Linear Simulations of Boussinesq Convection  
in a Deep Rotating Spherical Shell

PETER A. GILMAN

National Center for Atmospheric Research, Boulder, Colo. 80303
(Manuscript received 27 November 1974, in revised form 5 March 1975)

ABSTRACT

We present extensive linear numerical simulations of Boussinesq convection in a rotating spherical shell of finite depth. The motivation for the study is the problem of general circulation of the solar convection zone. We solve the marching equations on a staggered grid in the meridian plane for the amplitudes of the most unstable Fourier mode of longitudinal wavenumber $m$ between 0 and 24, for Taylor number $T$ between 0 and $10^7$, at a Prandtl number $P=1$, for a shell of depth $20\%$ of the outer radius. Stress-free, fixed-temperature boundary conditions are used at the inner and outer bounding surfaces. Modes of two symmetries, symmetric and antisymmetric about the equator, are studied. The principal results are as follows:

Increasing Taylor number $T$ splits the most unstable solutions for each $m$ into two classes: a broad band of high $m$ solutions which peak at or near the equator, and a small number of low $m$ solutions which peak at or near the poles. The equatorial modes are unstable at lower Rayleigh number $R$. The polar modes appear to be similar in many respects to plane-parallel convection with rotation parallel to gravity. Modes symmetric about the equator are unstable at lower $R$ than those which are antisymmetric, by a percentage which increases with $T$ in the range studied.

Equatorial modes of both symmetries propagate prograde (frequency $\omega>0$) at high $T$ and retrograde ($\omega<0$) at low $T$, in agreement with earlier work. Polar modes propagate, too, but very slowly.

Critical (first unstable) equatorial modes are shown to have or be closely approaching asymptotic dependence $R_c \sim T^{2/3}$, $m_c \sim T^{1/4}$, $\omega_c \sim T^{1/4}$ with increasing $T$, in agreement with analytical analyses of Roberts and Busse.

With increasing $T$, symmetric equatorial modes take on the form of rolls swirling about an axis parallel to the rotation axis and extending across both Northern and Southern Hemispheres in agreement with earlier results. Antisymmetric modes also assume a roll shape, but with swirl oppositely directed in the two hemispheres, together with fluid pumped across the equator parallel to the rotation axis. Polar modes become a ring of vortices more and more tightly arranged around the pole.

Outward radial heat flux peaks at the equator for symmetric equatorial modes, and at a low latitude for antisymmetric modes. Both are suppressed near the equator near the outer boundary at high $T$. Symmetric modes also transport heat toward the equator, while antisymmetric modes transport heat poleward at the lowest latitudes, equatorward at somewhat higher latitudes. Symmetric equatorial modes transport angular momentum radially inward at low $T$, radially outward at high $T$. These modes transport angular momentum toward the equator from higher latitudes at all $T$.

1. Introduction

It has been known for more than one hundred years that the surface of the sun rotates differentially, with equatorial regions moving faster than higher latitudes. This was first discovered by Carrington who tracked sunspots as they moved across the solar disk. Only in the last ten years or so has evidence accumulated indicating the sun really has a "general circulation" of which the differential rotation is only the long-term mean, with a wide spectrum of global-scale fluctuations in time, longitude and latitude also being present, signifying substantial departures from this mean. The evidence is of many types, including the movement of sunspots and other small features, Doppler shifts of the surface gas, global-scale movements of magnetic fields, and fluctuations of the rotation of the solar corona.

In parallel with the increase in observational evidence, there have been developed numerous mathematical models which attempt to provide explanations for some parts of the solar general circulation, emphasizing particularly the mean differential rotation and the gross features of the solar cycle. The literature on both observations and theory has been reviewed in Gilman (1974), Weiss (1971) and Stix (1974), to which the reader may turn for details. It appears that neither

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1 The National Center for Atmospheric Research is sponsored by the National Science Foundation.
the observations nor models are yet well enough developed to provide an unambiguous statement as to the nature and origin of the solar general circulation.

Perhaps the most important class of models that have been considered for producing the mean differential rotation are based on the theory of thermal convection in a rotating spherical shell, with uniform heating from below. This theory is, however, far from complete. Most previous work has focused on the limits of low or high Taylor number (Busse, 1970a, 1973; Durley, 1968, 1970, 1971; Heard, 1972), axisymmetric motions (Durley, 1968), thin shells (Busse, 1970a, 1973; Heard, 1972), full spheres in the high-rotation limit (Robert, 1968; Busse, 1970b), or hydrostatic motions (Yoshimura, 1971, 1974). Of these, Busse (1973) carries the problem the furthest.

From the above studies, important as they have been in opening up the problem, it is difficult to obtain information concerning the amplitude of the convection, induced heat flux and differential rotation for a typical case of finite but not asymptotically large Taylor number, finite supercritical Rayleigh number, and finite depth. At most, they examine nonlinear effects only as second-order initial tendency perturbations, with feedbacks not taken into account [except in Busse (1973)]. None of these calculations allow for several modes to be present at once. Calculations of Gilman (1972) go somewhat further into the nonlinear regime but they, too, are limited, in that they are only for an equatorial annulus, in which only the first unstable longitudinal wavenumber is allowed to grow to finite amplitude, change the mean state, and respond to those changes. Furthermore, results in Gilman (1972) pointed to an important discrepancy between all the convective shell models and the real sun; namely, that the models show a strong tendency to produce large differentials in heat flux with latitude, with a maximum at or near the equator, which is not observed. This effect leads to the need for studying the nonlinear, full-spherical-shell case in much greater depth, in order to see under what conditions the heat flux differentials can be largely eliminated, while retaining a reasonable differential rotation. This appears to us to require studying nonlinear interactions among several modes at Rayleigh numbers substantially above critical.

A feature common to virtually all the convection models referred to above is that they employ the Boussinesq approximation. That is, density variations are neglected except where they are coupled with gravity. For a gas, such an assumption holds only if the scale height is much larger than the depth of the convecting fluid. This is certainly not the case for the total depth of the solar convection zone, which stretches over many scale heights. However, in the deep part of this zone, the density does change relatively slowly with depth, and the departures from the Boussinesq conditions should be less important there. In any case, the main reason for making the Boussinesq approxima-

tion is simply that the theory of non-Boussinesq convection, for example that which would be described by the so-called anelastic equations, even without complicating factors such as spherical geometry and rotation, is not at all well developed compared to the Boussinesq theory. It is the belief of the author that more development of compressible Boussinesq deep-shell models should help a great deal when we are ready to include compressibility.

Before we can do a meaningful nonlinear, spherical-shell convection calculation even in the Boussinesq case, we need to explore as far as practical the linear solutions for the same model. The linear results themselves can provide much insight into the nature of spherical convection. They also should serve as important initial guides to the judicious truncation of the nonlinear calculation, which is needed if the amount of computing effort is to be kept within reasonable bounds while still capturing the important physical processes. In addition, the computer code for any such model will be long and complicated and can hide errors. Being able to test it with simpler but still physically meaningful calculations should (and has) helped to correct such errors. This paper then presents linear results, together with the formulation required to get these results.

Convection in a deep rotating spherical shell is complex, and we feel it appropriate to give a fairly detailed account of how large a Rayleigh number is needed to excite the various modes, how they propagate, how their structure varies with Taylor number and longitudinal wavenumber, the momentum and heat transport properties they possess, as well as, of course, physical interpretations of their properties. Some of the results overlap in certain limits with earlier work cited above. Nevertheless, to the author's knowledge the calculations presented here represent the only extensive linear study of convection in a rotating spherical shell of finite depth at finite Taylor number, i.e., at neither asymptotically small nor asymptotically large rotation rate.

2. Model assumptions and basic equations

In addition to making the Boussinesq approximation, for the reasons described above, we assume a central gravity (into which the weak centrifugal effects of the rotating system have been absorbed) acts on the fluid, and we ignore the self-gravitation of the fluid shell. The fluid is made unstable by a temperature difference $\Delta \theta$ imposed across its depth. Diffusive processes are represented by the (assumed constant) thermal diffusivity $\kappa$ and kinematic viscosity $\nu$. Anisotropy in these quantities can be easily introduced into the model, but we have not done so. The rotation rate $\Omega$ of the coordinate
system is assumed to be constant. The depth $d$ of the shell is also a constant, and is used to scale all lengths. Times are scaled by the thermal diffusion time $\delta^2/d\kappa$, velocities by $\kappa/d$, and temperatures by $\Delta \theta$. Four dimensionless parameters therefore characterize the system. These are $\beta$, the ratio of inner sphere radius to fluid shell depth; $P = \nu / \kappa$, the Prandtl number; $T = 4\nu \delta^2 / \rho^2$, the Taylor number; and $R = g_0 \delta \delta^2 / \nu$, the Rayleigh number, in which $g_0$ is gravity at the outer edge and $\alpha$ the coefficient of volume expansion. Note that even though we are examining global-scale convection in a rotating system, we do not assume the flow is either hydrostatic [as in, for example, Yoshimura (1974)], or geostrophic (heliostric for the sun). Hydrostatics is not appropriate for the motions because the most unstable modes have one horizontal dimension comparable to the depth, and because the radial component of Coriolis force is important. Not assuming geostrophic balance allows us to examine with the same calculations the cases of weak, moderate and strong rotational influence, as measured by the Taylor number.

We choose to write the linear governing equations relative to the rotating coordinate system in which we assume there is a basic state of rest. Then the basic initial temperature gradient is the conduction gradient, which, with the scaling we have chosen, is represented by $-\beta(\beta+1)/r^2$. Similarly, gravity is given by $g_0 \times (\beta+1)^2/r^2$. Then if we define $\lambda$ to be longitude, $\phi$ latitude, and $r$ the (dimensionless) radial coordinate, with $u$, $v$, $w$ the corresponding velocities measured relative to the rotating frame, $\theta$ the perturbation temperature, and $\pi = p/\rho$, in which $p$ is the constant mean density and $\rho$ the perturbation pressure, and with the assumptions and scalings given above, the governing equations of motion, thermodynamics and mass conservation are respectively as follows:

$$\frac{\partial u}{\partial t} = -\frac{1}{r \cos \phi} \frac{\partial \pi}{\partial \phi} - PT^4 w \cos \phi - PT^4 v \sin \phi$$

$$+ P \left\{ \frac{1}{r^2 \cos \phi} \left[ \frac{\partial v}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( u \cos \phi \right) \right] \right\}$$

$$+ \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \frac{\partial^2}{\partial r \partial \phi}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{r \phi} \frac{\partial \pi}{\partial \phi} - PT^4 u \sin \phi$$

$$+ P \left\{ \frac{1}{r^2 \cos \phi} \left[ \frac{\partial u}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( v \cos \phi \right) \right] \right\}$$

$$+ \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \frac{\partial^2}{\partial r \partial \phi}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{r} \frac{\partial \pi}{\partial r} + \left( \frac{\beta+1}{r} \right)^2 \left( \frac{\pi}{w} + PT^4 u \cos \phi \right)$$

$$+ P \left\{ \frac{1}{r^2 \cos \phi} \left[ \frac{\partial w}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} \left( w \cos \phi \right) \right] \right\}$$

$$+ \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial r} \left( w \cos \phi \right)$$

$$+ \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial \phi} \left( w \cos \phi \right)$$

$$\frac{\partial \theta}{\partial t} = \frac{\beta(\beta+1)}{r^2} w + \frac{1}{r^2} \frac{\partial \pi}{\partial r} + \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial \phi} \left( w \cos \phi \right)$$

$$+ \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial \phi} \left( w \cos \phi \right)$$

$$+ \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial \phi} \left( w \cos \phi \right)$$

$$+ \frac{1}{r^2 \cos \phi} \frac{\partial}{\partial \phi} \left( w \cos \phi \right)$$

In the present calculations we assume for boundary conditions that the top and bottom are stress-free and perfectly thermally conducting, so that $r = \beta$, $\beta+1$, we take

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial r} = w = \theta = 0.$$ (6)

Boundary conditions are also imposed at the pole and equator for computational purposes. These are described below when finite differences are introduced.

3. Numerical solution technique

Solution of the linear stability problem for spherical convection when rotation is absent has been done by Durney (1968). The problem is made straightforward by the fact that without rotation, individual spherical harmonics separate out to represent the longitudinal and latitudinal dependence. However, when rotation is added the neighboring latitudinal harmonics couple (because of the trigonometric dependence in the Coriolis forces) even without nonlinear inertial terms. The longitudinal dependence, which we represent by an (integer) wavenumber $m$, however, remains separable in the linear case. For this reason we decided to solve (1)-(5) by Fourier-analyzing all variables in longitude, but introducing a grid in the meridional plane on which we solve for the variables corresponding to each $m$. This also allows for easier comparisons with the earlier equatorial annulus work of the author (Gilman, 1972) which used the same technique. Since we began this work, an attractive scheme involving spherical harmonics has been put together by Young (1974) and applied to the nonlinear convection problem without rotation.

The layout of the grid is staggered in latitude and radius, similarly to Williams and Robinson (1973) [but derived independently], except that they made a thin
shell approximation (divergence of radii were ignored). The perturbation pressure $\pi$ as well as temperature $\theta$ and longitudinal motion $u$ are evaluated on a primary grid with spacings $\Delta \phi$ in latitude and $\Delta r$ in radius. Then the radial motion $w$ is evaluated at $r/2$ above and below the primary points, and the latitudinal motion $v$ is determined at $\Delta \phi/2$ north and south. The grid for $w$ coincides with the inner and outer boundaries, while that for $v$ coincides with the equator and the pole. When nonlinear terms are added they can be written in an energy-conserving form for this grid, as in Williams and Robinson (1973). The time differencing is taken to be leapfrog, with the diffusion terms lagged by one time step for stability.

If we define average and difference operators operating on any function $f$ as

$$\bar{f} = \frac{1}{\Delta} \left[ f(x + \Delta x/2) + f(x - \Delta x/2) \right]$$

and

$$\delta_x f = f(x + \Delta x/2) - f(x - \Delta x/2),$$

respectively, and assume each dependent variable has been expanded into a Fourier series of the form

$$F = \sum_{m=-\infty}^{\infty} f_m(\phi,r,t)e^{im\phi},$$

then the differential equations (1)-(5) above can be transformed into difference equations for the $m$th mode respectively as follows:

$$\delta^{-t}_{t} u_m = -\frac{im}{r \cos \phi} \pi_m + PT_i \tan \phi \cos \phi - PT_i \cos \phi \frac{r^2 \omega_m}{r \cos \phi} + P \left( \frac{1}{r^2 \cos \phi} \left[ -i m \omega_m + \delta_x (u_m \cos \phi) \right] \right) \frac{1}{r} + \delta_{r} (ru_m) - \frac{1}{r \cos \phi} i m \delta_{r} \omega_m \right)_{t=\Delta t} \tag{7}$$

$$\delta^{-t}_{t} v_m = -\frac{1}{r} \delta_{\phi} \pi_m - PT_i \omega_m \sin \phi + P \left( \frac{1}{r^2 \cos^2 \phi} \left[ \delta_x (im \omega_m \cos \phi) + m^2 v_m \right] \right) \frac{1}{r} + \frac{1}{r} - \delta_{r} \omega_m + \frac{1}{r} \delta_{r} v_m \right)_{t=\Delta t} \tag{8}$$

$$\delta^{-t}_{t} w_m = -\frac{(\beta+1)^2}{r^2} PR \delta_{\phi} \gamma + PT_i \gamma \cos \phi + P \left( \frac{1}{r^2 \cos \phi} \frac{1}{r} \left[ -i \delta_{x} (ru_m) - \frac{1}{r^2 \cos \phi} m^2 w_m \right] \right) \frac{1}{r^2 \cos \phi} \left[ \cos \phi (\delta_x \omega_m - \delta_x v_m) \right]_{t=\Delta t} \tag{9}$$

Eqs. (7)-(10), then, are marched forward in time, using velocities and temperatures from the previous two time steps. Each variable is averaged every 13 time steps according to a $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$ weighting for adjacent time steps to suppress two-step instability. The needed pressure is found by solving the two-dimensional inhomogeneous Helmholtz equation for it which is found by taking the finite-difference divergence of (7), (8) and (9), and setting the divergence at the new time step equal to zero. The pressure equation is actually solved by direct matrix inversion, using Choleski decomposition. Boundary conditions on pressure are taken from the equation of motion for the normal components of motion at the appropriate boundaries.

In finite-difference form, the boundary conditions at the inner and outer surfaces become

$$\delta_{4} u_m - \frac{1}{r} u_m - \frac{1}{r} v_m, \quad \omega_m = 0, \quad \theta_m = 0, \quad r = \beta, \quad \beta + 1. \tag{12}$$

It is computationally advantageous, particularly for the linear problem, to apply symmetry conditions at the equator, thus allowing integration of only a single hemisphere. The two choices are what we call symmetric conditions, for which all variables are symmetric across the equator except the north-south velocity $v$, which is antisymmetric; and antisymmetric boundary conditions, for which all variables have the opposite symmetry. Symmetric conditions then have the form

$$\delta_{4} u_m, \quad \delta_{4} v, \quad \delta_{4} \theta, \quad \delta_{4} \pi, \quad \nu = 0 \quad \text{at} \quad \phi = 0, \tag{13}$$

while antisymmetric conditions take the form

$$u, \quad w, \quad \theta, \quad \pi, \quad \delta_{4} \phi = 0. \tag{14}$$

It turns out that with rotation, the convection always sets in at a lower Rayleigh number in the symmetric case.

In nonlinear calculations, symmetric initial conditions would be preserved, but antisymmetric would not, resulting in interactions between symmetric and antisymmetric modes, the nature and importance of which only a nonlinear calculation can determine. We might expect that near the Rayleigh number needed for instability of the symmetric modes, antisymmetric modes, requiring a larger Rayleigh number to be excited, would not be important. On the other hand, at Rayleigh numbers significantly above that needed for instability
of both symmetric and antisymmetric modes, the antisymmetric modes could become important. This can be determined only by a nonlinear calculation. In preparation for that, we feel it necessary to study both.

With the staggering of the grid, problems with the singularity at the pole can be contained, with some care. Only \( v \) is evaluated right at the pole, and we set it equal to 0. This is completely general for all modes except \( m=1 \), which does allow for flow across the pole. However, we found that even with \( v=0 \) at \( \phi=\pm\pi/2 \) for \( m=1 \), we still obtained flow virtually across the pole, as if the constraint was a very weak one. All other variables required a relation between their values at the closest point to the pole \( (\pm\pi/2\mp\Delta\phi/2) \) and a set of points \( \Delta\phi/2 \) across the pole which are fictitious in the Fourier-analyzed system. For these, we applied the natural symmetry conditions implied in physical space implied by even and odd \( m \) patterns, with appropriate changes in sign for \( \Phi \), whose direction changes across the pole. In our notation, the resulting conditions, then, are

\[
\begin{align*}
\vec{u} & \cdot \delta_{\phi} \vec{u}, \delta_{\theta}, \delta_{\pi}, \delta_{\phi} \Phi, \Phi=0; \quad \Phi=\pm\pi/2 \text{ [even } m \text{]}, \\
\delta_{\phi} \Phi & \cdot \Phi, \Phi, \Phi, \Phi, \Phi, \Phi=0; \quad \Phi=\pm\pi/2 \text{ [odd } m \text{].} 
\end{align*}
\]

In actual practice, allowing larger \( m \) to exist at the pole would require extremely short time steps \( \Delta t \). For most cases, we simply removed a few points nearest the pole and imposed stress-free, perfect insulator boundary conditions at a high latitude, for \( m \) larger than a certain value depending on the equatorial symmetry and resolution used. In any case, the procedure works well only on modes which do not acquire significant amplitude near the pole with or without filtering. That is, modes which peak near the equator even when the polar points are removed are not affected, but we found some modest \( m \) modes which in the full sphere peaked near the equator, then peaked just outside the removed points after their removal, resulting in a substantially altered convection pattern for that mode.

Calculations were performed at two resolutions: 9 radial×79 latitudinal points (counted pole to pole and including boundary points), and 13×121. The higher resolution proved to be most important near the poles at higher Taylor number.

The typical numerical integration was performed as follows: The first integration in a series was started from simple sinusoidal forms in latitude and radius, with several profiles with different numbers of nodes in latitude given similar weight. Thereafter, the initial conditions for a new run were taken from the end of a previous run at a neighboring point in the parameter space, the ends of all runs being stored on magnetic tape. This allowed for a substantial saving in computer time, due to faster convergence to the linear result.

From the linear numerical simulations we are able to determine the growth (or decay) rate of the most unstable latitude and radial structure for a given \( m \), as well as its oscillation frequency due to rotation. By interpolation to zero growth rate, we can then determine the Rayleigh number and frequency for onset of instability for each longitudinal wavenumber \( m \).

Using this technique it is not, in general, possible to obtain information about other latitudinal and radial structures of modes with the same \( m \) that are less unstable, because they are soon swamped by the most unstable. The way to find these would be to attempt the numerical solution of the formal eigenvalue-eigenfunction problem. However, because the radial and latitudinal dependences do not separate when rotation is present, and because both of these structures would involve infinite series of higher harmonic functions, or coupled structures determined on a grid, this would be a major undertaking by itself and require a much different computer code than has been developed here, which code would not generalize to the nonlinear problem.

It is true that the additional, less unstable modes may become important in the nonlinear case, but rather than attempt the full eigenvalue problem, it seems better strategy to move to the nonlinear problem (which will be done in later papers) for which all modes can be excited.

4. Stability properties of the solutions

The model program is general enough that we can study almost any depth of spherical shell. We have chosen to concentrate on a depth of 20% of the outer radius (in our notation \( \beta = 4 \)), which is not unreasonable for the depth of the solar convection zone, although the actual depth is rather sensitive to the precise elemental composition which contributes to the opacity. We have also done a few calculations for depths of 10% and 40% \( \beta = 9 \) and 1.5. We have also concentrated on obtaining results for the case of Prandtl number \( \text{Pr} = 1 \). The Prandtl number of the quiet solar gas is many orders of magnitude smaller than that, but we are calculating global-scale modes, which are certainly influenced strongly by all the smaller scale convection as well as shearing instability which the large modes may generate. In a rough sense these processes should give rise to an effective Prandtl number based on turbulent viscosity and thermal diffusivity. Eventually, these should be replaced in the model by much more sophisticated nonlinear representations of the small-scale turbulence. Assuming a very small \( \text{Pr} \) for global scale modes would in our opinion be even less realistic than \( \text{Pr} = 1 \), as well as being computationally much more difficult.

a. Rayleigh numbers for instability

Figs. 1 and 2 display the Rayleigh number required for instability at several Taylor numbers \( \text{T} \) from 0 to \( 10^5 \), for \( m = 0 \) to 24. The mode unstable at the lowest Rayleigh number for each \( T \) is circled. This mode will
be referred to as the critical mode. At $T=0$ the stability curve is virtually flat out to $m=9$ reflecting the degeneracy of the nonrotating case for which a whole family of modes are unstable at the same $R$ (Durney, 1968). In general, they would all have the same latitudinal spherical harmonic index $l$ for $m \leq 9$, while for $m > 9$, they would have $m=l$, corresponding to the same structure in latitude for all large $m$. The slight wiggles in Figs. 1 and 2 are due to numerical truncation, we believe. The lowest $R$ is about 545, which is significantly lower than the 657.5 predicted in the plane-parallel case and in the spherical shell with constant gravity. The difference is due to the fact that our gravity increases inward and we have defined our Rayleigh number using gravity at the outer edge. As $T$ is increased, we can see that the stabilizing effect due to rotation is much more pronounced for low $m$, for which dissipation in longitude is less. Further, it is clear that the character of the instability is different at low $m$ from higher, as evidenced by the sharp break in the curve at $m=4$ for $T=10^2$, $m=2$ at $T=10^4$ and $3 \times 10^4$, and $m=1$ for $T=10^6$ and $10^8$ for the symmetric case. What we will see is that modes on the high $m$ side all peak at or near the equator, while modes on the low $m$ side peak at or near the poles. The flat portion of the low $m$ curves extend to higher $m$ in the antisymmetric case, because these boundary conditions are more stabilizing on the equatorial modes. The arcs defined on either side of the break undoubtedly continue, and for sufficient extension may represent structures which still peak at middle latitudes, but in each case they would no longer be the most unstable mode for that $m$.

We see also from Figs. 1 and 2 that, as we should expect, the mode $m$ unstable at lowest $R$ for each $T$ generally increases as $T$ increases, but there is a slight
"jog" in both symmetric and antisymmetric cases. Thus it appears that the transition from a low T regime to a high T regime is not completely smooth. We note also that for higher T, the curves are rather flat near the minimum, suggesting for the nonlinear case that a broad spectrum of modes may be excited not far above the stability boundary.

The calculations we have made for shells both thicker and thinner than reported above indicate curves similar to those in Figs. 1 and 2, but with \( m \) scaled down for the thicker shells, and scaled up for the thinner.

In determining what modes and equatorial boundary conditions to retain in the nonlinear case, it is important to note the difference in critical Rayleigh number \( R_c \) (circled modes) for instability in the symmetric and antisymmetric cases. Fig. 3 shows the ratio, and indicates that in the range of \( T \) we studied, \( R_c \) (antisymmetric) climbs steadily by comparison, reaching about 2.5 \( R_c \) (symmetric) by \( T = 10^5 \). We expect that eventually this ratio should approach a constant, with the same asymptotic dependence of \( R_c \) on \( T \) for both (this is discussed further below).

\textit{b. Oscillation frequencies of modes at instability onset}

Figs. 4 and 5 give the oscillation frequency for each mode at its Rayleigh number for onset of instability. The frequency is for the rate each mode propagates its own wavelength; to obtain the frequency for complete traversal around the sphere, we must divide \( w \) by \( m \). The principal features are that the frequencies are
virtually always negative at low \( T = 10^6, 10^9 \) [corresponding to retrograde propagation in longitude] and predominantly positive (prograde propagation) at high \( T \) (10⁹ and above). These results confirm earlier asymptotic results of Busse (1970a, b) and numerical results of Gilman (1972) for the equatorial annulus, and quantify the magnitude. We see also that all low \( m \) modes which peak near the pole (the flat parts of the curves in Figs. 1 and 2) also have much lower frequencies than the high \( m \) equatorial modes for the same \( T \) (at \( T \geq 10^9 \)).

As shown in Busse (1970b) and Gilman (1972), the prograde oscillation frequencies for symmetric equatorial modes at high \( T \) can be explained as arising from the convection assuming a locally cylindrical structure with the axis of the cylinder parallel to the axis of rotation. The oscillation results from the columns of fluid in the cylinder being forced to shrink as they approach the equator, inducing negative relative vorticity on the prograde side of existing negative vorticity, and lengthening, producing positive vorticity on the prograde side of existing positive vorticity as they move away. The decrease of \( \omega \) with increasing \( m \) is faster than expected from an inviscid Rossby wave, and is presumably due to damping by viscosity.

At low Taylor number, on the other hand, there is no tendency for the convection to assume a cylindrical structure, and latitudinal rolls are preferred. Nevertheless, retrograde oscillations occur because of some tendency for the radial component of vorticity to be conserved. Southward-moving particles induce positive relative vorticity on the retrograde side of already existing positive vorticity peaks, thus causing the pattern to move in the retrograde direction. The large vorticity component pointed in the latitudinal direction associated with the roll does not induce any propagation, because fluid particles moving in longitude-radius planes feel no change in the vorticity of the rotating frame.

Near the pole, the modes (as we shall see later) do not behave as columns of fluid, but a weak influence of the small variation of rotation vector with latitude is still felt, resulting in a small frequency there.

That the most unstable mode should also have nearly the peak frequency is reasonable, since similar longitude and radial dimensions are needed for optimum energy conversion; lengthening the longitudinal dimension (lowering \( m \)) decreases energy conversion and lowers the frequency because the longitudinal particle path becomes longer, during which no energy conversion takes place and the fluid column length does not change.

c. Separation to equatorial and polar modes

To demonstrate the evolution of the solutions into equatorial and polar modes as \( T \) increases, we have calculated the rms total velocity for each \( m \), averaged it in the radial direction, and normalized each to have amplitude unity at its peak. The result for symmetric modes is shown in Fig. 6. We see that for \( T = 0 \) there is a rather even spread of modes in latitude between \( m = 0 \) and 10, with all higher \( m \) peaking near the equator and having similar shapes. (Slight irregularities in the progression of the pattern with higher \( m \) are believed to be due to not running the calculation quite long enough for a single latitudinal structure to dominate.) By \( T = 10^9 \), a very clear break has developed, with wavenumbers \( \geq 5 \) peaking at the equator, wavenumber 4 near 60° latitude, and lower wavenumbers at still higher latitudes. By \( T = 3 \times 10^4 \), \( m = 3 \) and 4 have shifted over to the equator, and the spread of shapes of the equatorial modes has narrowed considerably. By \( T = 10^9 \), there is very little spread in shapes of \( m \geq 2 \), except right near the equator, where a dip has developed in all wavenumbers above about \( m = 6 \), with the peak for these modes occurring around latitude 20°. The dip comes from the fact that the modes are now peaking on a cylindrical surface concentric with the axis of rotation which cuts the outer boundary at a finite latitude.

The corresponding pictures (not shown) for the antisymmetric modes are somewhat more complicated, with more polar modes at higher \( T \), but the splitting is still quite pronounced. The subset of polar modes that occurs for both symmetric and antisymmetric equatorial boundary conditions occur at the same \( T \) and have the same structure.

These results suggest clearly that one of the important problems to be considered with a nonlinear model is how the gap between equatorial and polar modes gets filled in. One possibility among several is that modes, less unstable than those we have found at \( m \) near the break between polar and equatorial structures, and having amplitudes peaking at middle latitudes, become important.

d. Equatorial mode comparisons with earlier annulus calculation

Properties of the present solutions in equatorial regions can be compared with earlier results found by Gilman (1972) for an equatorial annulus of the same depth as the present shell, but extending only to about 40° latitude. In that calculation, gravity was taken as a constant, and curvature effects were ignored except in the Coriolis forces which were expanded about the equator. The critical Rayleigh numbers are compared in Fig. 7. We see that for both symmetric and antisymmetric modes, the shapes are very similar, with the equatorial annulus \( R_e \) being 15–20% larger at all \( T \). This difference can be explained as being due almost entirely to the increase of gravity with depth in the spherical shell, requiring, therefore, a somewhat smaller Rayleigh number (since it contains gravity at the outer boundary) to produce a given amount of buoyancy. Otherwise the curves are virtually the same. Fig. 8
Fig. 6. Total velocity amplitudes for all modes between \( m=0 \) and \( m=24 \) (labeled curves) normalized with respect to peak amplitude for each mode.

compares the frequency of the symmetric critical modes in the two cases, and indicates reasonable agreement of the prograde propagation at large \( T \), and the switch-over from retrograde to prograde occurring between \( T=10^4 \) and \( 10^5 \). However, the spherical shell calculations give frequencies about double the annulus case for \( T \lesssim 10^4 \). Most of these values in any case are quite small, and the technique used to find \( \omega \) in the annulus case was not as accurate as we used in this work.

e. Asymptotics for equatorial modes at large \( T \)

The large \( T \) results of Roberts (1968) and Busse (1970b) for a full sphere indicate that the critical Rayleigh number \( R_c \) should ultimately increase as \( T^{3/2} \), \( \omega_c \) as \( T^{1/2} \), and \( m_c \) as \( T^{1/6} \). We have tested our own solutions to see the degree to which they too show this dependence. The results are shown in Figs. 9 and 10 for the symmetric and antisymmetric modes. It appears from these results that in both cases \( \omega_c \) and \( m_c \) have already reached this \( T \) dependence by \( T=10^6 \), and that \( R_c \) is approaching it but has not reached it. This may be simply because \( T^{3/2} \) is a more rapidly varying function of \( T \) than is either \( T^{1/2} \) or \( T^{1/6} \). We note that accuracy of the grid does not appear to be a problem in this range of \( T \), since, for example, a 50% increase in resolution in both radial and latitudinal directions re-
sults in a less than 1% increase in $R_e$ at $T = 10^6$, and still smaller changes at smaller $T$.

f. Comparisons of polar modes with plane-parallel convection

Provided that their scale in latitude is not too large, we should expect fairly close correspondence in Rayleigh number for the polar modes with that for plane-parallel convection with rotation parallel to gravity (with similar applied boundary conditions and provided our variable gravity is scaled out). We find that the values are close at moderate $T$, but diverge for large $T$. We make the comparison by reducing all plane parallel $R$ values by a factor $R(m=0, T=0)/R$ (plane parallel, $T=0$) or approximately 0.838 for symmetric modes and 0.824 for antisymmetric modes. Fig. 11 shows a plot of the ratio of $R$ for $m=0$ and $1$ from our calculations with the scaled value of $R$ for the rotation parallel to gravity case, taken from Chandrasekhar (1961). We see that by $T = 10^4$, the low-resolution calculations are

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**Fig. 7.** Comparison of critical Rayleigh numbers for spherical shell and equatorial annulus of same depth [annulus curves taken from Gilman (1972)].

**Fig. 8.** Comparison of oscillation frequencies for critical modes under symmetric conditions for spherical shell and equatorial annulus. (Equatorial annulus data previously unpublished.)
producing too small an $R$ by 5–10%, and by $T=10^4$, give a grossly reduced $R$, only 40% of the plane-parallel value. The increased resolution of the 13×121 case clearly helps substantially, but at a cost of considerable computer time, especially since the increased resolution, at least for the linear problem, is needed only in polar regions. The reason why $R$ is lower than expected is simply because with increasing $T$ the modes want to become narrower in horizontal dimension and sharper in vertical gradients, and are eventually not fully resolved. One consequence is lower diffusion effects, therefore requiring a lower $R$ for instability.

From Chandrasekhar (1961, p. 120) we know the asymptotic dependence of the plane parallel convection with increasing $T$ is $R \sim T^{2/3}$ (as with the equatorial modes). One consequence of the deficient resolution near the pole is that this dependence is never reached. This is illustrated in the plot of $R/T^{2/3}$ for polar modes in Fig. 12. From this it is clear that increased resolution does help. This has implications for nonlinear and higher $T$ calculations in that, ideally, we should expect a fixed ratio between $R$ at the pole and $R$ at the equator to develop, since they both should behave like $T^{2/3}$. This results in a spuriously small ratio between $R$ (polar)/$R$ (equatorial), meaning that in the nonlinear calculations polar modes will probably appear at substantially lower $R$ than they should. It appears the 13×121 resolution is acceptable up to $T=10^5$, while the 9×79 resolution should probably not be used beyond $T=10^4$.

g. Effects of a polar cap

One of the early techniques by which we tried to retain computational stability near the pole was to remove a few points in latitude closest to the pole. We found that this worked well so long as a mode which before point removal peaked near the equator still did so. However, we found examples where a mode $m$ which peaked near the equator without point removal now peaked just equatorward of the removed points, and at a substantially lower $R$ than before. For example, $m=3$ at $T=10^4$, which normally peaked near the equator and became unstable for $R \approx 3240$ (from

![Fig. 11. Comparison of Rayleigh number for instability onset near poles, with Rayleigh number for plane-parallel convection with rotation parallel to gravity under same boundary conditions [latter from Chandrasekhar (1961)]. Comparisons are for modes $m=0$ and 1, at two resolutions (defined in text).]
Fig. 12. Asymptotic behavior of polar Rayleigh number for instability onset, for modes $m=0, 1$ at two resolutions, compared to plane-parallel convection case.

Fig. 1), now peaked just outside the polar wall and was unstable at $R=2340$, a reduction of about 28%. In addition, the frequency of the mode is greatly increased, in this particular case from about 1.8 to 10.3. At smaller $m$, the polar mode frequency is increased by a still larger factor, in some cases a factor of 50. These results indicate that care must be exercised in filtering near the pole to avoid those spurious effects. In the earlier stability and frequency plots (Figs. 1, 2, 4, 5) R and $\omega$ are plotted only for polar modes with no points removed, and equatorial modes which remain so when points are removed.

h. Overstability in polar modes

In plane-parallel convection under the influence of rotation parallel to gravity, it is well known (Chandrasekhar, 1961) that below a Prandtl number $P$ of about 0.677, the convection sets in first above a certain Taylor number as “overstability” or pairs of inertially oscillating modes rather than as stationary convection. Fig. 29 of Chandrasekhar (1961) illustrates the family of stability boundaries for different $P$. Davies-Jones and Gilman (1971) showed that if the fluid is instead confined to a channel, the Prandtl number upper limit is removed, and overstability occurs at all $P$, but with the needed Taylor number increasing with $P$. They actually got two complimentary modes of equal and opposite frequency, one propagating prograde along the inside edge of the rotating channel, and the other retrograde on the outside. What has happened in our present calculations is that the wall circling the pole allows one of these oscillations to occur in the sphere. The low frequencies found without the wall corresponded to stationary convection slightly modified by the curvature effects near the pole, while the large jump in frequency with peak amplitude just outside the wall represents the overstable mode propagating prograde around the edge of the wall. Had we added a second wall somewhat equatorward of the first, to create a channel circling the pole, we probably would have excited the second overstable mode which would have propagated retrograde, though at a slightly different frequency due to curvature effects.

Although we have not attempted to systematically study overstability in the sphere as a function of decreasing $P$, it is clear from the few calculations we have made for low enough $P$, that it definitely does occur even without putting in a wall near the poles. Because of the curvature effects, its nature is hard to sort out but it is clear from Fig. 29 of Chandrasekhar (1961) that the Rayleigh number needed near the poles for instability onset can be substantially reduced, which could make it more nearly equal to the $R$ values we have found for the equatorial modes. However, there is evidence from Rossby (1969) that the overstable modes are rather inefficient at transporting heat, so much so that in a laboratory experiment with mercury ($P=0.025$), the overstable modes changed into stationary convection well below the $R$ predicted by Chandrasekhar (1961). The degree to which these effects may contribute to smoothing out the heat flux with latitude can be studied in the nonlinear case by lowering the Prandtl number after a solution for a given $R$ and $T$ has been reached.

5. Mode structure

a. Equatorial symmetric modes

To show what the convection in the deep spherical shell looks like, particularly how it evolves from low to high Taylor number, we have produced computer plots for a few cases. Figs. 13, 14 and 15 illustrate the structure for the critical symmetric modes [circled $R(m)$ points in Fig. 1] successively for $T=0, 10^9$ and $3\times10^9$, which span from no rotation to strong rotational effects. We see from Fig. 13 that at $T=0$, the structure is a roll with north-south axis, with peak horizontal and vertical velocities near 25° latitude. Because of the symmetry conditions, there is no flow across the equator. With modest rotational influence ($T=10^9$) Fig. 14 shows that the peak in vertical motion is at the equator, and the predominantly east-west horizontal velocities near the top are turned sharply to their right, an amount which increases with latitude. The horizontal velocities near the bottom are only slightly turned. Now a fluid particle which rises at one latitude will, in general, sink back down at either a substantially lower or higher latitude.

By $T=3\times10^9$ (Fig. 15) Coriolis forces have so distorted the convection that nearly closed swirls are formed near the outer edge with centers at about 25° latitude. The bottom flow is also substantially turned. By $T=10^9$ (not shown) the swirls have become considerably tighter, so tight that very little horizontal flow occurs near the upper boundary in the neighborhood of the equator. On the other hand, the horizontal flow peaks at the equator near the bottom.

What is happening to the convection as $T$ is increased is that it is evolving from rolls with a local latitudinal axis into rolls with axis parallel to the axis of rotation, and extending across from one hemisphere
Fig. 13. Northern Hemisphere structure of critical ($m=9$) symmetric mode for $T=0$.
Top, horizontal flow near top boundary; middle, radial motion contours at mid-level (solid contours indicate upward motion, dashed contours downward motion); bottom, horizontal flow near bottom boundary. Arbitrary units. Each cross section spans 90° in longitude (marked every 10°). Dotted arcs signify 10° latitude contours, starting at equator for each section. Rotation of coordinate system is from left to right.
to the other. This axis cuts through the outer surface at a modest latitude. As $T$ gets quite large, the dimensions of the roll normal to its axis shrink, and the flow pulls away from the outer boundary near the equator. In the bottom sections, we are seeing the back side of the rolls, indicating they remain close to the inner boundary. The convection is attempting to satisfy the Taylor-Proudman constraint, of minimum variation of the variables in the direction of the rotation axis. This constraint must be broken, however, at the outer curved boundary since no flow is allowed normal to it. Consequently, the roll tries to be as long as possible by squeezing close to the inner boundary at the equator. It is probably also prevented by the Taylor-Proudman constraint from migrating past the cylinder tangent to the inner spherical boundary; if it were to do so, the fluid would be convecting partly in a stable region, and would be much more inefficient. The symmetry conditions applied at the equator allowing no flow across, together with the curved outer boundary, then force a flow along the axis of rotation which varies rather strongly with latitude, in violation of the Taylor-Proudman constraint. This nongeostrophic flow assumes the profile needed to keep fluid from escaping the region and therefore satisfies the boundary conditions. Examination of the pressure field (not shown) indicates that, while at $T=0$ the horizontal flow is always from high to low pressure, with increasing $T$ the flow is increasingly around centers of high and low pressure, in the geostrophic sense, with clockwise horizontal flow.
about highs in the Northern Hemisphere, etc. Corresponding effects are seen in the flow about highs and lows when one looks at longitude-radius sections (also not shown).

Evolving into a roll with axis parallel to the axis of rotation allows the convection to minimize the rotational constraint, and therefore become unstable at the lowest possible Rayleigh number. For a given $T$, larger scale modes, i.e., lower longitudinal wavenumber $m$, behave similarly, but are less and less efficient, and therefore become unstable at higher $R$, because the horizontal particle path lengths are increased, during which dissipation takes place but no energy conversion. In addition, the longitudinal pressure gradients, which must balance the Coriolis forces to keep the convection efficient, become weaker for smaller $m$. For $m=0$, the longitudinal pressure gradient disappears altogether. We suspect that at high enough $T$, all modes except $m=0$ become equatorial modes. The $m=0$ mode cannot, because no matter what the Rayleigh number, its lack of longitudinal pressure gradients prevents it from satisfying the Taylor-Proudman theorem by assuming an equatorial structure. It does so instead by becoming a tight vortex ring about the pole where the conservation of angular momentum has the least stabilizing effect.

The structure of the most unstable equatorial mode was also captured in the equatorial annulus approximation employed by Gilman (1972).

Fig. 15. As in Fig. 13 except for critical ($m=12$) symmetric mode for $T=3\times10^4$. 
b. Equatorial antisymmetric modes

The antisymmetric modes were not studied in detail in Gilman (1972) but may be important in the non-linear case, since in the range of $T$ studied they become unstable at no more than about 2.5 times the critical $R$ for the symmetric modes (see Fig. 3). Because of the symmetry conditions, their structure is quite different from the symmetric modes, particularly since they allow for flow across the equator. One example of their structure is given in Fig. 16 for $T = 3 \times 10^4$. We see that as in the case of the symmetric mode, swirls again develop near the top, and the horizontal flow near the equator largely disappears. But in contrast to the symmetric mode, near the bottom there is strong cross-equatorial flow, exchanging mass from the downflow regions of one hemisphere to the upflow regions of the other. Because of the antisymmetry the roll is more complicated with oppositely directed swirl about the axis in the two hemispheres. The antisymmetric pressure field associated with this configuration provides the nonequatorial pumping of fluid across the equator from high to low pressure. This takes place near the bottom or inner surface in order to allow the longest flow path and therefore the smallest variation of this
axial flow along the axis, thus coming as close as possible, under the imposed boundary conditions, to satisfying the Taylor-Proudman constraint. As with symmetric boundary conditions, the roll axis migrates toward the inner surface with increasing $T$, again to give the weakest variation in the swirl in the direction of the rotation axis.

The antisymmetric modes at lower $m$ for a given $T$ have similar structures but more elongated in longitude, and therefore stabilized, and at higher $m$ give structure that looks like lower $T$ modes, due to the increased friction breaking the rotational constraints.

c. Polar modes

The structure of polar modes is of interest even though they are not the most unstable in the spherical shell, because they peak at a much different location than the equatorial modes, and therefore could be expected to grow relatively unaffected by the equatorial modes until mid-latitudes are filled in with convection. The $m=0$ mode near the pole is easy to describe without figures. At $T=0$, it is a series of rings concentric with the polar axis, with flow confined to meridian planes. As $T$ is increased, these rings acquire an increasing azimuthal component of flow. They contract about the axis, and their cross section gets smaller. The modes $m=1$ and $2$ are harder to describe; horizontal velocity vectors near the outer boundary are shown for these in Fig. 17. The flow near the bottom (not shown) in each case looks almost the same, but with the direction of all the arrows reversed. The vertical motion (also not shown) peaks approximately at the points of divergence and convergence in the horizontal velocity vectors. The coordinate system is expanded so that $m=1$ is shown only poleward of about $75^\circ$, and $m=2$ poleward of $60^\circ$. From Fig. 17, we can see that with $m=1$, flow across the pole is simulated, even though to avoid the mathematical singularity there we set $\theta=0$ right at the pole in the integrations. For $T=0$, we see a point of divergence on the right-hand side near $80^\circ$ latitude, and a corresponding point of convergence on the left-hand side. At $T=10^4$, these vectors are all turned to their right by the Coriolis forces, resulting in two longitudinally elongated vortices of opposite senses on either side, with the centers moved closer to the pole by $3^\circ$ or $4^\circ$. By $T=3\times10^4$, the vortices have tightened still closer to the pole and in their own latitudinal dimension, with the suggestion of the formation of oppositely directed outer vortices appearing around $80^\circ$. At still higher $T$, these vortices get too tightly packed around the pole to be well resolved.

The $m=2$ mode shows a similar development of four vortices, but whose center for a given $T$ is at a somewhat lower latitude. Still higher $m$ show similar patterns, with more vortices placed still further out. However, even $m=3$, which comes in when antisymmetric boundary conditions are applied at the equator to suppress the equatorial modes, still has its ring of vortices occurring at very high latitudes. For a given $T$ from Figs. 1 and 2 all the polar modes seem to be unstable at about the same Rayleigh number. From the structure, what appears to happen is that as $m$ and therefore dissipation is apparently increased, the mode peaks simply spread a little further from the pole, increasing the physical distance in longitude associated with that $m$, and thereby resulting in about the same dissipation. This presumably can go on indefinitely, if equatorial modes are suppressed by some means.

d. Transition from polar to equatorial modes

As indicated earlier, the transition from polar to equatorial peaking modes occurs rather abruptly. How radically this changes the mode structure for the same $m$ is illustrated in Fig. 18, which shows the horizontal flow near the top for $m=3$ at $T=10^4$, which is a polar mode, and $T=10^4$, which is an equatorial mode. The $T=10^4$ plot is confined to $60^\circ$ and poleward, while the $T=10^4$ plot is for a full hemisphere, with latitude circles equally spaced. Comparison of the two structures indicates there is almost no overlap in significant amplitudes. At $T=10^4$, the vortices center at about $75^\circ$ latitude, while by $T=10^4$ they center near $25^\circ$, with long sweeps of flow all the way from the equator up to about $60^\circ$, and back, with a phase shift in latitude of more than half a longitudinal wavelength.

6. Transport properties

Although all of our simulations are for the linear system of equations, we can estimate at least the relative amplitudes of the nonlinear transports of heat and momentum to be expected from each mode interacting with itself. This gives us an idea of how the mean temperature field and differential rotation are likely to develop and change when the nonlinearities are included.

a. Heat transports

Under the influence of rotation the symmetric mode radial heat flux always peaks at the equator. This is because both the radial motion and temperature perturbations peak there. By $T=10^4$, the peak flux has shifted substantially toward the inner boundary, leaving a modest layer near the outer boundary where there is very little heat flux outward. This effect is more pronounced for some lower $m$ modes at the same $T$. As $T$ increases, the peak heat flux as a function of latitude changes from a spherical surface to one more nearly cylindrical, and parallel to the axis of rotation, thus reflecting the convective structure becoming a roll with axis aligned in that way.

The antisymmetric modes show heat flux also concentrated near the equator, but with the peak always between about $15^\circ$ and $20^\circ$ latitude. The symmetry
Fig. 17. Polar modes horizontal velocities near top for $m = 1, 2$; $T = 0$, $10^{-2}$, $3 \times 10^{-2}$. For $m = 1$, plot is from $75^\circ$ latitude circle to pole; for $m = 2$, from $90^\circ$ to pole (projection linear in latitude coordinate).
conditions require it to be zero at the equator, since both the radial motion and temperature perturbations change sign there. The locus of maximum heat flux also deforms as $T$ increases from being spherical to being nearly parallel to the rotation axis in these modes too, and by $T = 10^4$ a region of much reduced flux near the top near the equator also develops.

The radial heat flux associated with polar modes is, of course, concentrated within $10^6 - 15^6$ of the poles, and becomes more and more narrowly concentrated there for a given $m$ as $T$ increases.

All the modes transport heat in latitudinal as well as radial directions, although this does not directly release any potential energy. The symmetric modes quickly evolve by $T = 10^6$ into such a shape that they transport heat toward the equator. The magnitude of this flux is generally $10 - 30\%$ of the radial heat flux of the same mode. If one realizes in Figs. 13-15 for the structure of these symmetric modes that the temperature field is virtually in phase with the vertical motion field, then one can see immediately that warm fluid is predominantly moving toward the equator, and cold fluid toward higher latitudes. This is a natural consequence of the evolution with increasing $T$ of the convection into a roll with axis parallel to the rotation axis, since in the limit the heat flux should be perpendicular to this axis, resulting in an equatorward component at all latitudes away from the equator.

The latitudinal heat flux in the antisymmetric modes is rather different, with strong poleward flux developing nearest the equator and near the bottom, and equatorward flux confined to higher latitudes and levels. The poleward flux simply comes from the bottom flow crossing the equator converging into the warm regions where it then rises to the surface (as can be easily seen in Fig. 16, again noting temperature and radial motion fields nearly coincide). The equatorward flux at higher latitudes is again from the convection aligning itself there more along the axis of rotation.

As with their structure the heat transport properties of the symmetric equatorial modes are very similar to those found in the equatorial annulus case.

b. Angular momentum transports

As soon as rotation is added to the system, there is a reservoir of angular momentum which can be redistributed by the convection. The structure of the angular momentum transport by the first unstable modes is shown in Figs. 19 and 20. We see in Fig. 19 that the radial angular momentum transport is inward near the equator at $T = 10^6$ (and is for all lower $T$ also) but steadily evolves to outward transport at all latitudes by $T = 10^4$. What happens here is that with weak rotation the Coriolis forces push radially inward flow in the direction of rotation in longitude, and radially outward flow in the opposite direction, resulting in a net inward momentum flux. Another way of looking at this is that absolute angular momentum is nearly being conserved by the upward and downward moving parcels, so the rotation of the upward moving ones slows down, the downward moving one speeds up. As $T$ increases and the flow becomes more rotationally constrained, however, the pressure forces react and nearly balance Coriolis forces. These pressure forces provide an azimuthal torque, so the fluid no longer conserves angular momentum. To do this the pressure field must evolve in such a way that a tilt is retained in the iso-bars in radius-longitude planes such that they are displaced in the direction of rotation near the outer...
boundary compared to the inner boundary. (The opposite tilt would make pressure gradient and Coriolis forces reinforce each other.) As the flow increasingly parallels these isobars, fluid particles moving outward are now moving in longitude in the direction of rotation, and fluid particles sinking inward are moving in the opposite direction, thus leading to outward radial momentum transport. As with the heat transports, the radial momentum transport by $T=10^6$ tends to peak along a line parallel to the axis of rotation. Correlating the horizontal and vertical motion plots for $T=10^9$ and $3 \times 10^9$ in Figs. 14 and 15 verifies the change in direction of the transport with increasing $T$.

In the antisymmetric modes we can see that the effect of Coriolis forces in $r$-$\lambda$ planes producing downward momentum flux peaking at about latitude $15^\circ$ is even stronger, and is only partially overcome near the outer boundary by $T=10^6$. Of course, radial angular momentum transport by antisymmetric modes is always zero at the equator.

From Fig. 20, we see that the symmetric modes always transport angular momentum toward the equator in the upper half or more of the fluid shell. This also happens in the antisymmetric modes except for a weak patch of poleward transport right next to the equator. The large regions of equatorward flux are natural results of the Coriolis forces in longitude-latitude surfaces turning a predominantly east–west flow at low $T$ more into the north–south direction. From the flow near the top seen in Figs. 14 and 15, it is easy to see that vectors in the direction of rotation also have components toward the equator, while vectors pointed opposite to the rotation direction also have poleward components, thus giving a net equatorward flux.
As with the mode structure, we find that for a given T, modes with \( m \) greater than the first unstable mode tend to act like modes at lower T. Thus, for example, the upward momentum transport at \( T = 3 \times 10^6 \) seen near the equator shifts back to downward transport at higher \( m \). On the other hand, modes at lower \( m \) at that T show even more dominant upward transport. Typically, radial momentum transports are 10–30% of the latitudinal transport, which is the reverse of the ratio for heat transport.

Both radial and latitudinal momentum transport properties of the equatorial symmetric modes are similar to those found in the equatorial annulus.

Angular momentum transports by polar modes are both confined to polar regions and much weaker than the transport of equatorial modes, since they are so much closer to the axis, and so are not discussed here.

7. Implications for nonlinear studies

Perhaps the two major questions to be answered by nonlinear studies are 1) how quickly and in what way convective modes fill-in in latitude between the equatorial and polar modes we have found and how the resulting heat flux depends on latitude; and 2) what amplitude and structure of the differential rotation and meridional circulation are induced by the convective cells through their nonlinear transports of heat and angular momentum. The two questions are related in that, for example, the latitudinal and radial shears of the induced differential rotation should favor convection with longitudinal wavenumber \( m \) smaller than without the shears being present [see, e.g., Davies-Jones (1971) and earlier references cited therein]; this, in turn, may alter the momentum transports and feedback on the differential rotation. In addition, even without shear,

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**Fig. 20.** As in Fig. 19 except for latitudinal angular momentum flux T. Solid contours indicate poleward flux, dashed contours equatorward flux.
laboratory experiments (e.g., Willis et al, 1972) and theoretical models (e.g., Lipps and Somerville, 1971; Somerville and Lipps, 1973) indicate convection tends to increase in horizontal scale above critical, also suggesting we should see movement toward lower m. Breaking rotational constraints by nonlinear effects should also tend to reduce m.

The main goal of the nonlinear calculations vis-à-vis their application to the sun would be to find those conditions, if they exist, under which the heat flux becomes essentially uniform in latitude, while at the same time a differential rotation of about the right magnitude and profile is maintained by the convection.

With respect to the differential rotation, the momentum transports described in Figs. 19 and 20 indicate that if we take the Taylor number T large enough, we should get the maximum angular velocity occurring near the equator and near the outer angular boundary, since momentum is brought in from high latitudes and from below. This was confirmed in the equatorial annulus calculations of Gilman (1972), but these calculations also showed that as R increased, an equatorial deceleration began to appear in between two rather sharp jets symmetrically placed about and near the equator. This was due to nonlinear effects breaking the rotational constraints, but may have been amplified by restricting the calculation to a fixed and single longitudinal wavenumber. It is a feature we intend to study carefully, including the effects of several modes, and differential rotation inducing decreases in m.

The differential rotation of the sun appears to be predominantly (though not entirely) symmetric about the equator. In the spherical shell convection model, an antisymmetric differential rotation can be produced only by interactions between symmetric and antisymmetric modes of the same m. Our linear results suggest the symmetric modes, being unstable at lower Rayleigh number R, will likely dominate near the onset of instability. However, some unpublished nonlinear calculations for the equatorial annulus by the author suggest antisymmetric modes may become important at large R; this, too, needs to be studied.

Finally, of course, it will be very desirable to test the hydromagnetic dynamo properties of the nonlinear motions to see, for example, whether magnetic cycles are produced.

Acknowledgments. The author wishes to express his deepest appreciation to Jack Miller of the NCAR computing facility for his excellent programming of all aspects of the model presented here.

Friedrich Busse, Bernard Durney and David Williamson offered useful criticisms and insights.

REFERENCES