Updating Experiments with a Simple Barotropic Model

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(Manuscript received 2 February 1973, in revised form 6 August 1973)

ABSTRACT

Using a 12-component spectral model derived from the barotropic vorticity equation, updating experiments were performed at intervals ranging from one to six time steps, where one time step was equivalent to about 2.4 hr. The Monte-Carlo and stochastic dynamic methods were used to determine a representative initial error growth for the model. In the first groups of experiments, it was found that when all 12 components were updated at intervals of five time steps or less, the rms vorticity errors eventually converged to zero; but when the updating interval was increased to six time steps, there was no tendency for convergence. In the second group of experiments the effect of not updating the smallest scales (subsynoptic or mesoscale) was considered. The general result was that it was still possible to determine large-scale features rather well with fairly infrequent updating (three and four time steps) and model resolution to the intermediate (synoptic) scales, but intermediate-scale features could be recovered with good accuracy only when updating was done quite rapidly (every time step) or if smaller scale resolution was retained.

1. Introduction

In the past few years the concept has arisen of improving an analysis of a poorly observed meteorological parameter (e.g., wind) through the use of another parameter whose field is better observed (e.g., temperature). This idea has been tested almost exclusively within the context of numerical models. A usual procedure is to perform two numerical integrations with a model. The first run represents "true" atmospheric values; while the second run represents an "updated" field, which starts with hypothetical observations. For example, the initial winds may be chosen to be equal to the "true" values plus fairly large errors, while the initial temperatures may be taken as equal to the "true" values, so as to simulate a poorly observed wind field and an accurately defined temperature field, respectively.[2] The aim is to use the numerical model as an analysis tool by taking the record of "true" temperatures (which simulate observations) and periodically inserting them in the place of the forecast values as determined in the second run. It has been found after a period of time that the forecast wind field may consist of values which have less error than the "observed" winds. However, this is only necessarily true for the model and will depend on the rate and number of temperature insertions.

Most of the work thus far dealing with this subject, i.e., updating, has been oriented primarily toward improving the field of one meteorological parameter by updating another. Hence, any improvement due to updating will depend on how well one variable will adjust to another. This response of one parameter, such as wind, to the updating of another, such as temperature, would seem to be dependent in part on the natural growth of errors.

One may look at the adjustment of winds to an updated temperature field (or the inverse) as a two-part problem. The first part is the adjustment from unbalanced temperature and wind fields to balanced fields. If a model is allowed to run so that there are no effects from initial conditions, then at that time the temperatures and winds should be in near geostrophic balance. Now if the temperature field is corrected but the wind field is left unchanged (or vice versa) both these fields will consist of part Rossby waves and part gravity waves. After a sufficient time the gravity waves may be damped out by the model and the fields will consist of only the Rossby waves, which in general will not be equal to the "correct" fields. However, Williamson and Dickinson (1972) have shown that this value will approach the "correct" value as the number of updates approaches infinity. Therefore, in the absence of error growth, the geostrophic adjustment process should cause an eventual convergence of wind errors to zero, in response to updating with temperatures that have zero error. But results from the updating experiments of Jastrow and Halem (1970) and Williamson and Kasahara (1971) indicate a convergence of wind errors to an asymptote that is clearly greater than zero. This may well be due to the natural growth of errors between successive updates. Therefore,

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[2] These initial conditions are qualitatively similar to those at day 95 in the first experiment by Charney *et al.* (1969).

one must consider the growth of errors as a second part to the updating problem.

It is this second part of the problem, the relation between error growth and updating, that will be the main subject of concern here. Since a barotropic model is employed to investigate this problem, the only parameter which can be updated is wind. If the originally updated field does not converge to the "correct" values, then there would be no hope for any other parameter to adapt to its "correct" value. It should also be noted here that only part of the field may be updated at a particular time, since in a single-parameter model the updated variables "adjust" instantly to their "corrected" values, so if the entire field were to be updated at once all the errors would go to zero and there could be no chance for error growth between updates. This paper will first look at how the variation of updating intervals will affect the convergence of errors in the updated field. The other area of attention will be directed toward the effect that the inability to observe (and therefore update) smaller scales has on the convergence of errors in the larger updated scales.

2. Studies of the model

A spectral model based on the barotropic vorticity equation, with 12 components, was used for the updating experiments (Lorenz, 1972). The governing equation is

$$dY_j/dt=C(2Y_{j-2}Y_{j-1}-3Y_{j-1}Y_{j+1}+Y_{j+1}Y_{j+2}), \quad (1)$$

where j has been allowed to range from 0 to 11. It is assumed here that $Y_j=0$ for $j<0$ and $j>11$. The components have units of vorticity (sec^{-1}), and the jth component is supposed to be a representation of the vorticity of a "typical" wave vector of wavelength $2\pi D/2^{j/2}$, where $2\pi D$ is the wavelength of the longest allowable wave. The enstrophy ($\frac{1}{2}$ vorticity^2) of the jth component is $V_j=\frac{1}{2}Y_j^2$. The energy (kinetic) of the jth component is then given as $E_j=\frac{1}{2}D^2Y_j^2/2^j$. Since half-octave resolution has been chosen for this model, each component j may be viewed as corresponding to a scalar wavenumber, $N=2^{j/2}$. In addition, this component j actually represents a band of wavelengths whose width is equal to $2^{j/2}$ (Lorenz, 1972). Hence, it may be seen from the earlier definition that E_j will be proportional to N^{-2} if all the Y_j are equal, and furthermore that the energy per scalar wavenumber will be proportional to N^{-3}, since there are N wavenumbers per component. In this study the dimensionless constant c in (1) was taken as $\frac{1}{2}$. A more appropriate value might have been 0.38. This means that the entire field will evolve faster here than if $c=0.38$. Hence, the effect for this study is merely to cause the error growth to be about 30% faster than it should be when compared to real time.

The initial error growth for this model is dependent upon the initial distribution of the Y_j's as well as the initial error distribution. This may be seen by formulating a linearized governing equation for the initial error growth:

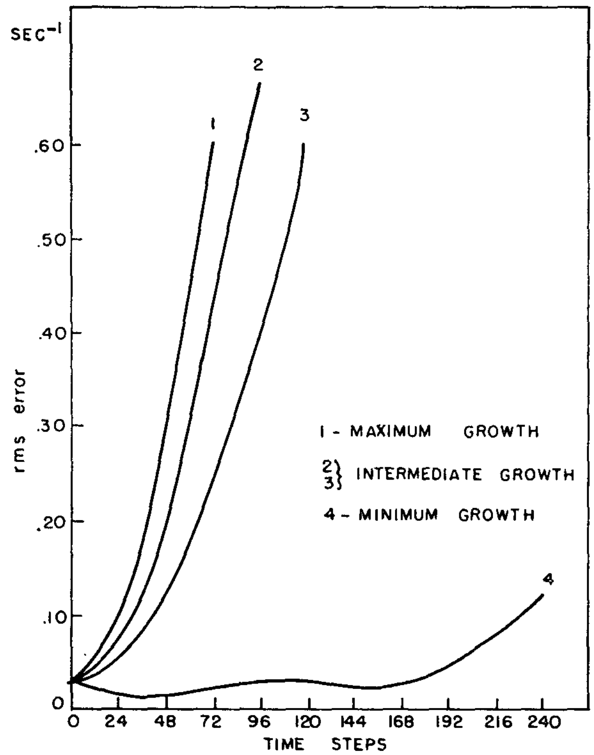

FIG. 1. Various error growth rates for different initial conditions.

$$de_j/dt=c[2Y_{j-1}e_{j-2}+(2Y_{j-2}-3Y_{j+1})e_{j-1} \\ +(-3Y_{j-1}+Y_{j+2})e_{j+1}+Y_{j+1}e_{j+2}], \quad (2)$$

where e_j represents an initial small error in Y_j. If one chooses all the Y_j's approximately equal to the same value Y, then

$$d(\sum_{j=0}^{11} e_j^2)/dt=6cY\sum_{j=0}^{11}(e_je_{j+2}-e_je_{j+1}). \quad (3)$$

Thus, the mean-square error growth is approximately proportional to R_2-R_1, where R_k is the "correlation with regard to j of members whose indices differ by k. In Fig. 1 four cases of different initial error growth are presented. In all the cases there is a "true" state (in which all components have an initial magnitude of 0.50×10^{-4}) and a "perturbed" state. The components of the "perturbed" state are either 0.47×10^{-4} or 0.53×10^{-4}, but in all cases there are an equal number of each magnitude. So the only initial difference is in the distribution and the sign of the errors. In the first case the errors are arranged so as to alternate in sign and thereby cause the maximum initial error growth ($R_1=-1$, $R_2=1$). In the fourth case the errors are arranged so that there are first two of one sign, then two of the other sign, and so on ($R_1=0$, $R_2=-1$).

This will not cause the absolute maximum decay, but the decay of errors will be quite close to the maximum decay rate. The other two cases represent a compromise between the two extremes.

It is quite apparent that there exist various possibilities for initial error growth rates. For this reason the stochastic dynamic and the Monte-Carlo methods were employed to try to determine a representative doubling time for small errors. This was done by looking at the growth of variances, which represent the mean-square uncertainty. A Monte-Carlo calculation starts with an ensemble of initial states and carries these states along with the governing equation. From these states a mean state and a variance may be computed at all times. The stochastic dynamic equations (this name was assigned by E. S. Epstein) in general represent a more complete set which include the governing equations as a subset. In this set the ensemble statistics are the variables. So if there are no uncertainties in the initial mean state, then the stochastic dynamic system will reduce to the basic governing equations. However, if there are uncertainties, then they must be considered in the prediction of the mean state. The chief problem in using the stochastic dynamic system of equations is in applying a closure assumption. Here the simple assumption was made that all third moments were zero for all times. This is not sufficient for many studies but was here, since only initial growth rates were considered. For more details on Monte-Carlo calculations and the stochastic dynamic system, the reader is referred to Fleming (1971).

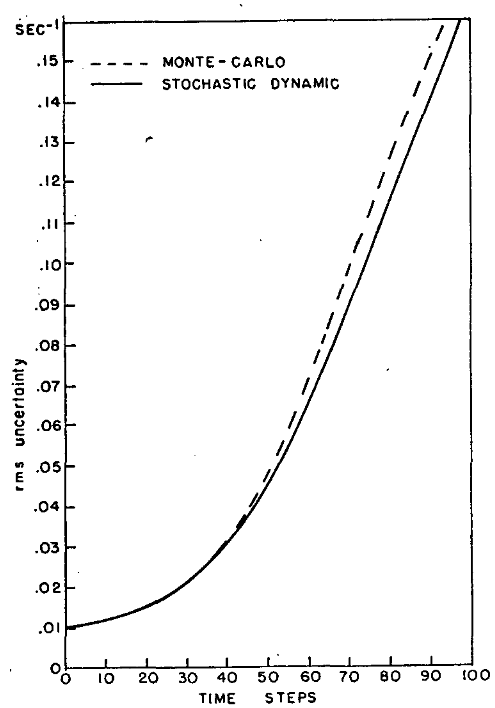

FIG. 2. The growth of uncertainties as studied through the Monte-Carlo and stochastic dynamic methods.

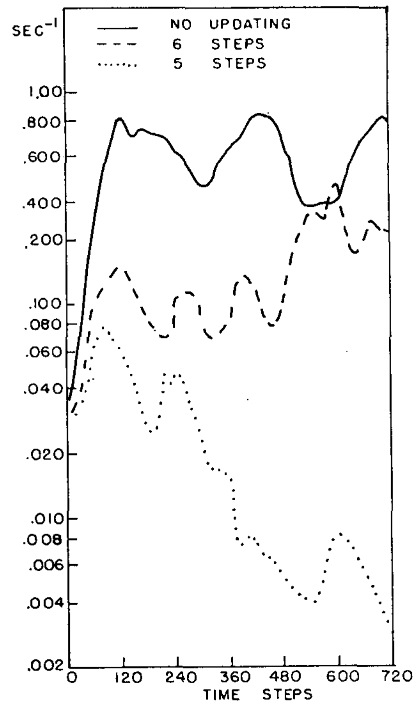

FIG. 3. Error growth for no updating, updating every six time steps, and updating every five time steps.

In Fig. 2 the square root of the variance is plotted against the number of the time steps. The Monte-Carlo method indicates an approximate doubling time for small errors of 16 time steps while the stochastic dynamic method indicates a doubling time of about 18 time steps. On the basis of these statistical results, the initial conditions that were chosen for the updating experiments gave a doubling time of slightly over 16 time steps. It should be noted, however, that the subsequent updating could possibly alter the error configuration so as to cause a change in the error growth rate. Therefore, care has been taken in the choice of an order in which components were to be updated, so as to attempt to prevent the occurrence of a minimum or maximum error growth rate configuration.

The length scale of the model has been arranged so that the spectral component j corresponds to the scalar wavenumber $2^{j/2}$ (Lorenz, 1972). Therefore, the components ranged from wavenumber 1 to approximately wavenumber 45. In the earth's atmosphere these components would correspond to scales ranging from about 30,000 km to 670 km, respectively.

One time step of the model was chosen to correspond to about 2.4 hr, if it is assumed that the total kinetic energy of the model is equal to the total kinetic energy of the atmosphere. Correcting for the constant of 0.38 as discussed earlier, it was found that the doubling time for small errors is a little over 2 days. This is within the lower end of the accepted range of doubling times for small errors in the atmosphere averaged over all scales down to the synoptic scale.

It should be noted here that these times are quite approximate. Later results will be discussed in terms of time steps or doubling times rather than real time.

3. Updating intervals versus error growth

In these experiments a perturbed state, as previously defined, was used to represent an observed atmospheric state with some initial observational errors. During the course of the time integration, the predicted values from the perturbed state were replaced with corresponding values generated from the previously described "true" state. This updating was done one component at a time with the interval between successive updates ranging from two time steps to six time steps. The order of the updating was arranged so that no adjacent components were updated in succession. (The updating here is not being performed so as to simulate any particular grid point updating, but rather just to update parts of the field while also attempting to avoid biasing error growth in any one component relative to the others.)

It was found that the interval between updates could be as large as five time steps and the rms error in the vorticity field would still converge. However, when the updating interval was increased to six time steps, the rms vorticity error showed no tendency to converge. In Fig. 3 the rms vorticity error is plotted against time (to 720 time steps) for normal error growth with no updating, updating every six time steps, and updating every five time steps.

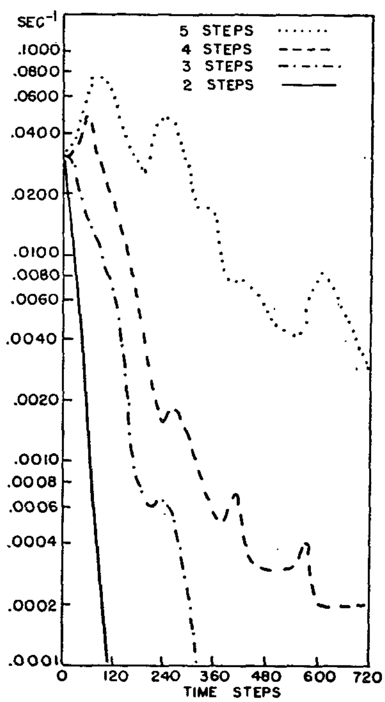

FIG. 4. Error growth for updating every two, three, four and five time steps.

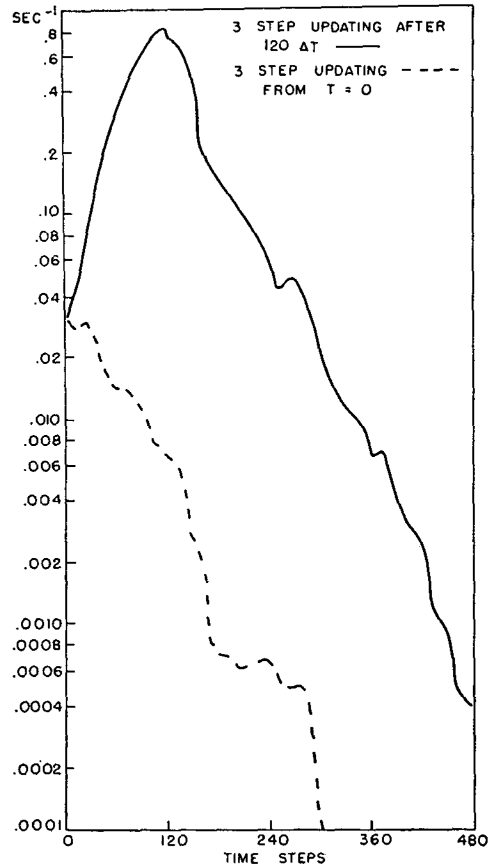

FIG. 5. Error growth after allowing normal error growth until the start of the nonlinear growth phase (120 time steps), then updating at an interval of three time steps. Three-step updating from $T=0$ is plotted as a comparison.

This updating was also done for time intervals of four, three and two time steps. As one would expect after observing the previous results, there was convergence of the rms vorticity errors for all cases. This may be seen in Fig. 4.

The question as to whether the size of the initial error can affect the convergence of errors was also investigated. It was found that the ability for the errors to converge in response to updating was not affected by the size of the initial error. The rate at which this convergence takes place was also unaffected. An example of this may be seen in Fig. 5. In this case the initial conditions were the same as in all the previous updating experiments. However, for this experiment updating was not started immediately as in the former cases. Normal error growth was allowed for 120 time steps (or 7.5 doubling times), bringing one to the nonlinear phase of the natural error growth, as seen in Fig. 3. After 120 time steps the perturbed state was updated every three steps with the corresponding components of the true state as was done previously.

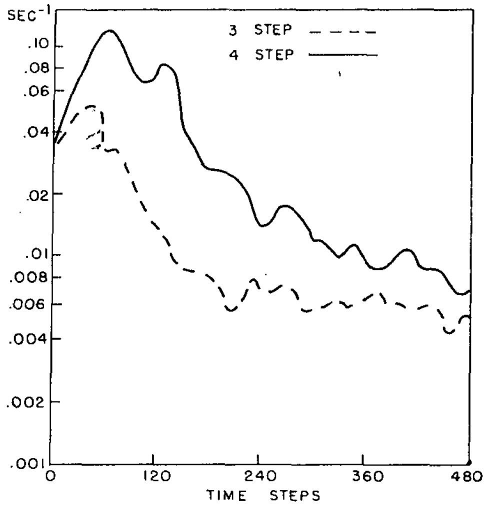

FIG. 6. Growth of the rms vorticity errors when updating is done with the "true" value plus or minus a random error of 0.005, for intervals of three and four time steps.

For a comparison, a plot of the previous three-step updating, which started with an initial error of 0.03 sec^{-1}, is included in Fig. 5. It is fairly clear that the rate of the convergence is not significantly altered by this large initial error, since the slopes of the curves are nearly equal.

One final experiment in which the updating covered the entire field of components, is included. Fig. 6 shows the effect of updating the perturbed state with the "true" values plus or minus a random "observational" error of 0.005. This was carried out for updating intervals of three time steps and four time steps. This experiment was done only for the case of maximum initial error growth (doubling time equals 13 time steps). However, the conclusion that the errors will converge to the "observational" error (0.005 here) should be generally applicable, at least for this model.

4. The effect of not updating the small scales

There is excellent evidence that errors initially confined only to the smallest scales of motion would eventually invade the largest scales of motion (Lorenz, 1969). This may be seen in Fig. 7 where rms wind error is plotted as a percentage of the mean wind value for the largest scales ($j=0, 1, 2, 3$), due to initial errors of about 2.5 m sec^{-1} in only the smallest scales ($j=10, 11$). It is seen that after about 12 doubling times (or 192 time steps) the errors in the largest scales are so large that the nonlinear growth phase has begun.

For this reason it seems quite reasonable to investigate the effect of not updating scales corresponding to the normally unobserved scales of the atmosphere, if there is to be any future hope of updating with real atmospheric data.

In the first group of experiments only components 0 through 7 were updated. This was performed for intervals of one, two and three time steps. The procedure was to update in a sequence involving only the eight components to be updated, rather than using the previous sequence of 12 and skipping the components that are not to be updated. Hence, the entire "observable" field is updated in 8, 16 and 24 time steps for 1, 2 and 3 time step intervals, respectively. In the second group, components 0 through 9 were updated for intervals of one, two, three and four time steps. The same basic updating procedure was followed for this case as was for the case of eight components updated. In the study of the previous section the point between convergence and divergence of errors was adequately seen by looking at the total vorticity error field. However, in this study it might be more advantageous to look at errors in separate scales. The reason is that now convergence of errors will depend on whether the propagation of errors from the small, non-updated scales to the larger scales, in addition to the natural error growth, can be overcome by the removal of errors through updating. Therefore, in some cases, errors may converge for larger scales and not for smaller scales. Also it would be expected that although errors might not become asymptotic to zero they could still converge to a value which is less than the difference be-

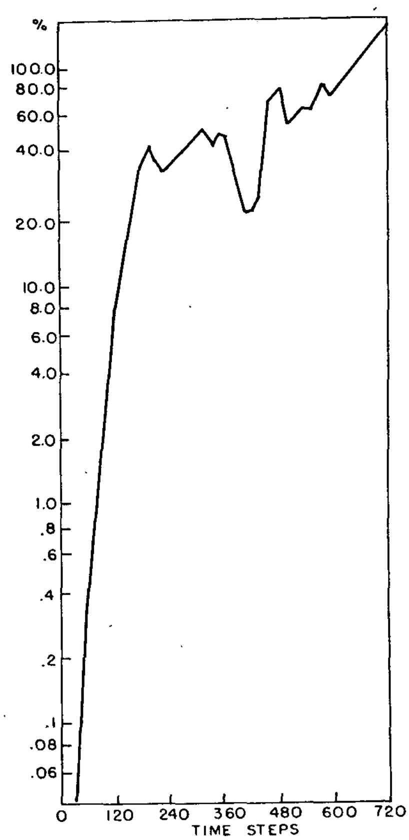

FIG. 7. Growth of percentage wind error in large scales ($j=0, 1, 2, 3$) due to initial error only in smallest scales ($j=10, 11$).

tween randomly chosen states. For this latest reason, in order to get a better feel for the magnitudes involved, it was decided to look at the square root of the mean energy errors (or rms wind errors) rather than rms vorticity errors, since wind is a quantity whose magnitude is more readily observed. This was not done in the previous section because all scales converged to zero and, hence, magnitudes were not as important. The rms wind errors will be given as a percent of the mean wind value in the scales considered, rather than the actual value.

When eight components were updated, the only case in which the rms wind error was clearly asymptotic to zero was for the large scales ($j=0, 1, 2, 3$) when updating was done every step. The large-scale rms wind error growth for eight-component updating may be seen in Fig. 8. For all three intervals (one, two and three time steps) there is an initial rapid decrease in the rms wind errors. Then the errors begin to grow again for two- and three-step updating, but remain asymptotic to zero for one-step updating. However, for both two and three steps the errors leveled off to rather small values. For two-time-step updating, the rms wind error divided by the mean wind oscillated between about 0.05% and 0.5%. The range for three-step updating was from about 0.05% to about 2%.

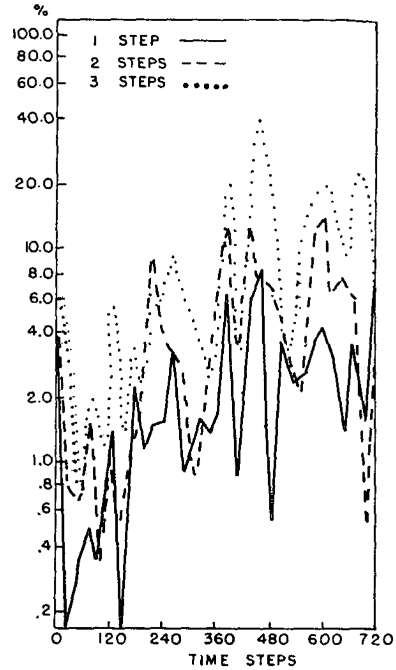

Fig. 9. Growth of percentage wind error in synoptic scales ($j=4, 5, 6, 7$) for updating eight components at intervals of one, two and three time steps.

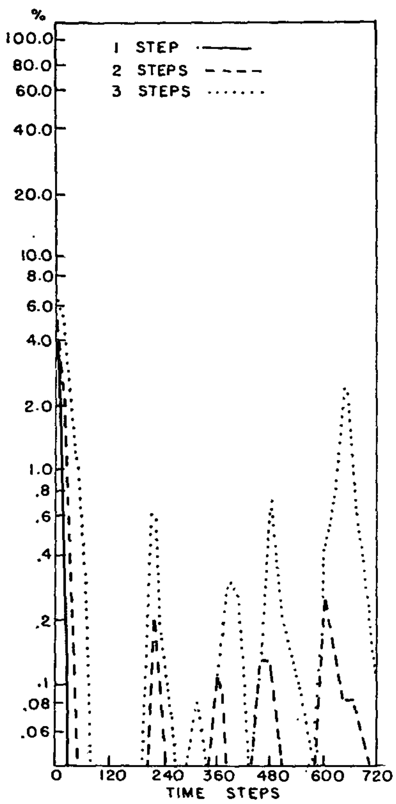

Fig. 8. Growth of percentage wind error in large scales ($j=0, 1, 2, 3$) for updating eight components at intervals of one, two and three time steps.

Intermediate-scale error growth for eight-component updating is observed in Fig. 9. The errors oscillated from about 1% to over 20% of the mean wind values for these scales ($j=4, 5, 6, 7$). After leveling off, the wind errors for three-step updating oscillated about an average of 10%, for two-step updating about 5%, and for updating every step about 3%. These errors are considerably greater than those for the larger scales.

As would be expected, error values for ten-component updating were less than those for only updating eight components. The percentage error in the large-scale wind field when ten components are updated can be observed in Fig. 10, for updating intervals of one, two, three and four time steps. All the rms wind errors remained quite small for this case. Updating every step and every other step caused errors to converge to zero quite rapidly. Three-step updating caused a leveling off of the rms wind error as it oscillated about 0.05% and for four steps the value appears to be just greater than 0.1%. Percentage wind errors for the intermediate scales in this group were also quite a bit better than their eight-component counterparts, as seen in Fig. 11. When updating was done every step, rapid convergence to zero took place once again. Mean error values for two-, three- and four-step updating intervals were about 0.2%, 0.6% and 3%, respectively.

A third study, updating components 0 through 10, produced only questionable results, and are not included here; but they may be found in the thesis upon which this paper is based (Stern, 1972).

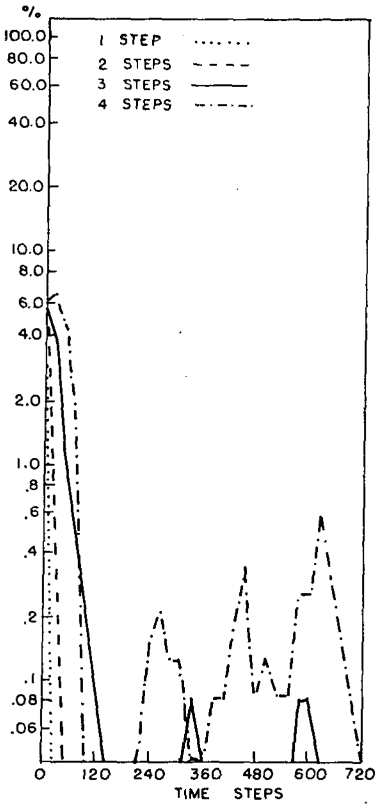

FIG. 10. As in Fig. 8 except for updating ten components and at intervals of one, two, three and four time steps.

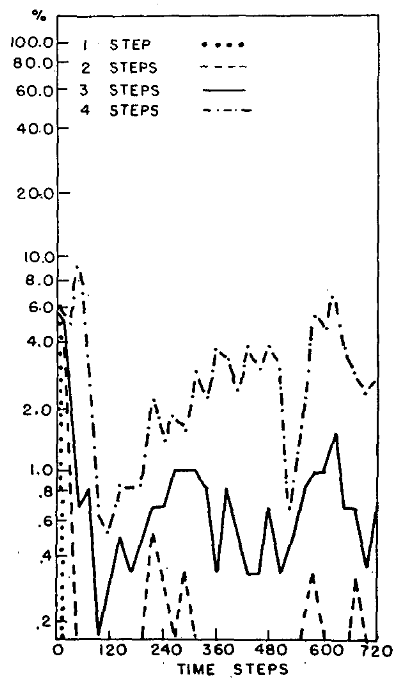

FIG. 11. As in Fig. 9 except for updating ten components and at intervals of one, two, three and four time steps.

5. Conclusions

The first group of experiments have given evidence that there may exist an upper limit to the allowable time interval between successive updates when updating is used to obtain a more accurate field of a desired parameter. This was seen earlier in Fig. 3 as the divergence between the five-step and six-step updating interval curves.

Another point to be derived from these results is the possibility of obtaining the total field of a given parameter even though only small portions of the field are available at any given time. For example, it was shown before that if one updated (with "true" values) as infrequently as one component every five time steps, then after an initial convergence time has elapsed, the *entire* "true" field could be recovered at any time step in the future.

The second group of experiments attempted to investigate the effect that the unobserved scales of motion could have on obtaining the synoptic and large scales through updating. It was found that the large-scale wind could be recovered quite consistently to within 1% of its "true" value with updating only the synoptic scales ($j=4$, 5, 6, 7) and large scales ($j=0$, 1, 2, 3) every three time steps or more often. However, in order to determine the synoptic-scale wind field to within 3% by updating only the synoptic scales or larger, the interval between successive updates had to be reduced to a single time step.

Including the sub-synoptic scales ($j=8$, 9) in the updating procedure, allowed the updating interval to be increased to four time steps for determination of the large-scale winds to within 1%. For 1% recovery of the synoptic-scale wind, an updating interval of no greater than three time steps is required.

From the preceding results, preliminary indications are that, given a good enough model which is mainly limited by its inability to resolve below synoptic scales, the large-scale features of the earth's circulation could be determined through updating with present day observations. One might even be able to use a present multilayer general circulation model. However, it is the author's opinion that accurate recovery of the synoptic-scale wind features would require either better resolution than is being achieved with the present observing systems, an increase in the present observing systems, or an increase in the present number of satellites with infrared sounders, so a high frequency of updates could be achieved, as the results in part four seem to imply. In any case, improvements in the modeling of smaller scale dynamics would probably also have to be made.

Acknowledgments. The author wishes to express his gratitude to Dr. Edward N. Lorenz for his helpful discussions which were of assistance in interpreting some of the results of the research performed. This research was supported by the National Science Foundation under Grant GA-28203X1.

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