

Method to Apply Homogeneous-Path Transmittance Models to Inhomogeneous Atmospheres

MICHAEL P. WEINREB AND ARTHUR C. NEUENDORFFER

National Environmental Satellite Service, NOAA, Washington, D. C. 20233

(Manuscript received 25 October 1972)

ABSTRACT

An accurate approximation to calculate atmospheric profiles of transmittance in instrumental spectral intervals is presented. The approximation assumes a known transmittance model; i.e., that the transmittance in the spectral interval of the instrument is known as a function of quantity of absorber, temperature, and total pressure in homogeneous paths. An inhomogeneous atmosphere is treated as a sequence of homogeneous layers. One applies the model by successively rescaling the quantity of absorber. The method is exact for monochromatic radiation. It avoids the computation of mean line strengths, and it is computationally fast. As an example, transmittance profiles are computed for two spectral intervals in which water vapor is the chief absorber. The results are in excellent agreement with the "exact" point-by-point calculation.

1. Introduction

To derive atmospheric temperature and moisture profiles from satellite radiation measurements, it is often necessary to know the transmittances between the satellite and a sequence of levels in the atmosphere in several spectral intervals. For example, if a profile is to be specified over 100 levels, we require knowledge of transmittances between the satellite and each of the 100 levels, as illustrated in Fig. 1. The process of deducing the profiles from satellite radiation data must include rapid calculations of these transmittances. The most accurate method is integration over the band, often called the point-by-point calculation (Drayson, 1966). In this procedure the absorption coefficients in very narrow spectral intervals are computed as sums of contributions from each of the lines in the band and integrated over the atmospheric path. The resulting transmittances for these narrow spectral intervals are then convoluted with the spectral response function of the instrument. This procedure, however, is too time-consuming. For homogeneous paths (constant temperature and pressure) rapid calculations of transmittances can be performed with transmittance models, which express transmittances over a finite spectral interval as a function of temperature, pressure, and quantity of absorber. Since atmospheric paths are not usually homogeneous, a technique is needed to apply the transmittance model to inhomogeneous paths. One well-known technique is the Curtis-Godson approximation (Curtis, 1952; Godson, 1955; Goody, 1964) in which one replaces the inhomogeneous path with an equivalent homogeneous path by defining effective values of pressure and quantity of absorber. This technique still requires the evaluation of mean line

intensities over the spectral response of the instrument at each point in the path.

An accurate and computationally fast approximation for applying transmittance models to inhomogeneous paths, which does not depend on calculation of mean line intensities, is described here, and preliminary results are presented.

2. Description of method

In the new method described herein we treat the problem of applying a known transmittance model to an atmosphere consisting of a sequence of M homogeneous layers, each layer having a specified value of total pressure P and temperature T . Fig. 1 can be used to represent this atmosphere. Each layer is bounded by two levels. The total quantity of absorber between level n and the top of the atmosphere is denoted by U_n . The quantity of absorber in the layer between levels $n-1$ and n is ΔU_n , i.e.,

$$\Delta U_n = U_n - U_{n-1}.$$

To compute the transmittance in the spectral interval of the instrument between the top of the atmosphere and levels 1, 2, . . . M , let τ denote transmittance in the spectral interval of the instrument. Then τ is related to $\tau(\nu)$, the spectral transmittance at wavenumber ν , by

$$\tau = \int_0^{\infty} \tau(\nu) \varphi(\nu) d\nu,$$

where $\varphi(\nu)$ is the normalized spectral response function of the instrument. Loosely speaking, τ is an average of $\tau(\nu)$, and we shall refer to it as such. The transmittance model provides τ for a homogeneous path as a known

function of P , T and U ; i.e.,

$$\tau = \tau(P, T, U).$$

Let P_1, T_1 and P_2, T_2 be the constant pressure and temperature in the first and second layers, respectively. Fig. 2 shows plots of $\tau(P, T, U)$ vs U (transmittance curves) for these two sets of conditions. If τ_1 is the transmittance in the spectral interval of the instrument at level 1, then

$$\tau_1 = \tau(P_1, T_1, U_1),$$

because we have traversed a single homogeneous layer. Thus, we move from A_1 to the point B_1 in Fig. 2. However, if τ_2 denotes the transmittance at level 2, $\tau_2 \neq \tau(P_2, T_2, U_2)$, because τ_2 must depend on both sets of conditions P_1, T_1 and P_2, T_2 . To evaluate exactly the absorption (the decrease in transmittance) occurring in the second layer we should start with the spectral transmittance $\tau_1(\nu)$ at level 1 and modify it with the additional absorption in the second layer to get $\tau_2(\nu)$. The transmittance τ_2 is the convolution of $\tau_2(\nu)$ with the instrument response function. Thus, we would move from B_1 to E in Fig. 2. The segment from B_1 to E does not lie on a transmittance curve for a single homogeneous path, because it describes absorption that occurs both under conditions P_1, T_1 and P_2, T_2 . In fact, B_1E can be computed only by the point-by-point calculation, which we wish to avoid.

The new approximation will use only transmittance curves that apply to the conditions in the M homogeneous layers, i.e., $\tau(P_n, T_n, U), n = 1, 2, \dots, M$. In evaluating the absorption occurring in the second layer we will move along a segment of the curve $\tau(P_2, T_2, U)$ until U has increased by ΔU_2 , the quantity of absorber in layer 2. The crucial step is to find the starting point on the curve $\tau(P_2, T_2, U)$ that leads to the most accurate absorption. Ideally, the start should be the point on $\tau(P_2, T_2, U)$ where the spectrum (i.e., spectral

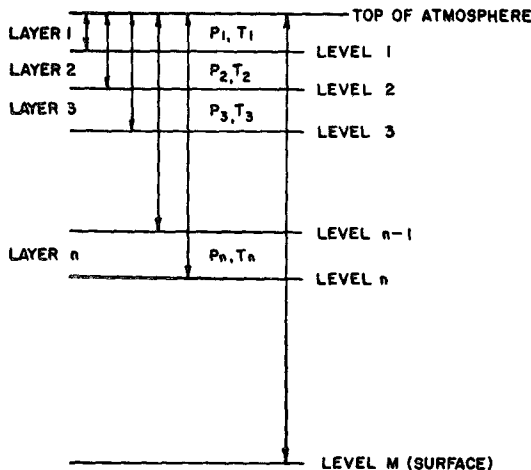


FIG. 1. Multi-level atmosphere.

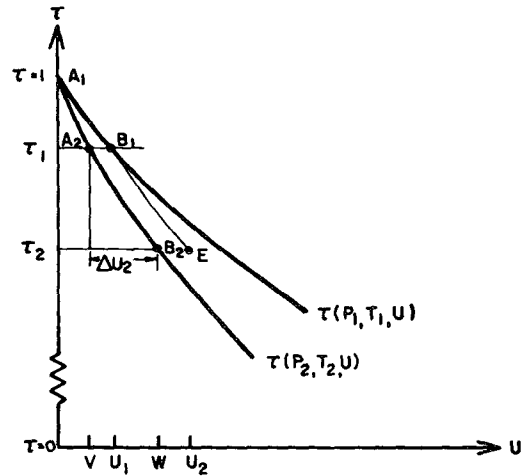


FIG. 2. Transmittance curves for atmospheric layers 1 and 2.

transmittance) is identical with the actual spectrum of the radiation that has traversed the first layer. However, as long as P_1, T_1 does not equal P_2, T_2 , such a point usually will not exist. Instead, in this approximation we select the point on $\tau(P_2, T_2, U)$ where the average of the transmittance spectrum is identical with the average of the spectrum computed exactly, i.e., where $\tau(P_2, T_2, U) = \tau_1$. This condition specifies the point A_2 . We arrive at B_2 , which specifies τ_2 , by moving a distance along $\tau(P_2, T_2, U)$ for which U increases by ΔU_2 . In an exact calculation the absorption occurring in the second layer is determined by the detailed structure of the transmittance spectrum as well as on P_2, T_2 and ΔU_2 . In this approximation the absorption depends on P_2, T_2 and ΔU_2 again, but only on the average of the transmittance spectrum.

This will be a good approximation when the spectrum at A_2 is similar to the exact spectrum, because then A_2B_2 will be nearly parallel to B_1E . Consequently, moving along both curves a distance for which U increases by ΔU_2 will give nearly the same value of τ_2 . If the spectrum at A_2 were identical with the actual spectrum, then the method would be exact.

Note that at B_2 the value of U is not U_2 . Instead it is W , which may be called the effective quantity of absorber for conditions P_2, T_2 , because radiation traversing a single homogeneous layer with conditions P_2, T_2 and W will suffer the same absorption as radiation traversing the first two layers in Fig. 2.

In general, to evaluate the transmittance at level n , given the transmittance at level $n-1$, we desire to select a starting point on the curve $\tau(P_n, T_n, U)$ where the spectrum is identical with the spectrum obtained from the exact calculation at level $n-1$. In this approximation, we select the starting point where $\tau(P_n, T_n, U)$ equals the computed transmittance τ_{n-1} . We then move along the curve a distance for which U increases by ΔU_n , the quantity of absorber in layer n .

The procedure is repeated until τ_M has been computed, and intermediate steps will have yielded $\tau_1, \tau_2, \dots, \tau_{M-1}$, the transmittances between the top of the atmosphere and levels 1, 2, $\dots, M-1$. The validity of the approximation at each level n depends upon how closely the spectrum at A_n matches the spectrum computed exactly at level $n-1$.

An outline of the actual calculations involved in the method now follows: Start with a transmittance model,

$$\tau = \tau(P, T, U).$$

For the transmittance through the first layer, evaluate $\tau(P, T, U)$ at the values P_1, T_1, U_1 . This gives τ_1 . The coordinates U_1, τ_1 specify B_1 in Fig. 2. Next, insert the values P_2, T_2 into the transmittance model and solve the equation

$$\tau(P_2, T_2, U) = \tau_1 \quad (1)$$

for U . If τ is a sufficiently simple function, this can be done exactly. If not, an approximation method, such as Newton's can be used. Denote the solution for U by V . The coordinates V, τ_1 specify A_2 in Fig. 2. Next, add ΔU_2 to V to define W :

$$W = V + \Delta U_2. \quad (2)$$

Finally, evaluate $\tau(P, T, U)$ at the values P_2, T_2, W . This gives τ_2 . The coordinates W, τ_2 specify B_2 in Fig. 2. Continue until all M transmittances have been calculated.

We now point out a limitation of the method. For a multi-layer atmospheric path the method will usually give different results when the direction of the calculation is reversed. That is, if the method is worked downward from level 1 to level M , we will obtain a transmittance from the top of the atmosphere to the surface that, in general, will differ from the transmittance computed upward from level M to level 1. The method differs in this from the exact calculation, which is independent of the direction of calculation. Since the approximation yields different results for different directions of calculation, one may ask which result is more accurate. The answer is that the approximation works best when applied in the direction of increasing rate of absorption. This can be visualized with the following example: Suppose the problem is to calculate the transmittance between the top of the earth's atmosphere and the surface in a spectral interval where water vapor is the chief absorber. If layers 1 through M are taken in order of decreasing height (increasing pressure) in the atmosphere, then the layers will be ordered in the direction of increasing rate of absorption. When the approximation method reaches layer n deep in the atmosphere, the true transmittance spectrum will be determined principally by the absorption which took place in the preceding few layers. The values of P and T in these layers will not be too different from their values in layer n . Therefore, the spectrum at A_n will be similar to the true spectrum. However, if the problem

is worked from the ground up, the layers will be ordered in the direction of decreasing rate of absorption. In considering level n , we will have a situation where most of the absorption will have occurred in layers far below, for which the values of P and T are considerably different. In this case the spectrum at A_n will be quite different from the true spectrum, and the approximation will not be so good.

For monochromatic radiation, the method is exact. In this case the entire relevant transmittance spectrum is the single value of τ_ν at frequency ν . Thus, the spectrum at each A_n will always be identical with the true spectrum. We demonstrate this mathematically for levels 1 and 2, noting in advance that the results of the exact calculation are

$$\begin{aligned} \tau_1 &= \exp[-K_\nu(P_1, T_1)U_1], \\ \tau_2 &= \{\exp[-K_\nu(P_1, T_1)U_1]\} \{\exp[-K_\nu(P_2, T_2)\Delta U_2]\}, \end{aligned}$$

where $K_\nu(P_1, T_1)$ and $K_\nu(P_2, T_2)$ are the known monochromatic absorption coefficients under conditions P_1, T_1 and P_2, T_2 , respectively. In applying the approximation method, we know the functions $\tau(P_1, T_1, U)$ and $\tau(P_2, T_2, U)$. These are given by

$$\begin{aligned} \tau(P_1, T_1, U) &= \exp[-K_\nu(P_1, T_1)U] \\ \tau(P_2, T_2, U) &= \exp[-K_\nu(P_2, T_2)U] \end{aligned} \quad (3)$$

Now, at point B_1 ,

$$\tau_1 = \exp[-K_\nu(P_1, T_1)U_1]. \quad (4)$$

To find A_2 we solve Eq. (1) for U . To do this we insert Eqs. (3) and (4) into Eq. (1) to obtain

$$U = [K_\nu(P_1, T_1)/K_\nu(P_2, T_2)]U_1 \equiv V.$$

Next, Eq. (2) yields

$$W = [K_\nu(P_1, T_1)/K_\nu(P_2, T_2)]U_1 + \Delta U_2.$$

Finally, we solve for τ_2 as

$$\begin{aligned} \tau_2 &= \tau(P_2, T_2, W) = \exp[-K_\nu(P_2, T_2)W] \\ &= \{\exp[-K_\nu(P_1, T_1)U_1]\} \{\exp[-K_\nu(P_2, T_2)\Delta U_2]\}, \end{aligned}$$

which is the exact solution.

This suggests that transmittances for a particular spectral interval can be obtained with more accuracy if the spectral interval is first divided into a number of narrower intervals, and then the transmittance is computed separately for each interval. As the number of divisions increases, the approximation will approach the exact result. This feature, the ability to approach continuously the exact result, is not possessed by scaling approximations, such as the Curtis-Godson method, which are not exact even in the monochromatic case.

Thus far we have demonstrated the new method with an atmosphere consisting of a sequence of homogeneous layers. If we wish to apply the method to the real

atmosphere, in which P and T vary continuously, we must first approximate the real atmosphere by a sequence of homogeneous layers which are sufficiently thin so that representative values of P and T can be found for each. It is emphasized that this is an additional approximation not inherent in this method.

3. Application

As an example of application of the method we calculate atmospheric transmittances in two of the spectral intervals sensed by the Vertical Temperature Profile Radiometer (VTPR). (The VTPR is a satellite instrument which sounds the atmosphere in eight intervals in the infrared to derive temperature and humidity profiles.) The first interval is centered at 535 cm^{-1} in a water vapor rotational band and has a bandwidth of about 8 cm^{-1} . The second interval, centered at 835 cm^{-1} with a bandwidth of about 5 cm^{-1} , senses in a region of continuous absorption and some weak water vapor rotational lines. [For details on the VTPR, see Falbel and Zink (1971).] In this example we treat only the transmittance arising from the water vapor spectral lines. Contributions from other atmospheric gases are assumed to be negligible.

For a transmittance model we used a modification of the polynomial representation of Smith (1969). The representation is

$$\ln(-\ln \tau) = \sum_{i=1}^{14} C_i(\nu) X_i, \quad (5)$$

where $X_1=1$, and

$$\begin{aligned} X_2 &= (0.1) \ln(UT/273), & X_3 &= \ln(P/1000), \\ X_4 &= \ln(T/273), \\ X_5 &= X_2 X_3, & X_6 &= X_2 X_4, & X_7 &= X_2^2, & X_8 &= X_4 X_7, \\ X_9 &= X_3 X_4, & X_{10} &= X_2 X_7, & X_{11} &= X_4 X_6, & X_{12} &= X_4^2, \\ X_{13} &= X_3 X_6, & X_{14} &= X_3 X_7. \end{aligned}$$

We determined the coefficients $C_i(\nu)$ by fitting Eq. (5) by least squares to point-by-point calculations of

TABLE 1. Coefficients for polynomial representation for spectral intervals centered at 535 cm^{-1} and 835 cm^{-1} .

	535 cm^{-1}	835 cm^{-1}
C_1	0.2476	-3.2645
C_2	6.1770	7.4604
C_3	0.5602	0.2524
C_4	1.8218	5.6018
C_5	0.4987	0.7095
C_6	-0.2877	-4.9476
C_7	1.2765	-4.3474
C_8	-1.5985	0.7378
C_9	0.0259	0.3412
C_{10}	2.3265	1.9356
C_{11}	-2.6686	-4.9977
C_{12}	0.5185	-0.9892
C_{13}	0.8032	1.1562
C_{14}	0.0630	-0.8650

TABLE 2. Mean pressures and temperatures at 50 levels, and quantity of absorber between top of atmosphere and these levels.

Level	Mean pressure (mb)	Mean temperature ($^{\circ}\text{K}$)	Total quantity of absorber U (precipitable cm)	
			535 cm^{-1}	835 cm^{-1}
1	100.50	199.8	0.0000	0.0000
2	107.26	200.4	0.0002	0.0004
3	114.32	201.6	0.0005	0.0008
4	121.72	202.9	0.0007	0.0013
5	129.45	204.0	0.0011	0.0018
6	137.52	205.3	0.0014	0.0024
7	145.95	206.6	0.0018	0.0030
8	154.74	208.1	0.0022	0.0037
9	163.91	210.1	0.0027	0.0044
10	173.45	212.0	0.0032	0.0053
11	183.38	214.0	0.0038	0.0063
12	193.72	215.9	0.0044	0.0074
13	204.46	218.3	0.0054	0.0090
14	215.63	221.0	0.0069	0.0115
15	227.22	223.8	0.0091	0.0151
16	239.25	226.5	0.0119	0.0198
17	251.73	229.2	0.0155	0.0258
18	264.66	232.0	0.0199	0.0331
19	278.07	234.8	0.0251	0.0418
20	291.95	237.5	0.0312	0.0519
21	306.32	240.2	0.0393	0.0655
22	321.19	242.5	0.0510	0.0851
23	336.57	244.9	0.0666	0.1110
24	352.46	247.4	0.0862	0.1437
25	368.89	249.8	0.1102	0.1836
26	385.85	252.2	0.1387	0.2312
27	403.37	254.4	0.1726	0.2877
28	421.44	256.4	0.2130	0.3550
29	440.09	258.3	0.2604	0.4339
30	459.32	260.2	0.3152	0.5253
31	479.14	262.2	0.3779	0.6297
32	499.57	264.2	0.4541	0.7568
33	520.61	266.2	0.5563	0.9271
34	542.27	268.3	0.6921	1.153
35	564.58	270.4	0.8635	1.439
36	587.53	272.5	1.072	1.787
37	611.14	274.5	1.320	2.200
38	635.41	276.7	1.610	2.683
39	660.37	278.9	1.943	3.237
40	686.03	280.8	2.321	3.868
41	712.38	282.7	2.769	4.614
42	739.46	284.3	3.314	5.522
43	767.26	285.7	3.960	6.601
44	795.79	287.3	4.715	7.858
45	825.08	288.7	5.582	9.303
46	855.13	290.3	6.591	10.98
47	885.96	292.3	7.789	12.98
48	917.56	294.2	9.197	15.32
49	949.97	296.1	10.82	18.04
50	983.19	298.0	12.68	21.13

transmittance for about 130 homogeneous paths. The conditions P , T , U for all paths were carefully selected to correspond to the range of conditions encountered in downward sensing of real atmospheres. Table 1 lists the $C_i(\nu)$ derived in this way. The standard deviations of these fits to $\ln(-\ln \tau)$ computed point by point were 0.0073 at 535 cm^{-1} and 0.0143 at 835 cm^{-1} .

For the VTPR temperature and humidity retrieval the atmosphere is partitioned into 100 layers distributed on a $P^{2/7}$ scale, where P is total atmospheric pressure. In computing water vapor transmittances we used the lowest 50 of these levels. We thereby approximated the lower portion of the real atmosphere by an atmosphere

TABLE 3. Transmittances calculated between top of atmosphere and indicated level for 835 cm^{-1} and 535 cm^{-1} intervals with conditions in Table 2.

Level	835 cm^{-1}		535 cm^{-1}	
	τ	τ error	τ	τ error
1	1.000	0.0000	1.000	0.0000
2	1.000	0.0000	0.9985	0.0001
3	1.000	0.0000	0.9973	0.0002
4	1.000	0.0000	0.9962	0.0002
5	1.000	0.0000	0.9952	0.0002
6	1.000	0.0000	0.9941	0.0003
7	1.000	0.0000	0.9931	0.0002
8	1.000	0.0000	0.9919	0.0002
9	1.000	0.0000	0.9907	0.0002
10	1.000	0.0000	0.9893	0.0002
11	1.000	0.0000	0.9878	0.0003
12	1.000	0.0000	0.9862	0.0003
13	0.9999	0.0000	0.9840	0.0003
14	0.9999	0.0000	0.9805	0.0004
15	0.9999	0.0000	0.9760	0.0005
16	0.9998	0.0000	0.9705	0.0008
17	0.9997	0.0000	0.9642	0.0010
18	0.9996	0.0000	0.9571	0.0013
19	0.9995	0.0000	0.9493	0.0015
20	0.9993	0.0000	0.9406	0.0019
21	0.9990	0.0001	0.9298	0.0021
22	0.9986	0.0001	0.9156	0.0023
23	0.9981	0.0001	0.8985	0.0025
24	0.9973	0.0001	0.8790	0.0025
25	0.9964	0.0002	0.8574	0.0025
26	0.9953	0.0002	0.8338	0.0023
27	0.9939	0.0002	0.8080	0.0021
28	0.9923	0.0002	0.7799	0.0016
29	0.9905	0.0002	0.7497	0.0009
30	0.9883	0.0002	0.7174	0.0003
31	0.9858	0.0003	0.6835	-0.0006
32	0.9828	0.0003	0.6456	-0.0015
33	0.9788	0.0004	0.5998	-0.0027
34	0.9737	0.0004	0.5461	-0.0037
35	0.9674	0.0004	0.4879	-0.0047
36	0.9601	0.0006	0.4276	-0.0051
37	0.9517	0.0007	0.3678	-0.0052
38	0.9423	0.0010	0.3101	-0.0048
39	0.9320	0.0014	0.2561	-0.0040
40	0.9208	0.0019	0.2068	-0.0028
41	0.9081	0.0026	0.1613	-0.0016
42	0.8938	0.0034	0.1199	-0.0003
43	0.8779	0.0043	0.0849	0.0006
44	0.8607	0.0051	0.0572	0.0009
45	0.8423	0.0057	0.0365	0.0008
46	0.8224	0.0063	0.0217	0.0004
47	0.8004	0.0063	0.0117	0.0001
48	0.7765	0.0058	0.0057	-0.0002
49	0.7510	0.0044	0.0025	-0.0003
50	0.7242	0.0021	0.0009	-0.0001

which is a sequence of 50 homogeneous layers ($M=50$). In Table 2 are shown the mean pressures and temperatures for these layers. Also shown are the quantities of water vapor used in the calculations. These large quantities are typical of paths with large zenith angles. The quantity of absorber in the calculations at 835 cm^{-1} was chosen to be larger than the amount in the calculations at 535 cm^{-1} so that the weak absorption at 835 cm^{-1} would yield transmittances perceptibly different from unity.

The "exact" transmittances that served as the standard for comparison were computed point by point through the layered atmosphere in the computer program provided by H. J. Bolle¹ and were convoluted over the spectral response functions of the VTPR.

The results for both spectral intervals are shown in Table 3, which lists the transmittances computed by the approximation method at each level and their deviations from the "exact" transmittances. In the case of 535 cm^{-1} , the average of the absolute errors over levels 2-50 is 0.0014, and the maximum error is 0.0052. For 835 cm^{-1} , the average of the absolute errors over levels 26-50 is 0.0013, and the maximum error is 0.0063. This example demonstrates that the approximation method can be used to reproduce accurately the "exact" transmittances computed point by point.

¹ Private communication.

REFERENCES

- Curtis, A. R., 1952: Discussion of "A statistical model for water-vapour absorption" by R. M. Goody. *Quart. J. R. Meteor. Soc.*, **78**, 638-640.
- Drayson, S. R., 1966: Atmospheric transmission in the CO₂ bands between 12 μ and 18 μ . *Appl. Opt.*, **5**, 385-391.
- Falbel, G., and D. Zink, 1971: A sequential filter radiometer for sensing the vertical temperature profile of the earth's atmosphere. *Proc. IERE Conf. on Infrared Techniques*, University of Reading, 231-255.
- Godson, W. L., 1955: The computation of infrared transmission by atmospheric water vapor. *J. Meteor.*, **12**, 272-284.
- Goody, R. M., 1964: *Atmospheric Radiation, I. Theoretical Basis*. Oxford, Clarendon Press, 236-243.
- Smith, W. L., 1969: A polynomial representation of carbon dioxide and water vapor transmission. ESSA Tech. Rept. NESC 47, National Environmental Satellite Center, Washington, D. C., 20 pp.