A Comparison of Formulations of Stochastic Collection

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ABSTRACT

A general formulation of the collection process is derived and the relative merits of the Berry and the Kovetz-Olund collection schemes assessed.

1. Introduction

In the past few years interest in stochastic formulations of the collection process has risen because it appears to explain the initial processes by which warm rain is produced. Most of this interest, however, has focused upon the "stochastic collection equation" as presented by Berry (1967). This note is to point out that, indeed, there are other formulations; and one specific formulation, that of Kovetz and Olund (1969, hereafter referred to as K–O), is more attractive when considering multiple, dependent processes.

At this time the K–O model has been harshly criticized by Reinhardt (1972) and List (1974). Reinhardt states that the model is incorrect because 1) it does not reduce to the "stochastic collection equation" and 2) it suffers from excessive "anomalous spreading." List simply states that the K–O model suffers from "faulty numerical procedures." Neither author presents specific details to justify his claims. We feel that, since the K–O technique is becoming more widely used (e.g., Silverman and Glass, 1973; Takahashi, 1974) that some specifics regarding the relative accuracy of the K–O technique are warranted.

So that we may gain a perspective regarding the different models of collection, we first present the formulations in a single framework, without treating the problem in great detail or with great rigor.

2. A generalized collection equation

We consider a system of interacting cloud particles and calculate the expected or mean number of particles in a given size range. At the outset we realize that this treatment contains many unqualified assumptions that have been considered, more or less, by several workers (see, for example, Warshaw, 1967; Gillespie, 1972; Drake, 1973). We intend to broaden the concept slightly by considering a generalized breakup term and use primarily a finite-difference approach, an approach that appears more physically sound because of the discontinuous nature of the collection process.

In this approach we consider that it is only *possible* to have M total particles and each individual particle, regardless of its size, is tagged with a different index number i. The collection problem is pictured as a death and birth process. That is, when two particles coalesce, they are destroyed and another, larger particle with a new index is created. If two particles undergo partial coalescence and break up, the interaction can produce a large number of particles with new identities and different index numbers. We presume that there are a sufficient number of indices to tag each particle that is present or will appear during the course of the experiment. Of course, normally the size index will increase with the particle size but one can imagine labeling the particles according to their probability of interaction, or according to any other physical criterion such as electrical charge or impurity content. If, indeed, one labels the particles according to their interactions, then it is possible to consider first those particle interactions which are expected to occur first and then correct for alterations in the population due to the interactions *as they occur* in the finite time interval Δt. Here we proceed in the standard way and assume that the number of interactions in Δt involves a relatively small number of the total population so that no such correction is necessary.

Let:

- α_{ij} the probability that particle i interacts with particle j in unit time.
- β_{ijk} the probability that particle i interacts with particle j and produces particle k in unit time.
- $p_i(t)$ probability that particle i exists at time t.

Following standard procedure, α_{ij} is the collection

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probability given by

$$\alpha_{ij}=\pi(R_i+r_j)^2E_{ij}(V_i-v_j)/V, \tag{1}$$

where:

- R_i radius of the larger particle i
- r_j radius of the smaller particle j
- E_{ij} appropriate collection efficiency of particle i for particle j
- V_i terminal velocity of the larger particle
- v_j terminal velocity of the smaller particle
- V the total cloud volume.

Note that with electrical effects (and other influences) the values of α_{ij} could be altered due to changes in the collection efficiency E_{ij}. These changes may stem from variations in the trajectories (collision efficiency) and/or changes in the coalescence efficiency. Also, α_{ij} might change due to variations in terminal velocities resulting from charge-electrical field interactions and influences such as turbulence. The β_{ijk} are conceived as generalized breakup probabilities which form a tensor which is related to the redistribution function used by K–O. It will be called the "redistribution probability" and, generally speaking, would include the effects of coalescence and breakup. It is justified by the recent measurements of List (1974) which indicate that breakup occurs only during particle interactions.

We assume that we have a homogeneous, well-mixed volume V of M possible particles and note that the probability that a particle exists at time $t+\Delta t$ is equal to the probability that it existed at time t *and* did not interact with any other particles *or* that the particle is produced by interactions of other particles:

$$p_k(t+\Delta t)=p_k(t)(1-\sum_{i=1}^{M}p_i(t)\alpha_{ik}\Delta t)+\sum_{i=1}^{M}\sum_{j=1}^{M}p_i(t)p_j(t)\beta_{ijk}\Delta t, \tag{2}$$

where the α_{ik} and β_{ijk} are zero if any two indices are the same.

Usually, however, a size distribution is represented by a histogram with each interval containing a range of sizes (m,n). Thus we are interested in $\xi_{mn}(t)$, the expected number of particles in the size interval (m,n):

$$\xi_{mn}=\sum_{k=m}^{n}p_k(t). \tag{3}$$

The dynamic equation of concern describes the way ξ_{mn} changes with time. This is given by

$$\begin{aligned}\Delta\xi_{mn}(t)/\Delta t&=[\xi_{mn}(t+\Delta t)-\xi_{mn}(t)]/\Delta t\\&=\sum_{k=m}^{n}\sum_{i=1}^{M}\sum_{j=1}^{M}p_i(t)p_j(t)\beta_{ijk}-\sum_{k=m}^{n}p_k(t)\sum_{i=1}^{M}p_i(t)\alpha_{ik}.\end{aligned} \tag{4}$$

Interchanging the order of summations and rewriting the first term we have

$$\frac{\Delta\xi_{mn}(t)}{\Delta t}=\sum_{i=1}^{M}\sum_{j=1}^{M}p_i(t)p_j(t)\sum_{k=m}^{n}\beta_{ijk}-\sum_{k=m}^{n}p_k(t)\sum_{i=1}^{M}p_i(t)\alpha_{ik}, \tag{5}$$

which can be considered a general collection equation. If we now assume that the total possible number of particles M is divided into N categories [the kth category of which contains the interval (m,n)], that in each category i there are Ω_i single particle indices, and that every particle in a category has the same probability of existence and the same interaction coefficients, then

$$\frac{\Delta\xi_{mn}(t)}{\Delta t}=\sum_{i=1}^{N}\sum_{j=1}^{N}\Omega_ip_i(t)\Omega_jp_j(t)\Omega_k\beta_{ijk}-\Omega_kp_k(t)\sum_{i=1}^{N}\Omega_ip_i(t)\alpha_{ik}, \tag{6}$$

where here the indices i, j, k represent the number of the category.

From this expression we can derive the equations of Kovetz and Olund and the continuous collection equation of Berry if we use the assumption of uniformity, divide both sides of the equation by V, and take $f_i(t)=\Omega_ip_i(t)/V$ as the expected number of particles of size i per unit volume. Then

$$\frac{\Delta f_k(t)}{\Delta t}=\sum_{i=1}^{N}\sum_{j=1}^{N}f_i(t)f_j(t)R_{ijk}-f_k(t)\sum_{i=1}^{N}f_i(t)V_{ik}, \tag{7}$$

where V_{ik} is the collection kernal $[=\alpha_{ik}V]$ and R_{ijk} the redistribution kernel $[=\Omega_k\beta_{ijk}V]$, which is identical to the formulation of K–O if the unwanted terms in V_{ik} and R_{ijk} are set equal to zero and $R_{ijk}\equiv B_{ijk}P_{ij}$ (K–O's notation).

It is easy to see that the continuous analog of (7) is the equation given by Silverman and Glass [1973, Eq. (10)]:

$$\frac{df_x(t)}{dt}=\int_{x_0}^{x_m}\int f_y(t)f_z(t)R_{yzx}dydz-f_x(t)\int_{x_0}^{x_m}f_y(t)V_{xy}dy, \tag{8}$$

where x, y, z are particle masses; x_0 is the mass of the smallest particle and x_m the mass of the largest particle; and again R_{yzx} and V_{xy} are defined so that undesirable interactions have zero values. If, in addition, we assume that every interaction creates a single particle that has the properties of one of the given categories, i.e.,

$$R_{yzx}=\tfrac{1}{2}\delta(x-z,\,y)V_{yz}, \tag{9}$$

TABLE 1. Mass distribution function [G(ln r)].*

Radius (μm)	T=10 min Berry	T=10 min Golovin	T=10 min K–O	T=20 min Berry	T=20 min Golovin	T=20 min K–O
17.7	1.05	1.035	0.999	0.645	0.647	0.634
19.8	1.00	0.978	0.932	0.736	0.733	0.708
22.3	0.858	0.830	0.790	0.816	0.806	0.765
25.0	0.634	0.608	0.595	0.871	0.853	0.796
28.1	0.379	0.362	0.388	0.882	0.856	0.788
31.5	0.168	0.161	0.213	0.830	0.798	0.735
35.4	4.83×10^{-2}	4.76×10^{-2}	9.68×10^{-2}	0.704	0.671	0.634
39.7	7.55×10^{-3}	7.87×10^{-3}	3.57×10^{-2}	0.516	0.488	0.498
44.5	4.91×10^{-4}	5.76×10^{-4}	1.07×10^{-2}	0.306	0.290	0.350
50.0	9.09×10^{-6}	1.32×10^{-5}	2.67×10^{-3}	0.134	0.129	0.215
56.1	2.79×10^{-8}	5.93×10^{-8}	5.67×10^{-4}	0.038	0.038	0.114
63.0	1.09×10^{-11}	2.63×10^{-11}	1.05×10^{-4}	0.0060	0.0063	0.052

* Units: g m^{-3} per unit ln radius interval.

where δ is the delta function, then Eq. (8) reduces to

$$\frac{df_x(t)}{dt}=\frac{1}{2}\int_{x_0}^{x_m} f_{x-z}(t)V_{x-z,z}dz-\int_{x_0}^{x_m} f_x(t)f_y(t)V_{xy}dy, \quad (10)$$

which is the continuous collection equation as presented, for example, by Mason (1971) or Bleck (1970).

3. The different viewpoints

As the two models were conceived, they differ in their basic approaches to the problem. The Berry model [Eq. (10)] considers a *continuum* of particles and looks at the rates at which particles enter and leave a given infinitesimal size interval. Interactions with all other particles remove particles of this size but only interactions of specific particles can create particles of the given size. In the numerical computations with a realistic collection kernel, it is necessary to choose a finite number of particle classes, using a logarithmic form. As a result, the specific particles generally are not of a chosen class so that their numbers and collection properties must be inferred by *interpolation* between particle classes. In essence, the model looks *backward* to find the particles necessary to create particles of a given size.

In contrast, the K–O model [Eq. (7)] looks at the *entire distribution* and allows it to evolve in a *forward* sense. At the outset the particle distribution is discrete

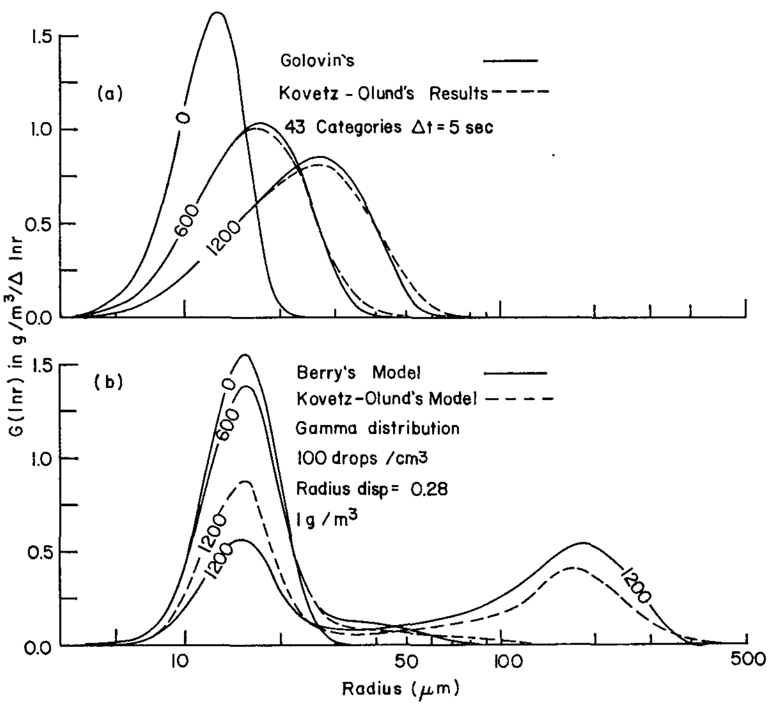

FIGURE 1.

and the numerical scheme simply quantifies the physical interactions. Every combination of particle interactions is considered and the particles that coalesce are removed from the appropriate classes. Then the newly formed particles are simply put into the appropriate new class. The problem is that, again, the newly formed particles do not have exactly the size of a given class. To overcome this problem, the particles are simply redistributed into the nearest two classes conserving both number and mass.

4. Comparison of results

The results of a three-fold comparison of the two techniques are shown in Fig. 1 and Table 1. In Fig. 1a the solid line presents the analytical solution of Golovin (1963) with his initial distribution (1 gm m^{-3}, $\nu=0$, $R_{fs}=10$ μm as given by Reinhardt) and his artificial sum of masses collection kernel, $V_{ij}=10^3(x_i+x_j)$. The Berry solution agrees with the Golovin solution within a few percent as shown in Table 1. However, the K–O solution shows a peak value about 6% lower after 20 min and a slight extension of the tail. It is significant that this spread in the distribution, referred to as "anomalous dispersion," is not associated with an increase in the total rainwater.

When one considers the case of a realistic collection kernel, the situation is not so good. Fig. 1b compares the results of the Berry calculation with the K–O results for the case of a real kernel with aerodynamic collection efficiencies as given by Berry (1967). Here we see little "anomalous dispersion" but the peak in the rainwater size is approximately 25% lower than given by Berry.

5. Discussion

In terms of autoconversion, this discrepancy is serious. However, in terms of actual rain development, it may be of little consequence because there is little need to carry the computation beyond 15–20 min. This is because of the extremely limited physical basis of all the formulations. To be general, they should include condensation, breakup, effects of electrical charging, effects of ice characteristics, and many other processes. Such extensions are difficult with the Berry model (see Leighton and Rodgers, 1974). However, if one accepts the possible errors, such extensions are relatively simple with the K–O model. In fact, we have just completed the development of a stochastic electrical model based on the K–O model (see Scott and Levin, 1974a,b) and presently have preliminary results from a model which also contains condensation, breakup and ice phase interactions within a Lagrangian framework.

To the credit of the K–O model is its ability to preserve mass precisely within the limits of the computer ($\sim$one part in 10^{10}). Indeed, it is just this problem which detracts credibility from the Berry model. In use, the Berry model requires continuous renormalization and filtering.

Also, it can be argued that the Kovetz–Olund model has merit and is more physically reasonable because it permits one to use a finite-difference formulation directly. True, the spreading of the distribution is probably due to the forcing of some mass into higher categories by the redistribution technique, but the treatment does not ignore the discontinuous character of the collection process. That is, it does not consider that a particle k at time t is produced by interactions of particles i and j *also at time* t when, in fact, we know that the two interacting particles must have existed at an earlier time $t-\delta t$ and the concentrations of particles i and j took a *discontinuous* jump at time t.

In summary, the authors hope that they have displayed the relative merits and flaws of the different formulations for collection. Reinhardt (1972) criticized the Kovetz–Olund treatment because of the abovementioned "anomalous spreading" and also because he found that it would not reduce to the "stochastic collection equation." Indeed, we see that it should not so reduce because the continuous collection equation comes from a different branch in the derivation and, in a sense, contains different assumptions. At present, it appears that the K–O model has a definite use in further studies. Indeed, a variation of the K–O model may meet the need for a new, more general formulation of the collection process as stated by List (1974).

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