inversion has been filled in by heating, the inversion base generally grows linearly with time in the morning, but is proportional to the square root of time in the afternoon. The variation of Δ with time differs greatly from case to case.

1. Introduction

Planetary boundary layers with an upward heat flux at the surface generally are capped by an inversion. As the boundary layer heats up during the course of a sunny day, the inversion base gradually rises because the turbulence in the boundary layer entrains the warm air above the inversion base. This process is called penetrative convection. It is evident that the dispersion of atmospheric pollutants is limited by the height of the inversion "lid"; the problem of predicting the rate of rise of the inversion base thus is of great practical concern. Several investigators have formulated equations that govern the entrainment rate (i.e., the rate of rise of the inversion base). For example, Lilly (1968) predicted that the entrainment rate is inversely proportional to the difference in potential temperature across the inversion (the "strength" of the inversion). However, he did not describe how the strength of the inversion varies in time. This is an important issue, because it would seem that the inversion strength might decrease with time as the air in the boundary layer heats up. Deardorff *et al.* (1969) formulated an expression in which the rate of rise is inversely proportional to the lapse rate of potential temperature in the stable air above the inversion. On first sight, this formulation—which was developed independently by Summers (1965) and Tennekes (1973)—appears to be more attractive than the one proposed by Lilly, because penetrative convection into an ambient atmosphere with a stable lapse rate is capable of maintaining the inversion strength (Ball, 1960). In this way, the potential for conflict that is inherent to Lilly's model might be avoided. On the other hand, the inversion strength itself plays no role whatsoever in Deardorff's

equation, which leads one to wonder again what happens to it as time passes. Clearly, this problem calls for a model, however simplified, which contains not only a differential equation for the height of the inversion base, but also a differential equation for the inversion strength.

In this paper, such a model is developed. For the sake of clarity and simplicity, the conditions in and above the convective boundary layer will be assumed to be homogeneous in the horizontal plane, so that horizontal advection and large-scale subsidence do not enter the picture. It appears that these effects can be accounted for in a fairly straightforward way if so desired (Deardorff, 1972a). Also, radiation effects will be ignored. Since a radiative heat flux tends to decrease the net heating rate of the air in the boundary layer, it can be incorporated into the model by making a (generally small) reduction in the surface heat flux (see Section 2). Finally, it should be stressed that the analysis pertains to dry convection: in this model there is no release of latent heat. The presence of water vapor can be accounted for, if necessary, through the use of the virtual potential temperature instead of the "dry" potential temperature that appears in the equations.

2. Formulation of the equations

Schematic profiles of the potential temperature and of the vertical turbulent heat flux in and above a convective boundary layer are given in Fig. 1. The air above the inversion base (located at height h) is stable; its vertical gradient of potential temperature (γ) is taken to be independent of z and t (all symbols that are not defined have their conventional meanings.) The inversion strength is called Δ ; it is only fair to note that the problem of defining the inversion strength in a temperature profile obtained from observations or

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soundings is not nearly as simple as it appears here. The potential temperature in the upper part of the boundary layer is assumed to be independent of z ; in reality, the potential temperature tends to increase a little as the inversion base is approached (Deardorff, 1967). The dependence of the turbulent heat flux on the height z is taken to be linear; this assumption, which is in excellent agreement with numerous observations and calculations (e.g., Deardorff, 1967, 1972b), makes the heating rate of the air in the boundary layer independent of z . If the heating rate is uniform, the temperature profile in Fig. 1 shifts to the right as time increases, without changing its overall shape. This again is in accord with observations; it also ensures that the temperature gradient in the upper part of the boundary layer remains zero as long as convection (and the associated rapid mixing of enthalpy) dominates the dynamics inside the boundary layer.

Fig. 1 shows a downward heat flux at the inversion base. This is an immediate consequence of the entrainment of warm air into the cooler boundary layer. As the inversion base rises, the air that enters the boundary layer through entrainment (note the shaded area in the temperature plot of Fig. 1) becomes appreciably cooler. If the entrainment rate is dh/dt , the rate of decrease of enthalpy in the newly entrained air is equal to $c_p \rho \Delta dh/dt$ per unit area in the horizontal plane. This enthalpy loss has to be carried away by a heat flux; since there is no turbulence above the boundary layer that can sustain a heat flux, there must be a downward turbulent flux of heat into the boundary layer. The downward heat flux at the inversion base is $c_p \rho (\overline{\theta w})_i$ per unit area; if this flux is equated to the enthalpy loss $c_p \rho \Delta dh/dt$, there results (Ball, 1960; Kraus and Turner, 1967; Lilly, 1968)

$$-(\overline{\theta w})_i = \Delta \frac{dh}{dt}. \tag{1}$$

A few comments are in order. First, radiative fluxes probably are relatively unimportant if the penetrative convection is sufficiently vigorous: as long as $|\overline{\theta w}|_i$ is large enough, the turbulent flux will dominate the radiative one. Second, if $\Delta \rightarrow 0$, then $(\overline{\theta w})_i \rightarrow 0$ (unless $dh/dt \rightarrow \infty$, which is unlikely because convective elements have a finite rise velocity even if there is no inversion). On the other hand, if $(\overline{\theta w})_i$ is maintained at a finite value by the convective turbulence in the boundary layer (as proposed by Lilly, 1968), $dh/dt \rightarrow \infty$ as $\Delta \rightarrow 0$. Obviously, it is extremely important to know how Δ behaves. Third, the downward heat flux near the inversion base probably is the major cause of the small positive temperature gradient in the upper part of the boundary layer that is observed frequently (e.g., Deardorff, 1967).

The key issue is the kinematic behavior of the inversion strength Δ . On one hand, the inversion strength tends to decrease as the boundary layer warms up. Per

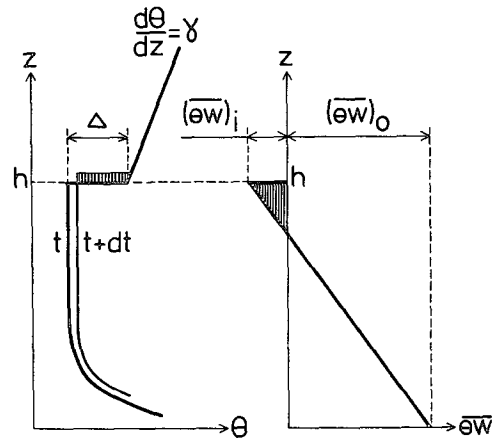


FIG. 1. The vertical distributions of potential temperature and turbulent heat flux in and above a convective boundary layer.

unit time, this inversion-filling effect is equal to $\partial \Theta / \partial t$ in the boundary layer (see Fig. 1). On the other hand, the inversion strength tends to increase as the entrainment into the stable air above the inversion base progresses. Per unit time, the entrainment increases Δ by an amount $\gamma dh/dt$. The net rate of change of Δ thus must be given by

$$\frac{d\Delta}{dt} = \gamma \frac{dh}{dt} - \left(\frac{\partial \Theta}{\partial t} \right)_{bl}, \tag{2}$$

where the subscript *bl* refers to the situation inside the boundary layer. It is of interest to observe here that the danger inherent in (1) as the limit $\Delta \rightarrow 0$ is taken is now removed by the first term on the right-hand side of (2): if Δ threatens to become small, dh/dt probably becomes large, so that the inversion strength increases again. Therefore, $\Delta \rightarrow 0$ is a special case that can occur only if the air above the inversion base happens to be exactly adiabatic.

The heating rate in the boundary layer is controlled by the enthalpy equation

$$c_p \rho \left(\frac{\partial \Theta}{\partial t} \right)_{bl} = - \frac{\partial}{\partial z} (c_p \rho \overline{\theta w}). \tag{3}$$

For the model described by the curves in Fig. 1, the gradient of $\overline{\theta w}$ in the vertical is independent of height, so that the heating rate becomes

$$\left(\frac{\partial \Theta}{\partial t} \right)_{bl} = \frac{(\overline{\theta w})_o}{h} - \frac{(\overline{\theta w})_i}{h}. \tag{4}$$

Note that the heating rate is increased by the entrainment of warm air into the boundary layer. As given by (1), the entrainment is associated with a downward heat flux near the inversion base, which makes a positive contribution to the net rate heating rate in (4).

Also, the divergence of the radiative heat flux generally would tend to decrease the net heating rate (Deardorff, 1967), but that effect can be accounted for by replacing $(\overline{\theta w})_0$ in (4) by an effective heat flux which is somewhat smaller than the turbulent heat flux near the surface.

Substitution of (4) into (2) yields

$$\frac{d\Delta}{dt} = \gamma \frac{dh}{dt} - \frac{(\overline{\theta w})_0}{h} + \frac{(\overline{\theta w})_i}{h}. \quad (5)$$

This equation describes the kinematics of the inversion strength Δ ; it is evident that $d\Delta/dt$ can be either positive or negative, depending on the situation. Together with (1), Eq. (5) completely describes what will happen to Δ and h as time proceeds, if, in addition, information on $(\overline{\theta w})_0$ and $(\overline{\theta w})_i$ is available.

It is extremely fortunate that the system (1), (5) allows a simple quadrature. Substitution of (1) into (5) yields

$$\frac{d\Delta}{dt} = \gamma \frac{dh}{dt} - \frac{(\overline{\theta w})_0}{h} - \frac{\Delta}{h} \frac{dh}{dt}. \quad (6)$$

With some rearrangement, there results an equation that is related to one given by Kraus and Turner [1967, Eq. (6)], i.e.,

$$\frac{d\Delta}{dt} + \Delta \frac{dh}{dt} = \gamma h \frac{dh}{dt} - (\overline{\theta w})_0. \quad (7)$$

This immediately integrates to

$$h\Delta - h_0\Delta_0 = \frac{1}{2}\gamma(h^2 - h_0^2) - \int_0^t (\overline{\theta w})_0 dt. \quad (8)$$

In (8), the subscripts 0 refer to the situation at $t=0$ (which may be thought of as a time in the vicinity of sunrise). In order to simplify the analysis of the consequences of (8), the heating history of the boundary layer [the last term of (8)] is approximated by means of a time-averaged value of $(\overline{\theta w})_0$ such that

$$\int_0^t (\overline{\theta w})_0 dt = [\overline{\theta w}]_0 t. \quad (9)$$

Strictly speaking, this approximation is valid only if $(\overline{\theta w})_0$ does not change too rapidly during the course of the day. In many cases of practical interest, however, the approximation will be fairly good if the origin of time is chosen as a few hours after sunrise. The kinematics of the inversion in the early hours after sunrise requires a different approximation, in which the surface heat flux is assumed to increase linearly with time. That approximation is discussed in Section 5.

If the approximation of steady heating is substituted

into (8), there results

$$h\Delta - h_0\Delta_0 = \frac{1}{2}\gamma(h^2 - h_0^2) - [\overline{\theta w}]_0 t. \quad (10)$$

At times, it will be convenient to use a nondimensional form of (10). We recommend

$$\frac{h\Delta}{h_0\Delta_0} - 1 = \alpha \left(\frac{h^2}{h_0^2} - 1 \right) - \tau, \quad (11)$$

where

$$\alpha = \frac{\gamma h_0}{2\Delta_0} \quad (12)$$

is the nondimensional lapse rate above the inversion base, and

$$\tau = \frac{[\overline{\theta w}]_0 t}{h_0\Delta_0} \quad (13)$$

is the nondimensional time. This clock runs slower if the surface heat flux is less—a desirable feature.

3. Some special cases

One of the most important features of (10) is that the effects of the initial conditions (h_0, Δ_0) , of the conditions in the stable air above the inversion base (γ), and of the heating history of the boundary layer can be evaluated without a detailed knowledge of the convective turbulence that drives the entrainment and the resulting rise of the inversion base. For example, if $\gamma=0$ (i.e. $\alpha=0$), (11) becomes

$$\frac{h\Delta}{h_0\Delta_0} = 1 - \tau. \quad (14)$$

This states that the inversion fills up completely when $\tau=1$, i.e., when

$$t = \frac{h_0\Delta_0}{[\overline{\theta w}]_0}. \quad (15)$$

A few numbers may be helpful. If $h_0=200$ m, $\Delta_0=5$ K and $[\overline{\theta w}]_0=0.2$ K m sec⁻¹, then $t=5000$ sec, or about an hour and a half. This shows again that a positive value of γ is required to maintain a distinct inversion (one with finite Δ) during the course of the day.

Another example is the case for which both $\gamma=0$ and $(\overline{\theta w})_0=0$ (as might happen on a day where a cloud deck, entering the scene around sunrise, prevents heating). In this case

$$\Delta h = \Delta_0 h_0, \quad (16)$$

so that the situation is determined entirely by the initial

conditions. Here, the inversion strength gradually weakens as the inversion base rises. Even a boundary layer with zero heat flux at the surface can entrain an inversion, even though this probably goes quite slowly (see also Section 4).

A third case of interest is the generation of a frictional inversion of finite strength by a boundary layer whose surface heat flux is zero. Suppose the initial strength Δ_0 of the inversion is zero. In this case, (10) reduces to

$$h\Delta = \frac{1}{2}\gamma(h^2 - h_0^2). \tag{17}$$

Here the strength of the inversion increases as h increases. This process probably is quite slow (see also Section 4).

Due to the entrainment aloft, the boundary layers described by (16) and (17) warm up as time proceeds, though the surface heat flux is zero. This shows how difficult it is to find a truly neutral (adiabatic) boundary layer: boundary layers with zero heat flux at the surface are likely to have a downward heat flux (with the greatest value of the negative flux occurring at the inversion base) throughout their interior. On basis of their flux Richardson numbers, such layers would have to be classified as stable.

By far the most interesting special case occurs when the left-hand side of (10) formally—and without justification—is uncoupled from its right-hand side. In this case (which is not likely to be physically realistic), we have

$$h\Delta = h_0\Delta_0, \tag{18}$$

and

$$h^2 = h_0^2 + \frac{2}{\gamma}[\overline{\theta w}]_0 t. \tag{19}$$

The second of these expressions is virtually identical with the equations derived by Summers (1965), Dardorff *et al.* (1969), and Tennekes (1973); it was also used in an air pollution study by Leahey and Friend (1971). The derivation of (19) in those papers was based on a similarity theory in which the kinematics of Δ was ignored. We now see that that theory omits the left-hand side of (10). This procedure is justifiable as a first approximation only in conditions such that the two terms of (18) are both much smaller than the terms of (19). Those circumstances do occur (see Section 4); we shall see that the error involved in the use of (19) then amounts to some 20%. It should be pointed out that (19) should not be used in conjunction with (18), because (19) is based on the premise that (18) can be ignored. In other words, on occasions where (19) is a useful, though crude, approximation, Eq. (18) is completely off the mark.

Because $\frac{1}{2}dh^2/dt$ defines an “exchange” coefficient (Tennekes and Lumley, 1972, Chaps. 2 and 7), the exchange coefficient K_i for inversion entrainment ac-

ording to (19) is given by

$$K_i = \frac{[\overline{\theta w}]_0}{\gamma}. \tag{20}$$

If $\gamma = 5\text{K km}^{-1}$ and $[\overline{\theta w}]_0 = 0.2\text{K m sec}^{-1}$ are representative numbers, we find that K_i is of the order of $40\text{ m}^2\text{ sec}^{-1}$. Taking a typical value for h , i.e., $h = 1000\text{ m}$, we obtain $dh/dt = 4\text{ cm sec}^{-1}$ or 144 m hr^{-1} . The entrainment rate is thus appreciably smaller than the turbulent velocity fluctuations in a convective boundary layer, which are easily of the order of 1 m sec^{-1} .

4. The heat flux at the inversion base

The problems discussed in the preceding section force us to consider the downward heat flux at the inversion base. This is also clear from nature of the basic equation (8): it contains two unknowns, so that an additional assumption or postulate is required to make the problem determinate. In simple terms, the issue is this: how can the turbulent motion inside the boundary layer maintain a downward heat flux near its top? Warm air is transported down into the boundary layer; since warm air prefers to rise, this implies that a certain amount of power has to be expended in order to carry the entrained air into the boundary layer. The requirement for power expenditure suggests that the equation for the kinetic energy of the turbulence should be consulted [we note that Kraus and Turner (1967) used what appears to be a somewhat different approach]. In an approximation suggested by Lumley and Panofsky (1964, p. 75), the energy budget near the inversion base reads

$$\frac{g}{T_0} \overline{\theta w} = -\frac{\partial}{\partial z} \left(\frac{1}{2} \overline{q^2 w} \right) + \epsilon, \tag{21}$$

where g/T_0 is the buoyancy parameter, $\frac{1}{2}\overline{q^2 w}$ the turbulent flux of kinetic energy, and ϵ the rate of dissipation of kinetic energy. Near the outer boundaries of a turbulent flow the dissipation rate generally is relatively small (Tennekes and Lumley, 1972, Chap. 4), so that the last term of (21) may be ignored for our present purposes. A similar assumption was made by Ball (1960). As (21) shows, there needs to be a flux convergence of kinetic energy to maintain a downward heat flux, just as we had expected. Considering the dimensions of this flux convergence, we are led to the postulate that it should be proportional to σ_w^3/h , where σ_w is the standard deviation of vertical velocity in the bulk of the boundary layer and h , as before, is the height of the inversion base. With this assumption, the heat flux at the inversion base becomes (apart from an unknown numerical coefficient)

$$-(\overline{\theta w})_i = \frac{T_0 \sigma_w^3}{g h}. \tag{22}$$

This expression cannot be recommended when h becomes so small that the dynamics of the surface layer dominates that of the entire boundary layer. A tentative lower limit for h in this context is put at 100 m.

If the boundary layer is in a state of free convection, the kinetic energy of the turbulence is maintained almost exclusively by buoyancy. In this situation, the following estimate has proved to be quite accurate (Tennekes, 1970; Deardorff, 1972; and several other sources):

$$\sigma_w^3 = 0.2 \frac{\overline{(\theta w)}_0 g h}{T_0} \quad (23)$$

Substitution of (23) into (22) yields

$$-(\overline{\theta w})_i = 0.2 (\overline{\theta w})_0 \quad (24)$$

This simple result agrees quite well with the available experimental data (Deardorff *et al.*, 1969); it also confirms the result obtained by Lilly (1968), which is identical to (24) except for the value of the coefficient (0.1 in Lilly's expression).

With (24), the problem of the description of the kinematics of an inversion above a strongly unstable boundary layer becomes determinate. It turns out that an explicit solution of the system (1), (7), (24) leads to extremely complicated analytical expressions. However, it is relatively easy to obtain an approximate solution of the problem in an implicit form. If the term $h d\Delta/dt$ occurring in (7) is written as $h(d\Delta/dh)(dh/dt)$, there results, with some rearrangement

$$\left(h \frac{d\Delta}{dh} + \Delta - \gamma h \right) \frac{dh}{dt} = -(\overline{\theta w})_0 \quad (25)$$

On the other hand, substitution of (24) into (1) yields

$$5\Delta \frac{dh}{dt} = (\overline{\theta w})_0 \quad (26)$$

Combining (25) and (26), we obtain

$$\frac{d\Delta}{dh} + 6\Delta - \gamma h = 0 \quad (27)$$

The solution of (27) is

$$\Delta h^6 = \Delta_0 h_0^6 + \frac{\gamma}{7} (h^7 - h_0^7) \quad (28)$$

Because of the large powers of h occurring in (28), this equation rapidly loses its dependence on the initial conditions. This seems reasonable because the entrainment is controlled by the dynamics of the turbulence and not so much by the initial conditions. If $h = 2h_0$, for example, h^7 is 132 times as large as h_0^7 . Therefore,

an excellent approximation to (28) for all times such that $h > 2h_0$ is

$$\Delta \approx \frac{\gamma h}{7} \quad (29)$$

This is the proper time to discuss the effect of the numerical coefficient in the estimate (24) of the heat flux at the inversion base. If that were 0.1 for example, the term 6Δ in (27) would become 11Δ , and the final approximate solution would be identical to (29), except for a factor 12 in the denominator where (29) has a 7. Therefore, the general behavior of (29) is independent of the value of the coefficient, but the numerical value of Δ differs if a different coefficient is used. Since it is unlikely that the coefficient in (24) would ever be less than 0.1 or larger than 0.2 (Deardorff *et al.*, 1969), the numerical uncertainty is quite limited.

It is consistent with the approximations involved in (29) to ignore the initial conditions in (10) when we want to compute the entrainment rate of a strongly unstable boundary layer. Neglecting $h_0 \Delta_0$ and h_0^2 in (10), we obtain

$$h\Delta \approx \frac{1}{2} \gamma h^2 - [\overline{\theta w}]_0 t \quad (30)$$

Substitution of (29) into (30) yields

$$h^2 \approx \frac{14}{5} \frac{[\overline{\theta w}]_0}{\gamma} t \quad (31)$$

This differs only in the numerical coefficient from the "asymptotic" form of (19):

$$h^2 \approx 2 \frac{[\overline{\theta w}]_0}{\gamma} t \quad (32)$$

We conclude that the general nature of the relation between h and t is not affected by the assumptions which need to be made about the heat flux at the inversion base. In other words, (19) is not a poor approximation as far as the parabolic increase of h with t is concerned. Clearly, the way in which h increases with time is quite insensitive to the sign and the magnitude of the rate of change of Δ . This is a great practical advantage, because it implies that equations like (31) or (32) can be used without worrying too much about the rate of change of the inversion strength. The major difference between (31) and (32) is that the relatively strong downward heat flux at the inversion base predicted by (24) increases the entrainment rate by some 20% above the value predicted by (19). It is nevertheless worthwhile to point out that (29) does not lead to unrealistically large values of Δ . If $\gamma = 5\text{K km}^{-1}$ and $h = 1.4\text{ km}$, the inversion strength predicted by (31) is equal to 1K; that seems a reasonable figure.

Because an equation for the downward heat flux a

the inversion base is available, we cannot resist the temptation to take a minor detour and discuss the entrainment caused by boundary layers whose surface heat flux is zero. In that case, the kinetic energy of the turbulence is of order u_*^2 , where u_* is the surface friction velocity. Substituting this estimate into (22), we obtain

$$-(\overline{\theta w})_i = A \frac{T_0 u_*^3}{g h} \tag{33}$$

The unknown coefficient A presumably is of order one. Substitution of (33) into (1) yields

$$h \frac{dh}{dt} = A \frac{T_0}{g} u_*^3 \tag{34}$$

This relation was obtained first (on basis of dimensional analysis) by Kato and Phillips (1969), who studied the rate of entrainment associated with thermocline erosion in the ocean. Their conclusions confirmed the results obtained in an earlier experiment by Turner (1968). The laboratory data obtained by Kato and Phillips suggest that the coefficient A should be about 2.5; we shall use that number for lack of appropriate atmospheric information. We note that in liquids the fractional temperature jump Δ/T_0 occurring in (34) has to be replaced by the fractional density jump $\Delta\rho/\rho_0$.

When the surface heat flux is zero, (8) reduces to

$$h\Delta = h_0\Delta_0 + \frac{1}{2}\gamma(h^2 - h_0^2) \tag{35}$$

The system (34), (35) can easily be integrated. For our purposes an integration for the special case in which $h_0\Delta_0 = \frac{1}{2}\gamma h_0^2$ will be quite sufficient. This case is obtained also if $h_0 = 0$, and the solution obtained this way is asymptotically equal to the full solution of (34), (35). With $h_0\Delta_0 = \frac{1}{2}\gamma h_0^2$, we find that

$$\frac{1}{2}\gamma h^2 \frac{dh}{dt} = 2.5 \frac{T_0}{g} u_*^3 \tag{36}$$

This integrates to

$$h^3 - h_0^3 = u_*^3 \left(\frac{15T_0 t}{\gamma g} \right) \tag{37}$$

At large values of h/h_0 , the height h is given by (cf. Kato and Phillips, 1969)

$$h \approx u_* \left(\frac{15T_0 t}{\gamma g} \right)^{\frac{1}{3}} \tag{38}$$

This is a surprising result, because h is not proportional to the square root of t , as in most of the other examples we discussed, but proportional to the cube root of t . The inversion strength associated with (38) is given by

$$\Delta \approx u_* \left(\frac{7.5T_0}{g} \right)^{\frac{1}{3}} \left(\frac{\gamma}{2} \right)^{\frac{2}{3}} t^{\frac{2}{3}} \tag{39}$$

it increases quite slowly with time. The implicit expression for Δ is not so complicated; it reads

$$\Delta = \gamma h / 2 \tag{40}$$

This shows that the inversion strength is proportional to the lapse rate γ of the stable air above the inversion base and to the height of the boundary layer—a realistic result that is quite similar to (29). Clearly, continued entrainment increases the strength of this frictional inversion. The entrainment rate predicted by (38) is

$$\frac{dh}{dt} \approx \frac{1}{3} \frac{h}{t} \approx u_* \left(\frac{5T_0}{9\gamma g} \right)^{\frac{1}{3}} t^{-\frac{2}{3}} \tag{41}$$

indicating again that a boundary layer with zero surface heat flux entrains very slowly.

The effects of the initial conditions are demonstrated most clearly by considering a situation in which $\gamma = 0$. At first sight, this appears to be a rather artificial case. However, if Δ is large, so that the entrainment rate is small, h^2 remains close to h_0^2 , so that

$$\Delta_0 h_0 \gg \frac{1}{2}\gamma(h^2 - h_0^2) \tag{42}$$

In this case, (35) may be approximated by

$$h\Delta = h_0\Delta_0 \tag{43}$$

This equation thus refers to a situation where the initial value of $h_0\Delta_0$ is so large that the lapse rate above the inversion base plays no role. If we now substitute (34), the entrainment rate of a neutral boundary layer, into (43), we obtain

$$\frac{dh}{dt} = 2.5 \frac{T_0 u_*^3}{g \Delta_0 h_0} \tag{44}$$

This integrates to

$$h = h_0 + 2.5 \frac{T_0 u_*^3}{g \Delta_0 h_0} t \tag{45}$$

If $T_0 = 300\text{K}$, $u_* = 0.1 \text{ m}^2 \text{ sec}^{-2}$, $g = 10 \text{ m sec}^{-2}$, and $\Delta_0 h_0 = 250\text{K m}$, this becomes

$$h = h_0 + 0.03 u_* t \tag{46}$$

Clearly, the entrainment rate dh/dt obtained this way is extremely small: if $u_* = 0.3 \text{ m sec}^{-1}$, $dh/dt = 0.01 \text{ m sec}^{-1}$ approximately. This means that the inversion base rises only about 100 m over a period of 3 hr.

The inversion strength decreases extremely slowly in cases of this type. Substituting (46) into (44), we obtain

$$\Delta = \frac{\Delta_0}{1 + 0.03 u_* t / h_0} \tag{47}$$

The inversion strength is reduced to half its initial value in a time equal to $33 h_0 / u_*$. Thus, if $h_0 = 100 \text{ m}$, $u_* = 0.3$

m sec⁻¹, it then takes 3 hr before $\Delta = \frac{1}{2}\Delta_0$. This conclusion agrees with experience: if there is no surface heat flux and if the initial inversion strength is large, it is unlikely that the inversion erodes away before the end of the day. In this context, the limitations of (38) also become apparent: unless it happens that $h_0\Delta_0 = \frac{1}{2}\gamma h_0^2$, solutions of the type (38) will be obtained only if $\Delta_0 = 0$ and h_0 is small enough. In other words, the inversion-filling process described by (46) and (47) is so slow that the boundary layer may not reach the next stage of entrainment [the one described by (38)] during the course of the day.

5. Convection in the morning hours

During the first few hours after sunrise, (10) is not a good approximation to (8), because the surface heat flux starts from zero at a time near sunrise, and increases roughly linearly thereafter (Deardorff, 1967). Let us approximate $(\overline{\theta w})_0$ in (8) by

$$(\overline{\theta w})_0 = \frac{(\overline{\theta w})_n}{\mathcal{T}} t, \quad (48)$$

where $(\overline{\theta w})_n$ is the surface heat flux at noon and \mathcal{T} is the time scale of the initial increase of flux with time. From Fig. 9 in Deardorff's paper (1967) it appears that \mathcal{T} will be about 3 hr in many cases. Integration of (8) with the aid of (48) yields

$$h\Delta - h_0\Delta_0 = \frac{1}{2}\gamma(h^2 - h_0^2) - \frac{1}{2} \frac{(\overline{\theta w})_n}{\mathcal{T}} t^2. \quad (49)$$

Alerted by the results obtained at the end of Section 4, we realize that the first issue is that of the time it takes before the nocturnal inversion (small h_0 , but large Δ_0 and a fairly large value of $\Delta_0 h_0$) that exists at sunrise is filled by heating from below. Clearly, penetrative convection with an appreciable entrainment rate can occur only after most of the initial inversion has been filled in. During the initial heating, Δ is fairly large, so that the entrainment rate is slow [this is evident from (26), the entrainment equation of a convective boundary layer]. Therefore, $h^2 - h_0^2$ remains small during the initial heating, and the term with γ in (49) may be neglected. In other words, if the initial product $h_0\Delta_0$ is large enough, the lapse rate above the inversion base is immaterial. The initial filling of the nocturnal inversion—which we propose to call the “morning transient”—is thus described by

$$\frac{h\Delta}{h_0\Delta_0} = 1 - \frac{(\overline{\theta w})_n}{2\mathcal{T}h_0\Delta_0} t^2. \quad (50)$$

The time it takes to remove this inversion is

$$t = \left[\frac{2\mathcal{T}h_0\Delta_0}{(\overline{\theta w})_n} \right]^{\frac{1}{2}}. \quad (51)$$

This equation defines the time scale of the morning transient. If $h_0\Delta_0 = 100\text{K m}$, $\mathcal{T} = 10^4 \text{ sec}$ ($\sim 2.5 \text{ hr}$), and $(\overline{\theta w})_n = 0.5\text{K m sec}^{-1}$, this time scale is 2000 sec—slightly more than half an hour. However, the time scale increases rapidly as the initial inversion becomes stronger. For example, with $\Delta_0 = 10\text{K}$, $h_0 = 100 \text{ m}$, it takes 2 hr to fill it up [we again took $(\overline{\theta w})_n = 0.5\text{K m sec}^{-1}$]. Also, the time scale increases as the surface heat flux decreases; therefore, the morning transient in winter is slower than the one in summer. Let us take a fairly extreme case: with $\Delta_0 h_0 = 1000\text{K m}$, $\mathcal{T} = 10^4 \text{ sec}$, $(\overline{\theta w})_n = 0.1\text{K m sec}^{-1}$, the time scale becomes $\sim 4 \text{ hr}$. This problem is of great relevance to air pollution: strong inversions occurring in wintertime may never reach the stage where penetrative convection can increase the height of the mixed layer appreciably. In such a case, the concentration of pollutants—which depends very strongly on the mixing depth h —is likely to reach extreme values.

The morning-transient problem defined by (51) obviously is so important that it is necessary to define $h_0\Delta_0$ in such a way that it can be computed from a temperature sounding, without having to rely on the schematic state of affairs postulated in Fig. 1. Now, in the model of Fig. 1, $c_p\rho h_0\Delta_0$ is the enthalpy deficit of all the air below the inversion base relative to the air above [the lapse rate γ above the inversion base is taken to be zero, to conform with the approximation involved in Eq. (50)]. If the potential temperature above the inversion base is denoted by Θ_a , we may thus define an enthalpy-deficit integral (per unit horizontal area) such that

$$c_p\rho \int_0^{z>h} (\Theta_a - \Theta) dz \equiv c_p\rho\Delta h. \quad (52)$$

If the integral is taken at $t=0$, there results

$$\Delta_0 h_0 \equiv \int_0^{z>h_0} (\Theta_a - \Theta)_{t=0} dz. \quad (53)$$

With the aid of (53), the value $\Delta_0 h_0$ can be computed easily if a temperature sounding is available (Fig. 2). Note that the upper limit of integration may be any z that is higher than the inversion base h_0 , because the integrand of (53) is zero for $z > h_0$.

The procedure described above is valid only if $h_0\Delta_0 \gg \frac{1}{2}\gamma h_0^2$. In cases where this inequality does not hold, the following procedure is recommended. Extrapolate the potential temperature profile above the inversion base downward, to define an ambient temperature $\Theta_{a,0}$ at the surface. Then,

$$\Theta_a = \Theta_{a,0} + \gamma z. \quad (54)$$

If we now integrate $\Theta_a - \Theta$ with respect to z , that integral becomes independent of the location of the upper in-

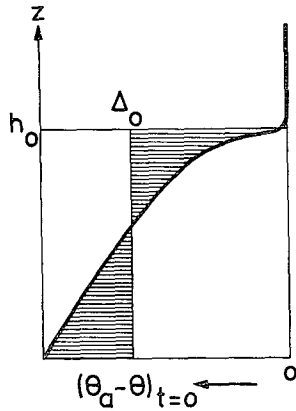


FIG. 2. The calculation of $h_0\Delta_0$ from a temperature sounding if the lapse rate above the inversion base is negligibly small. The two shaded portions have equal areas; the calculation of h_0 and Δ_0 individually is possible only if h_0 can be defined from the sounding.

tegration limit. Therefore, we may define

$$\Delta_0 h_0 - \frac{1}{2} \gamma h_0^2 \equiv \int_0^{z>h_0} (\Theta_a - \Theta)_{t=0} dz. \quad (55)$$

The geometry involved here is sketched in Fig. 3. Note that the individual values of Δ_0 and h_0 cannot be found this way if the height of the inversion base is not fairly distinct; however, that problem is not crucial as far as the integration of (8) is concerned.

It is worthwhile to observe that relations such as (52), (53) and (55) shed some additional light on the nature of the basic equation (8). Apparently, (8) states that the enthalpy-deficit integral of the air in the boundary layer is equal to the sum of its initial value, the enthalpy deficit gained by entrainment into the stable air above the inversion base, and the enthalpy-deficit decrease caused by heating from below.

Let us now return to convection in the morning hours. After the morning transient has run its course, Δ has become small. Therefore, for penetrative convection in the morning we may assume that Δ_0 is negligible if it takes only a short time before the morning transient has decayed. Under these conditions, (49) becomes

$$h\Delta = \frac{1}{2} \gamma (h^2 - h_0^2) - \frac{1}{2} \frac{(\overline{\theta w})_n}{\mathcal{T}} t^2. \quad (56)$$

The associated entrainment rate probably is fairly large, because (56) suggests that h should be approximately proportional to t . As soon as $h > 3h_0$ (say), the term $\frac{1}{2} \gamma h_0^2$ in (56) may be neglected. Also, the solution (28) of the convective entrainment equation (26) simplifies to (29) if $h > 3h_0$. Therefore, we may ignore the initial conditions in (28) and (56) if we are willing to restrict our interest to times for which $h > 3h_0$. In this limit, (28) becomes (29), and the height h of the

inversion base is governed by

$$h \approx \left[\frac{7 (\overline{\theta w})_n}{5 \gamma \mathcal{T}} \right]^{1/2} t. \quad (57)$$

If $(\overline{\theta w})_n = 0.5 \text{ K m sec}^{-1}$, $\gamma = 5 \text{ K km}^{-1}$ and $\mathcal{T} = 10^4 \text{ sec}$, the entrainment rate is about 12 cm sec^{-1} , i.e., $\sim 300 \text{ m hr}^{-1}$. It is interesting that the parabolic dependence of h on t at later times during the day is preceded by a period in which the dependence is linear. Probably by coincidence, this behavior is similar to that of the classical dispersion problem. If we define a velocity scale v by

$$v^2 = \frac{7 (\overline{\theta w})_n}{5 \gamma \mathcal{T}}, \quad (58)$$

and if we put $(\overline{\theta w})_n = [(\overline{\theta w})_n]_0$ (for demonstration purposes only!), then (31) and (57) may be written as

$$h^2 \approx \begin{cases} v^2 t^2 & (\text{morning}) \end{cases} \quad (59)$$

$$\begin{cases} 2v^2 \mathcal{T} t & (\text{afternoon}) \end{cases} \quad (60)$$

These expressions are formally identical with those of dispersion in isotropic turbulence (Monin and Yaglom, 1971). The occurrence of the time scale \mathcal{T} of the daily heating cycle also gives a clue as to the applicability of (59) and (60): the morning-convection equation (59) should be used only for times smaller than \mathcal{T} , and the afternoon-convection equation (60), strictly speaking, is an asymptotic form that pertains to times large compared to \mathcal{T} .

An illustration of the results obtained here is given in Fig. 4. During the morning transient, the inversion base lifts very slowly, while the strength of the inversion decreases (period A). After the initial inversion has been filled in to a point where penetrative convection can begin, the height of the inversion base begins to rise linearly with time, at a relatively fast rate (assuming that the morning transient has decayed well before

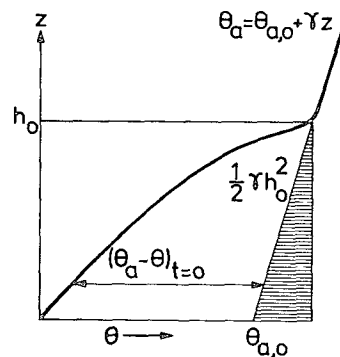


FIG. 3. The calculation of $h_0\Delta_0 - \frac{1}{2} \gamma h_0^2$ from a temperature sounding. The shaded area is equal to $\frac{1}{2} \gamma h_0^2$; if the inversion base h_0 is not clearly recognizable in the sounding, only $h_0\Delta_0 - \frac{1}{2} \gamma h_0^2$ can be computed with (55), but not Δ_0 and h_0 individually.

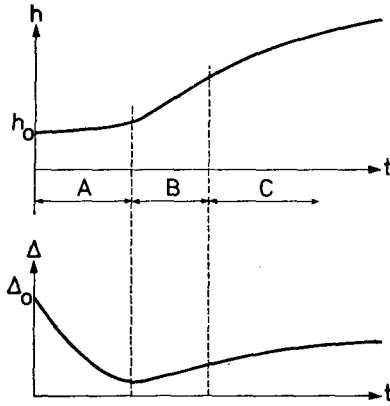


FIG. 4. The variation of h and Δ during the course of a typical day. The period (A) is the morning transient; it is followed by "morning convection" (B), and later by "afternoon convection" (C).

noon). Later in the day, the height of the inversion base continues to increase, but at a rate approximately proportional to the square root of time. During the entire period of penetrative convection (B+C) the inversion strength Δ is proportional to γh , as shown by (29). This sequence of events is quite similar to that observed by Deardorff (1967).

6. Concluding remarks

The results obtained in this paper show that the system of equations for the description of inversion dynamics is capable of explaining a wide variety of situations. Within the scope of this paper it is impossible to make an exhaustive catalogue of all cases that might occur; it is recommended that a digital computer be enlisted for this task. In order to treat cases in which the boundary layer is not in a state of free convection, an interpolation formula between (24) and (33) is needed. A convenient interpolation, based on (22), is

$$-(\overline{\theta w})_i = \Delta \frac{dh}{dt} = 2.5 \frac{T_0 u_*^3}{gh} + 0.2 (\overline{\theta w})_0. \quad (61)$$

This type of formula is similar to those used for the wind shear in unstable surface layers (Lumley and Panofsky, 1964). Empirical adjustment of the numerical coefficients in (61) may be necessary to obtain accurate results. As was remarked earlier, approximations such as (22) and (61) should not be used if h is too small. Also, the entrainment velocity dh/dt cannot become infinite; we recommend limiting dh/dt to $0.2 \sigma_w$ [the same σ_w that was used in (22)], because that is the rate at which a boundary layer with $\gamma=0$ and $\Delta=0$ entrains the air aloft (Tennekes and Lumley, 1972, Chaps. 4 and 5).

Together with Eq. (8), rewritten here for convenience

$$h\Delta - h_0\Delta_0 = \frac{1}{2}\gamma(h^2 - h_0^2) - \int_0^t (\overline{\theta w})_0 dt, \quad (62)$$

the entrainment equation (61) completely determines the history of the inversion, provided the daily cycle of $(\overline{\theta w})_0$, the friction velocity u_* , the ambient lapse rate γ , and the initial conditions are known. The value of $\Delta_0 h_0 - \frac{1}{2}\gamma h_0^2$ can be determined with the aid of (55); if the initial inversion is sufficiently distinct to permit an estimate for the initial height h_0 of the inversion base, Δ_0 can be evaluated from (55). As our analysis has shown, the exact values of h_0 and Δ_0 themselves are relatively unimportant after the morning transient (Section 5) has decayed.

The system (61), (62) predicts not only the daily course of h , but also, through (4), the daily cycle of the temperature in the boundary layer. Our theory, therefore, is related to the methods that are used to predict the maximum temperature on sunny days with light winds (e.g., Myers, 1958).

From a mathematical point of view, the system (61), (62) poses very interesting problems because several different time scales are involved and because it is apparently convenient to shift the origin of time such that the equations describing any particular stage in the history of the inversion become relatively simple in first approximation. This situation calls for an asymptotic analysis; it is probable that the occurrence of several time scales will require the employment of two-variable expansions with shifted time origins (Cole, 1968).

It is necessary to stress that this is an exploratory paper. The number of issues raised by the system of equations developed here is so large that it will take considerable time before all of the results have been compared with observations. We hope to present detailed verifications of this theory at a later time. A paper by Carson (1973), based on a theory quite similar to the one presented here, contains several case studies of interest.

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