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The Development of Thermal Anomalies in a Coupled Ocean-Atmosphere Model

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ABSTRACT

A simplified but nonlinear model of large-scale air-sea interaction is formulated which involves the interaction between a finite-amplitude cyclone wave, the large-scale atmospheric temperature field, and sea-surface temperatures. The system interacts through a simplified model of air-sea heat exchange and consequent alterations of wind-driven advection of the sea-surface temperature field. It is shown that in cases when there is a large heat release to the atmosphere and long-term storage of heat in the mixed layer that small "seed" anomalies can grow by a finite-amplitude feedback instability.

1. Introduction

In mid-latitudes the dominant physical process which governs the dynamics of the atmosphere is the rectified transport of heat, vorticity and momentum by unstable baroclinic waves (cyclone waves). These waves spontaneously grow on the available potential energy associated with the equatorward gradient of temperature and therefore the strength and activity of the waves is sensitive to the large-scale or zonally averaged temperature field and the processes which maintain it. Namias (1963, 1973) has forcefully noted the correlation between long-term and large-scale departures in sea-surface temperature and atmospheric flow patterns and temperature fields. The long thermal memory of the ocean and its coupling by heat transfer to the atmosphere ensures that thermal anomalies in the upper mixed layer of the ocean will persistently affect the large-scale atmospheric flow and temperature fields and therefore the embedded cyclone wave field. At the same time the resulting change in the wind pattern alters the wind stress field which drives the oceanic circulation and hence the distribution of oceanic thermal anomalies.

Since the relatively small-scale cyclone waves also alter and determine the large-scale thermal and wind fields in the atmosphere, the coupled ocean-atmosphere system and its evolution over long time scales of months and years involves a mixture of dynamical elements of widely differing intrinsic time and space scales. The departure of the coupled system from a statistically "normal" pattern can manifest itself in several ways including long-term changes in the wavenumber regimes of the global atmospheric flow pattern.

A key conceptual question at the heart of the air-sea coupling problem is to determine the process which maintains and even intensifies an oceanic sea surface temperature anomaly. As Favorite and McClain (1973) have shown, the anomalies in the Pacific propagate across the ocean on the oceanic advective time scale and the anomalies seem to be intense in the eastern Pacific. This suggests a slow instability in the coupled air-sea system which favors the enhancement of "seed" thermal anomalies, which may be introduced in the western oceanic regions and grow by interaction with the atmosphere as they advect eastward through the large-scale oceanic subtropical gyre.

The purpose of this paper is to present a simple analytical model of air-sea interaction which displays a positive feedback mechanism which will intensify "seed" sea-surface temperature anomalies on the advective time scale of the oceanic gyre. The model is highly idealized but contains the three essential participants in the coupled dynamics. They are 1) a time varying atmospheric zonal flow in which is embedded 2) a quasi-geotrophic finite-amplitude baroclinic wave and 3) a variable sea-surface temperature field. All three elements interact through simplified laws of heat exchange between the ocean and atmosphere, varying wind-driven circulation of the ocean, and rectified transports of vorticity and heat by the cyclone wave disturbance. The detailed description of the interaction process is deferred to subsequent sections of the paper, but the essential interaction mechanism is as follows. In the presence of small oceanic thermal anomalies which tend to strengthen the latitudinal atmospheric temperature gradient, enhanced cyclone activity results. The rectified potential vorticity transport of the cyclone wave in regions of relatively strong mean temperature gradients tend to be balanced by vortex tube stretching in the atmosphere driven by heat exchange with the ocean. In an attempt to balance this heat exchange, meridional advection of sea-surface temperature by the wind-driven circulation is required. This in turn requires a change in the large-scale atmospheric thermal wind. The spatial phase in latitude of this change is such as to intensify the horizontal temperature gradient in which the cyclone wave is embedded, which leads to a further intensification of the cyclone wave amplitude and sea-surface temperature anomaly. This constitutes a positive feedback in the coupled air-sea system.

The model is formulated so that the growth of the anomaly is strictly temporal, while the anomaly is homogeneous in longitude. The relationship of this problem to the more relevant (and technically difficult) longitudinally dependent growth is discussed in the final section of this paper.

2. The model

The model used for the atmospheric component of the air-ocean system is a variant of the standard quasi-geostrophic dynamics suitable for mid-latitudes (Pedlosky 1970).

The *dimensionless* space coordinates are x, y and z which measure distance eastward, northward and upward, respectively. The dimensionless time coordinate is t. The dimensionless horizontal velocity vector is $\mathbf{q}$, while the vertical velocity is w. Pressure, density, temperature and potential temperature departures from a standard atmosphere are p, ρ, T and θ, respectively. The relationship between these dimensionless variables and their dimensional counterparts (denoted by asterisks) is

$$\left.\begin{array}{l} \begin{pmatrix} x_* \\ y_* \\ z_* \end{pmatrix} = L\begin{pmatrix} x \\ y \\ (D/L)z \end{pmatrix} \\ \begin{pmatrix} \mathbf{q}_* \\ w_* \end{pmatrix} = U\begin{Bmatrix} \mathbf{q} \\ (D/L)w \end{Bmatrix} t_* = (L/U)t \\ p_* = p_s(z) + \rho_s(z) f_0 U L p \\ \rho_* = \rho_s(z) + \rho_s(z) f_0 U L/gD\rho \\ \theta_* = \theta_s(z) + \theta_s(z) f_0 U L/gD\theta \\ T_* = T_s(z) + T_s(z) f_0 U L/gDT \end{array}\right\}, \tag{2.1}$$

where p_s, ρ_s, T_s and θ_s are the standard atmosphere distributions of the thermodynamic variables. The Coriolis parameter, $f=2\Omega\sin\theta$, is linearized in the standard β-plane fashion around a central latitude θ_0 such that

$$f = f_0\left(1+\frac{U\beta}{f_0 L}y\right), \tag{2.2a}$$

where

$$\beta = \frac{2\Omega\cos\theta_0}{R_e U}L^2, \tag{2.2b}$$

L and D are appropriate horizontal and vertical spatial scales, U is a scale for the horizontal velocity, and R_e is the earth's radius.

The equations governing the dependent variables in (2.1) are

$$\epsilon\left(\frac{d\mathbf{q}}{dt}+w\frac{\partial\mathbf{q}}{\partial z}\right)+(1+\epsilon\beta y)\mathbf{k}\times\mathbf{q} = -\nabla p/(1+\epsilon\mathfrak{F}\rho), \tag{2.3a}$$

$$\rho = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s p), \tag{2.3b}$$

$$\left(\frac{d\rho}{dt}+w\frac{\partial\rho}{\partial z}\right)+(1+\epsilon\mathfrak{F}\rho)\left(\nabla\cdot\mathbf{q}+\frac{\partial w}{\partial z}\right)+\frac{w}{\rho_s}\frac{\partial\rho_s}{\partial z}=0, \tag{2.3c}$$

$$\epsilon\frac{d\theta}{dt}+(1+\epsilon\mathfrak{F}\theta)S(z)w+\epsilon w\frac{\partial\theta}{\partial z} = \frac{\dot{Q}_*}{\rho_s c_p T_s \epsilon\mathfrak{F} f_0}\frac{(1+-\mathfrak{F}\theta)}{(c+\epsilon\mathfrak{F}T)}, \tag{2.3d}$$

$$\ln(1+\epsilon\mathfrak{F}\theta) = \frac{c_v}{c_p}\ln\left(1+\frac{\rho_s}{p_s}Uf_0Lp\right)-\ln(1+\epsilon\mathfrak{F}\rho), \tag{2.3e}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla,$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y},$$

$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the x, y, z directions, and c_p and c_v are the specific heats for air at constant pressure and volume, respectively.

The dimensionless parameters which appear in (2.3) are

$\epsilon = \dfrac{U}{f_0 L}$ the Rossby number

$S = \dfrac{g}{\theta_s}\dfrac{\partial\theta_s}{\partial z}D/(f_0^2L^2)$ the stratification parameter

$\mathfrak{F} = f_0^2L^2/(gD)$ the rotational Froude number.

The *dimensional* variable $\dot{Q}_*$, the rate of heat addition per unit volume of air, will be related below to the difference between atmospheric and sea-surface temperatures. It is convenient to define

$$\dot{Q} = \frac{\dot{Q}_*}{\epsilon^2\mathfrak{F} f_0 \rho_s c_p T_s}, \tag{2.4}$$

with $\dot{Q}$ being subsequently considered an O(1) variable.

Under the scaling presumptions

$$\left.\begin{array}{c} \epsilon \ll 1 \\ \mathfrak{F} = O(\epsilon) \end{array}\right\}, \tag{2.5}$$

each variable is expanded in a series in ϵ, e.g.,

$$w=w^{(0)}+\epsilon w^{(1)}+\cdots.$$

Substitution into (2.3) then yields the following dynamical equations for the lowest order variables

$$\mathbf{k}\times\mathbf{q}^{(0)}=-\nabla p^{(0)},\quad w^{(0)}=0, \tag{2.6a}$$

$$\rho_s\rho^{(0)}=-\frac{\partial}{\partial z}(p^{(0)}\rho_s), \tag{2.6b}$$

or

$$\theta^{(0)}=\frac{\partial p^{(0)}}{\partial z}, \tag{2.6c}$$

and

$$\frac{d}{dt}(\zeta^{(0)}+\beta y)=\frac{1}{\rho_s}\frac{\partial\rho_s w^{(1)}}{\partial z}, \tag{2.7a}$$

$$\frac{d\theta^{(0)}}{dt}+w^{(1)}S=\dot{Q}, \tag{2.7b}$$

where

$$\zeta^{(0)}=\mathbf{k}\cdot\nabla\times\mathbf{q}^{(0)}.$$

Elimination of $w^{(1)}$ between (2.7a, b) leads to the potential vorticity equation

$$\frac{d}{dt}\left[\zeta^{(0)}+\beta y+\frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\rho_s\theta^{(0)}}{S}\right]=\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\dot{Q}}{S}\right), \tag{2.8}$$

but for the purposes of this study it is more convenient to work with (2.7a, b).

To produce the maximum simplification in the model the continuous (in the vertical) system (2.7a, b) is replaced by a two-layer system (Fig. 1) in which all velocities are defined at layers 1 and 2 while the temperature field is defined at the mid-level, M, in the usual way. The set (2.7a, b) then becomes in the finite-difference version:

$$\left[\frac{\partial}{\partial t}+u_1^{(0)}\frac{\partial}{\partial x}+v_1^{(0)}\frac{\partial}{\partial y}\right]\times\left[\rho_{s_1}(\zeta_1^{(0)}+\beta y)+\frac{\rho_{sM}}{S_M/4}(p_2^{(0)}-p_1^{(0)})\right]=\left[\rho_s\frac{w^{(1)}}{\frac{1}{2}}\right]_0-\frac{\rho_{sM}}{\frac{1}{2}}\frac{\dot{Q}_M}{S_M}, \tag{2.9a}$$

$$\left[\frac{\partial}{\partial t}+u_2^{(0)}\frac{\partial}{\partial x}+v_2^{(0)}\frac{\partial}{\partial y}\right]\times\left[\rho_{s_2}(\zeta_2^{(0)}+\beta y)+\frac{\rho_{sM}}{S_M/4}(p_1^{(0)}-p_2^{(0)})\right]=\rho_{sM}\frac{\dot{Q}_M}{\frac{1}{2}S_M}-\frac{\rho_{s4}}{\frac{1}{2}}w_4^{(1)}. \tag{2.9b}$$

Subscripts 0, 1, M, 2, 4 refer to variables at the top, upper-level, mid-level, lower level and surface, respectively, while u and v are the velocities in the x and y directions. In order to dynamically close the system, relations between the vertical velocities at levels 0 and 4 and the horizontal velocities at levels 1 and 2 must be given. I assume that at the surface

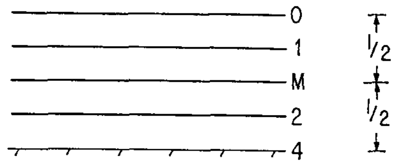

FIG. 1. The layer model. Dynamic variables are carried explicitly at levels 1 and 2. The atmospheric temperature is defined at the mid level M, proportional to the pressure difference between levels 2 and 1.

$$w_4^{(1)}=\frac{\delta}{\epsilon D}\zeta_2^{(0)}, \tag{2.10}$$

which is the Ekman pumping condition and δ is the Ekman layer depth. For purposes of symmetry only, in order to ease the subsequent algebraic calculations, a similar relationship is assumed at the top, i.e.

$$w_0^{(1)}=\frac{-\delta}{\epsilon D}\zeta_1^{(0)}. \tag{2.11}$$

It would not be difficult in principle to simply set $w_0^{(1)}$ to zero but the resulting system becomes algebraically much more complex.

The vortex tube stretching provided by the heating term $\dot{Q}_M$ in (2.9a, b) depends on the heat released to the atmosphere from the sea. I assume that $\dot{Q}_{M*}$ can be approximated by the relation

$$\dot{Q}_{M*}=-c_D U_*(T_{M*}-T_{w*})c_p\rho_M/D, \tag{2.12}$$

where c_D is a drag coefficient, U_* a typical surface atmospheric velocity, and T_{M*} and T_{w*} are the dimensional temperatures of the atmosphere and sea surface, respectively. If $\dot{Q}_{M*}$ is zero in the standard atmospheric state it follows that

$$\dot{Q}_M=-2\frac{c_D U_*}{U}\frac{L}{D}(p_1-p_2-2T_w), \tag{2.13}$$

where

$$T_{w*}=T_s(1+\epsilon\mathfrak{F}T_w). \tag{2.14}$$

Then (2.9a, b) can be rewritten as

$$\left[\frac{\partial}{\partial t}+\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\psi_1+\beta y+F(\psi_1-\psi_1)]=-r\nabla^2\psi_1+\kappa\{\psi_1-\psi_2-\theta_w\}, \tag{2.15a}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial\psi_2}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi_2}{\partial y}\frac{\partial}{\partial x}\right][\nabla^2\psi_2+\beta y+F(\psi_1-\psi_2)]=-r\nabla^2\psi_2-\kappa\{\psi_1-\psi_2-\theta_w\}, \tag{2.15b}$$

where

$$\left.\begin{array}{l}\psi_n \equiv p_n^{(0)} \\ \mathbf{q}_n^{(0)} = \mathbf{k}\times\nabla\psi_n\end{array}\right\}, \quad n=1,2 \tag{2.16a}$$

$$\theta_w = 2T_w. \tag{2.16b}$$

In reaching (2.15a, b) the Boussinesq approximation has been used, i.e., the differences between ρ_{s_1}, ρ_{s_2} and ρ_{s_M} have been ignored. The parameters appearing in (2.15a, b) are

$$\left.\begin{array}{l}F = 4/S_M \\ r = 2\delta/\epsilon D \\ \kappa = c_D \dfrac{U_*}{U}\dfrac{L}{D}F\end{array}\right\}. \tag{2.17}$$

The system (2.15a, b) is identical to the layer model I used in an earlier study of finite-amplitude baroclinic waves (Pedlosky, 1970) with the addition of the heating term, which in this model is proportional to the difference between the single mid-level temperature resolvable in the model and θ_w, a measure of the sea-surface temperature.

To finally close the system an equation governing the evolution of θ_w must be added to the system. On time scales of the order of months to a year or so, the heat extracted or added to the ocean at the surface is stored in the surface mixed layer. To model this storage and keep the coupled problem as simple as possible I propose the following law for the evolution of θ_w:

$$\frac{d\theta_w}{dt} = H(x,y,t) + \lambda(\psi_1 - \psi_2 - \theta_w), \tag{2.18}$$

where H is a given heating function that is responsible for the alteration of the sea-surface temperature by processes other than direct heat exchange with the atmosphere. That process is modeled by the last term in (2.18), where λ can be related in a rough way to the physical properties of the system by

$$\lambda = c_D \frac{U_*}{U}\frac{c_p}{c_{pw}}\frac{\rho_M}{\rho_w}\frac{L}{h}, \tag{2.19}$$

where ρ_w, c_{pw} and h are the density, the specific heat of the water in the oceanic mixed layer, and a typical depth of the mixed layer. This modeling assumption is subject to severe criticisms. In the real ocean the process of heat storage and vertical mixing is considerably different depending on whether the oceanic layer is heated (in which case the depth of penetration is relatively shallow) or cooled (in which case there is much deeper penetration of the thermal anomaly). This nonlinear asymmetry is absent in the present model simply in order to keep the model tractable. In the same vein the coefficient λ should really depend on the variable surface winds. The constant λ approximation is made, in spite of its deficiencies, to allow a study of the simplest model that allows air-sea coupling by heat exchange.

The dominant temperature gradient in the model to be considered below is the equator-to-pole gradient. I therefore write

$$\frac{d\theta_w}{dt} = \frac{\partial\theta_w}{\partial t} + v_s\frac{\partial\bar{\theta}_w}{\partial y}, \tag{2.20}$$

where $\partial\bar{\theta}/\partial y$ is the mean north-south temperature gradient. The quantity v_s is the advective velocity in the upper layer of the ocean. In general it can be thought of as some combination of the velocity associated with the surface Ekman flux and the geostrophic velocity associated with the Sverdrup transport. In the present case, I consider only the latter, although to consider both would again not be difficult in principle. Now the Sverdrup flow in the northward direction is proportional to the wind stress curl and I assume that this can be taken as proportional to the vorticity of the lower layer of the atmosphere so that

$$v_s = +\alpha_1\nabla^2\psi_2, \tag{2.21}$$

where, in terms of dimensional quantities, α_1 can be approximately represented as

$$\alpha_1 = \frac{c_D U_*}{\beta_* L^2}\frac{L}{h_T}\frac{\rho_M}{\rho_w},$$

where $\beta_* = 2\Omega\cos\theta_0/R_e$, and h_T is a depth, of the order of the oceanic thermocline depth, which is used to relate the meridional Sverdrup transport to a meridional oceanic speed. The stress itself is assumed proportional to the drag coefficient multiplied by the velocity squared [only the linear, variable part appears in (2.21)] and the air density. α_1 then provides the proper parametric representation of the Sverdrup relation which relates the northward oceanic transport to the wind stress curl/$\rho_w\beta^*$. Then (2.18) becomes

$$\frac{\partial\theta_w}{\partial t} = H + \lambda(\psi_1 - \psi_2 - \theta_w) - \alpha_1\nabla^2\psi_2\frac{\partial\bar{\theta}}{\partial y}. \tag{2.22}$$

The closed dynamical set consists of (2.15a, b) and (2.22). As boundary conditions I take (Pedlosky 1070)

$$\left.\begin{array}{l}\dfrac{\partial\psi_n}{\partial x} = 0 \\ \displaystyle\int_{-\infty}^{\infty}\frac{\partial^2\psi_n}{\partial y\partial t}dx = 0\end{array}\right\}, \quad y=0,1; \quad n=1,2. \tag{2.23}$$

The coupling coefficients λ and α_1 are small [$O(10^{-2})$–(10^{-3})], which implies that on an atmospheric advective time scale there is little change in θ_w forced by the atmosphere. The coupling occurs on a much longer time scale and the purpose of this paper is to present the results of a study of the evolution of this system on the (nondimensional) time scale λ^{-1}. On that time scale the atmosphere and sea are coupled in this model by the heat exchange between air and sea and by the advection of sea-surface temperature by the changing atmospheric flow patterns.

Steady zonal solutions of the system for $H=0$ exist, i.e.,

$$\psi_1=-U_1y,\quad \psi_2=-U_2y,\quad \theta_w=-(U_1-U_2)y,\quad (2.24)$$

representing a zonal state with a uniform thermal wind in the atmosphere and a matching horizontal temperature gradient in the ocean. Values of U_1-U_2, the atmospheric thermal wind, will be chosen to be near the critical value for cyclogenesis and the subsequent long-term evolution of the coupled system will be examined in response to initial perturbations in either the atmospheric or oceanic temperature gradients. Although linear stability theory defines the critical values of the temperature gradient for cyclogenesis, the system evolves on such a long time scale that finite-amplitude baroclinic wave theory must be used to describe the interaction process between the cyclone wave, the atmospheric zonal flow, and the sea-surface temperature for by the time the interaction terms in (2.22) become important, the cyclone wave will have reached finite amplitude.

The first task, however, is to find the atmospheric linear stability criterion. This involves examining the response of (2.15a, b) on time scales sufficiently short so that θ_w is effectively held constant and the wave-like solutions of (2.15a, b) are decoupled from (2.22).

3. Linear stability theory

To examine the linear stability properties of the basic state (2.24) to small perturbations, let

$$\left.\begin{aligned}\psi_n&=-U_ny+\phi_n(x,y,t),\quad n=1,\ 2\\ \theta_w&=-(U_1-U_2)y+\theta_w'(x,y,t)\end{aligned}\right\}.\quad (3.1)$$

Then the free problem ($H=0$) becomes

$$\left[\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right][\nabla^2\phi_1-F(\phi_1-\phi_2)]+\phi_{1x}[\beta+F(U_1-U_2)]$$
$$=-r\nabla^2\phi_1+\kappa(\phi_1-\phi_2-\theta_w'),\quad (3.2a)$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2-F(\phi_2-\phi_1)]+\phi_{2x}[\beta-F(U_1-U_2)]$$
$$=-r\nabla^2\phi_2-\kappa(\phi_1-\phi_2-\theta_w'),\quad (3.2b)$$

$$\frac{\partial\theta_w'}{\partial t}=\lambda(\phi_1-\phi_2-\theta_w')+\alpha_1\nabla^2\phi_2(U_1-U_2).\quad (3.2c)$$

Since λ and α_1 are assumed to be small it is sufficient for the purposes of the linear stability study to let θ_w' be zero, which is equivalent to looking for the wavelike, *homogeneous*, solutions to (3.2a, b).

Let

$$\phi_n=A_ne^{ik(x-ct)}\sin m\pi y+{}^*,\quad n=1,\ 2,\quad (3.3)$$

where * represents the complex conjugate solution. In (3.3), k is the x wavenumber, c the complex wave speed, and the y dependence has been chosen to automatically satisfy the boundary conditions on $y=0$, 1. Substitution of (3.3) into (3.2a, b) yields two equations for A_1 and A_2, namely

$$A_1\left[(c-U_1)(a^2+F)+\beta+F(U_1-U_2)+i\left(\frac{a^2r}{k}+\kappa/k\right)\right]$$
$$=A_2[(c-U_1)F+i\kappa/k],\quad (3.4a)$$

$$A_2\left[(c-U_2)(a^2+F)+\beta-F(U_1-U_2)+i\left(\frac{a^2r}{k}+\kappa/k\right)\right]$$
$$=A_1[(c-U_2)F+i\kappa/k],\quad (3.4b)$$

where

$$a^2=k^2+m^2\pi^2.$$

If A_1 and A_2 are eliminated between (3.4a) and (3.4b), a quadratic equation for c results, whose solution is

$$c=\frac{U_1+U_2}{2}-\frac{\beta(a^2+F)}{a^2(a^2+2F)}-\frac{ir}{k}\frac{(a^2+F)}{(a^2+2F)}-\frac{i\kappa}{k(a^2+2F)}$$
$$\pm[2a^2(a^2+2F)]^{-1}\left[4\beta^2F^2-(U_1-U_2)^2a^4(4F^2-a^4)\right.$$
$$\left.-4a^4\left(\frac{rF}{k}-\frac{\kappa}{k}\right)^2+i8\beta F\left(\frac{rF}{k}-\frac{\kappa}{k}\right)a^2\right]^{\frac{1}{2}}.\quad (3.5)$$

Marginal stability occurs when $c_i=0$. Thus, for a given k and m, a critical shear $(U_1-U_2)=U_c$ is defined by the relation

$$c_i(k,m\pi,\beta,F,r,\kappa,\ U_1-U_2=U_c)=0,\quad (3.6)$$

where

$$\frac{U_c^2}{4}=\frac{1}{(2F-a^2)}\left(\frac{r^2a^2}{k^2}+\frac{2\kappa r}{k^2}\right)$$
$$\times\left\{1+\beta^2F^2\left[\frac{ra^2(a^2+F)+\kappa a^2}{k}\right]^{-2}\right\}.\quad (3.7)$$

There are several interesting features of this stability criterion. First, if $\kappa\neq0$, *all* shears are unstable (for $a^2<2F$) in the limit $r\to0$. Thus in the presence of thermal damping ($\kappa\neq0$), the β effect is not capable of stabilizing weak shears. Small thermal damping introduces an energy-releasing phase shift in otherwise stable waves. Second, even in the absence of thermal damping, the critical shear at any wavelength in the limit of small viscous dissipation ($r\to 0$) is

$$\frac{U_c^2}{4}=\frac{\beta^2F^2}{a^2(a^2+F)^2(2F-a^2)},\tag{3.8}$$

which is to be compared to the inviscid criterion (Pedlosky, 1964)

$$\frac{U_c^2}{4}=\frac{\beta^2F}{a^4(4F-a^2)}.\tag{3.9}$$

The ratio of (3.8) to (3.9) is $a^2/(a^2+2F)$, i.e., always less than 1, so that the presence of viscous dissipation also *destabilizes* small shears. In view of the relatively ineffective role of β in providing a stability constraint, *I will henceforth ignore β with regard to its role in the dynamics of the wave field*. This is done to take advantage of the resulting algebraic simplifications without altering the basic mechanics of the baroclinic wave or its rectified transports in finite amplitude. With $\beta=0$

$$\frac{U_c^2}{4}=\frac{1}{(2F-a^2)}\left(\frac{r^2a^2+2\kappa r}{k^2}\right),\tag{3.10}$$

and U_c becomes infinite at $a^2=2F$ and at $a^2=m^2\pi^2$ ($k=0$). The length scale associated with the *minimum* critical shear is

$$a^2_{\min}=\left[\left(\frac{2\kappa}{r}+m^2\pi^2\right)\left(\frac{2\kappa}{r}+2F\right)\right]^{\frac{1}{2}}-2\kappa/r.\tag{3.11}$$

In the limit $\kappa/r\to 0$, $a^2_{\min}\to(2F)^{\frac{1}{2}}m\pi$, while in the limit $\kappa/r\to\infty$, $a^2_{\min}\to F+m^2\pi^2/2$, so that increasing thermal damping shifts the wavelength for minimum critical shear to smaller scales.

On the neutral curve, for all k

$$c=\frac{U_1+U_2}{2},\tag{3.12}$$

while the amplitude ratio becomes

$$\gamma\equiv\frac{A_2}{A_1}=\frac{(\alpha^2-F)U_c-2i[(ra^2+\kappa)/k]}{FU_c-2i\kappa/k}.\tag{3.13}$$

It is a simple matter to show that $|\gamma|=1$.

The plan of the subsequent analysis is to consider the finite-amplitude evolution of a slightly unstable baroclinic wave and the role it plays in altering the mean, zonal atmospheric flow and the long-period interplay of these atmospheric elements with the evolving sea-surface temperature. In particular, the goal is to determine under what conditions, if any, small perturbations in θ_w will naturally grow on the long time scales due to the feedback between the cyclone wave, the changing zonal atmospheric flow, and the changing sea-surface temperature.

4. Finite-amplitude air-sea coupling theory

Consider a small but finite amplitude disturbance superimposed upon the basic state given by (2.24). That is, let

$$\left.\begin{aligned}\psi_n&=-U_ny+a_0\phi_n(x,y,t)\\ \theta_w&=-(U_1-U_2)y+a_0\theta(x,y,t);\quad U_1-U_2=U_c\end{aligned}\right\},\tag{4.1}$$

where a_0 is a nondimensional amplitude measure. It is understood that

$$a_0\ll 1.\tag{4.2}$$

A new "long" time variable (Pedlosky, 1970)

$$T=a_0^2t\tag{4.3}$$

is introduced. The scaling in (4.3) anticipates that at least one evolution time scale for the cyclone wave leads to a balance between linear growth and finite-amplitude equilibration. The dependent variables ϕ_n and θ are then functions of both t and T. If (4.1) and (4.3) are inserted into (2.15a, b) and (2.22), then

$$\left[\frac{\partial}{\partial t}+a_0^2\frac{\partial}{\partial T}+(U_2+U_c)\frac{\partial}{\partial x}\right][\nabla^2\phi_1-F(\phi_1-\phi_2)]+\phi_{1x}FU_c+r\nabla^2\phi_1-\kappa(\phi_1-\phi_2-\theta)=-a_0J[\phi_1,\nabla^2\phi_1-F(\phi_1-\phi_2)],\tag{4.4a}$$

$$\left[\frac{\partial}{\partial t}+a_0^2\frac{\partial}{\partial T}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2-F(\phi_2-\phi_1)]-\phi_{2x}FU_c+r\nabla^2\phi_2+\kappa(\phi_1-\phi_2-\theta)=-a_0J[\phi_1,\nabla^2\phi_1-F(\phi_1-\phi_2)],\tag{4.4b}$$

$$\left[\frac{\partial}{\partial t}+a_0^2\frac{\partial}{\partial T}\right]\theta=\frac{H}{a_0}+\lambda(\phi_1-\phi_2-\theta)-\alpha_1\nabla^2\phi_2\frac{\partial\bar{\theta}}{\partial y},\tag{4.4c}$$

where the symbol

$$J(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.\tag{4.5}$$

The variables ϕ_n and θ are now expanded in a series in a_0, *viz.*

$$\left.\begin{aligned}\phi_n&=\phi_n^{(1)}+a_0\phi_n^{(2)}+a_0^2\phi_n^{(3)}+\cdots\\ \theta&=\theta_1+a_0\theta_2+a_0^2\theta_3+\cdots\end{aligned}\right\},\qquad(4.6)$$

whose substitution into (4.4) yields a sequence of linear problems. In carrying out the expansion it is assumed for the sake of economy of exposition that $\lambda=O(\alpha_1)=O(a_0^2)$, although subsequently the ratios λ/a_0^2, α_1/a_0^2 will be considered small numbers. Similarly H is written

$$H=a_0^2\lambda h(y,T),\qquad(4.7)$$

i.e., the external heating of the mixed layer is presupposed to be essentially zonal and occurring on the long time scale and is of sufficiently small amplitude, $O(\lambda)$, to lead to a change of θ_2 on the λ^{-1} time scale.

The O(1) problem which arises from the leading order terms of (4.4a, b, c) and (4.6) is

$$\left[\frac{\partial}{\partial t}+(U_c+U_2)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(1)}-F(\phi_1^{(1)}-\phi_2^{(1)})]$$

$$+FU_c\frac{\partial\phi_1^{(1)}}{\partial x}+r\nabla^2\phi_1^{(1)}-\kappa[\phi^{(1)}-\phi_2^{(1)}]=-\kappa\theta_1,\qquad(4.8\text{a})$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2^{(1)}-F(\phi_1^{(1)}-\phi_2^{(1)})]$$

$$-FU_c\frac{\partial\phi_2^{(1)}}{\partial x}+r\nabla^2\phi_2^{(1)}+\kappa[\phi_1^{(1)}-\phi_2^{(1)}]=\kappa\theta_1,\qquad(4.8\text{b})$$

$$\frac{\partial\theta_1}{\partial t}=0.\qquad(4.8\text{c})$$

In principle, the solutions of (4.8a, b, c) consist of two elements. With θ_1 independent of the "fast" time t, one solution to (4.8a, b) can be written as a *homogeneous* travelling wave

$$\begin{Bmatrix}\phi_1^{(1)}\\ \phi_2^{(1)}\end{Bmatrix}=\begin{Bmatrix}A(T)\\ \gamma A(T)\end{Bmatrix}e^{ik(x-ct)}\sin m\pi y+{}^*,\qquad(4.9)$$

where

$$\left.\begin{aligned}c=\frac{U_1+U_2}{2};\quad \frac{U_c^2}{4}=\frac{ra^2+2\kappa r}{k^2(2F-a^2)};\quad a^2=k^2+m^R\pi^2\\ \gamma=\frac{(\alpha^2-F)U_c-2i(ra^2+\kappa)/k}{FU_2-2i\kappa/k}\end{aligned}\right\},\qquad(4.10)$$

and where $A(T)$ is the yet to be determined evolving amplitude of the wave.

To this solution should be added a solution forced by θ_1. This solution can at most be a function of x, y and T alone. Subsequent analysis shows that θ_1 must decay on the λ^{-1} time scale and it is *a priori* useful, consistent and convenient to simply let $\theta_1=0$. Then (4.9) alone is the O(1) solution. It is important to note that the cyclone wave field here is represented by a *single* wave. In the real atmosphere, of course, several waves of differing wavelengths coexist simultaneously. In this model, the cooperative action of the total wave field is represented by only one of its members. The underlying assumption is that the gross effect of the wave field on the mean state is similar for the total wave field and each of its cyclone scale members.

The $O(a_0)$ problem is

$$\left[\frac{\partial}{\partial t}+(U_c+U_2)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(2)}-F(\phi_1^{(2)}-\phi_2^{(2)})]$$

$$+FU_c\frac{\partial\phi_1^{(2)}}{\partial x}+r\nabla^2\phi_1^{(2)}-\kappa[\phi_1^{(2)}-\phi_2^{(2)}]$$

$$=-\kappa\theta_2-J[\phi_1^{(1)},\nabla^2\phi_1^{(1)}-F(\phi_1^{(1)}-\phi_2^{(1)})]$$

$$=-\kappa\theta_2-ikm\pi(\gamma^*-\gamma)F|A|^2\sin 2m\pi y,\qquad(4.11\text{a})$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2^{(2)}-F(\phi_1^{(2)}-\phi_1^{(2)})]$$

$$-FU_c\frac{\partial\phi_2^{(2)}}{\partial x}+r\nabla^2\phi_2^{(2)}+\kappa[\phi_1^{(2)}-\phi_2^{(2)}]$$

$$=\kappa\theta_2-J[\phi_2^{(1)},\nabla^2\phi_2^{(1)}-F(\phi_2^{(1)}-\phi_1^{(2)})]$$

$$=\kappa\theta_2+ikm\pi(\gamma^*-\gamma)F|A|^2\sin 2m\pi y,\qquad(4.11\text{b})$$

$$\frac{\partial\theta_2}{\partial t}=0,\qquad(4.11\text{c})$$

where the O(1) solutions have been used to explicitly evaluate the Jacobian nonlinear forcing terms on the right-hand sides of (4.11a, b). The $\phi_n^{(2)}$ solutions forced by the $|A|^2$ terms in (4.11a, b) are functions of y and T alone. They represent the alteration of the mean field due to the $O(a_0^2)$ receified northward transport of potential vorticity by the $O(a_0)$ wave field. A second solution is forced by θ_2 which is, by (4.11c), independent of t. It will be seen below that it is consistent to take θ_2 to be a function of y and T alone, in which case (4.11a,

b) reduce to

$$r\frac{d^2\Phi_1}{dy^2}-\kappa(\Phi_1-\Phi_2)=-\kappa\theta_2(y,T)-2km\pi\gamma_iF|A|^2\sin 2m\pi y, \quad (4.12a)$$

$$r\frac{d^2\Phi_2}{dy^2}-\kappa(\Phi_2-\Phi_1)=\kappa\theta_2(y,T)+2m\pi\gamma_iF|A|^2\sin 2m\pi y, \quad (4.12b)$$

where $\Phi_n\equiv\phi_n^{(2)}$. The function $\theta_2(y,T)$ which represents the large-scale sea-surface temperature drifting on the "long" time T is as yet indetermined. To find equations for $A(T)$ and $\theta_2(y,T)$ it is necessary to consider the $O(a_0^2)$ problem which can be written as

$$\left[\frac{\partial}{\partial t}+(U_c+U_2)\frac{\partial}{\partial x}\right][\nabla^2\phi_1^{(3)}-F(\phi_1^{(3)}-\phi_2^{(3)})]+\frac{\partial\phi_1^{(3)}}{\partial x}FU_c+r\nabla^2\phi_1^{(3)}-\kappa[\phi_1^{(3)}-\phi_2^{(3)}]$$
$$=-\kappa\theta_3-J[\phi_1^{(1)},\nabla^2\phi_1^{(2)}-F(\phi_1^{(2)}-\phi_2^{(2)})]-J[\phi_1^{(2)},\nabla^2\phi_1^{(1)}-F(\phi_1^{(1)}-\phi_2^{(1)})], \quad (4.13a)$$

$$\left[\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x}\right][\nabla^2\phi_2^{(3)}-F(\phi_2^{(3)}-\phi_1^{(3)})]-\frac{\partial\phi_2^{(3)}}{\partial x}FU_c+r\nabla^2\phi_2^{(3)}+\kappa[\phi_1^{(3)}-\phi_2^{(3)}]$$
$$=\kappa\theta_3-J[\phi_2^{(1)},\nabla^2\phi_2^{(2)}-F(\phi_2^{(2)}-\phi_1^{(2)})]-J[\phi_2^{(2)},\nabla^2\phi_2^{(1)}-F(\phi_2^{(1)}-\phi_1^{(1)})] \quad (4.13b)$$

$$\frac{\partial\theta_3}{\partial t}=Ae^{ik(x-ct)}\left[\frac{\lambda}{a_0^2}(1-\gamma)-\frac{\alpha_1}{a_0^2}a^2\gamma U_c\right]\times\sin m\pi y+^*, \quad (4.13c)$$

whose θ_3 is the sea-surface temperature fluctuation induced by the passage of the cyclone wave due both to the heat transfer to and from the sea in the passing temperature field of the wave and due to the wavelike disturbance of the geostrophic advection of sea-surface temperature. It is a relatively small temperature fluctuation $[O(\lambda)]$ because the forcing by the passing cyclone wave has such a high frequency $[k(U_1+U_2)]$ compared to the natural inverse time (λ) of thermal adjustment of the sea-surface layer.

The right-hand side of (4.13a, b) contains terms which are proportional to, or have projections on, the $O(1)$ free solutions (4.9). Such forcing terms (for example, $\kappa\theta_3$) would give rise to a secular growth of $\phi_n^{(3)}$ in t unless a particular combination of such forcing terms is chosen to vanish. This procedure of eliminating secular terms is similar to the process described in Pedlosky (1970). The result is the amplitude evolution equation, which may be written in the form

$$\frac{i}{k}\frac{dA}{dT}\left[\gamma F-(a^2+F)-\gamma\left(\frac{FU_c-2i\kappa/k}{FU_c+2i\kappa/k}\right)[F-\gamma(a^2+F)]\right]$$
$$-\frac{\kappa}{k^2c}A\left[\frac{\lambda}{a_0^2}(1-\gamma)-\frac{\alpha_1}{a_0^2}a^2\gamma U_c\right]\left[1+\gamma\left(\frac{FU_c-2i\kappa/k}{FU_c+2i\kappa/k}\right)\right]$$
$$+2m\pi A\int_0^1 dy\sin 2m\pi y\left\{\left[\frac{d^2\Phi_1}{dy^2}-F(\Phi_1-\Phi_2)\right]-\Phi_1[\gamma F-(a^2+F)]-\gamma\left(\frac{FU_c-2i\kappa/k}{FU_c+2i\kappa/k}\right)\right.$$
$$\left.\times\left[\gamma\left(\frac{d^2\Phi_2}{dy^2}-F(\Phi_2-\Phi_1)\right)-\Phi_2[F-\gamma(a^2+F)]\right]\right\}=0. \quad (4.14)$$

To close the system an equation for the evolution of $\theta_2(y,T)$ must be obtained. This is accomplished by considering the $O(a_0^3)$ sea-surface temperature equation, i.e.,

$$\frac{\partial\theta_4}{\partial t}=-\frac{\partial\theta_2}{\partial T}+\frac{\lambda}{a_0^2}h(y,T)+\frac{\lambda}{a_0^2}(\Phi_1-\Phi_2-\theta_2)+\frac{\alpha_1}{a_0^2}\frac{d^2\Phi_2}{dy^2}U_c. \quad (4.15)$$

The right-hand side of (4.15) is independent of t and hence to avoid a secular growth of θ_4, the right-hand side of (4.15) must vanish yielding the desired equation governing the long-time behavior of θ_2,

$$\frac{\partial\theta_0}{\partial T}=\frac{\lambda}{a_0^2}h(y,T)+\frac{\lambda}{a_0^2}(\Phi_1-\Phi_2-\theta_2)+\frac{\alpha_1}{a_0^2}\frac{d^2\Phi_2}{dy^2}U_c. \quad (4.16)$$

An examination of (4.12a, b) reveals after adding the equations that

$$r\frac{d^2}{dy^2}(\Phi_1+\Phi_2)=0, \quad (4.17)$$

or

$$\Phi_1+\Phi_2=c_1(T)+c_2(T)y. \quad (4.18)$$

Since

$$\frac{\partial^2\Phi_n}{\partial y\partial T}=0 \quad \text{on} \quad y=0,1, \quad (4.19)$$

it follows that aside from a *uniform* barotropic flow it is possible, without loss of generality, to write

$$\Phi_1=-\Phi_2=\Phi(y,T). \quad (4.20)$$

In which case our system reduces after much algebra to the three coupled equations,

$$\frac{dA}{dT}-n_1A+A\int_0^1\left(M\frac{d^2\Phi}{dy^2}-N\Phi\right)\sin 2m\pi y dy=0, \tag{4.21a}$$

$$r\frac{d^2\Phi}{dy^2}-2\kappa\Phi=-\kappa\theta_2-2km\pi\gamma_i F|A|^2\sin 2m\pi y, \tag{4.21b}$$

$$\frac{\partial\theta_2}{\partial T}=\frac{\lambda}{a_0^2}h(y,T)+\frac{\lambda}{a_0^2}(2\Phi-\theta_2)-\frac{\alpha_1}{a_0^2}U_c\frac{d^2\Phi}{dy^2}, \tag{4.21c}$$

where

$$M=-\frac{2m\pi k}{F}\frac{\left\{\gamma_r+\gamma_i\frac{\kappa}{rF}\left[\frac{2-a^2/F}{(a^2/F)+(2\kappa/rF)}\right]^{\frac{1}{2}}\right\}}{\left\{\gamma_i(1+a^2/F)+\frac{\kappa}{rF}[1-\gamma_r(1+a^2/F)]\left[\frac{2-a^2/F}{(a^2/F)+(2\kappa/rF)}\right]^{\frac{1}{2}}\right\}}, \tag{4.22a}$$

$$N=2m\pi k\frac{\left\{\gamma_r\left(\frac{a^2}{F}-1\right)-1+\gamma_i\frac{\kappa}{rF}\left(\frac{a^2}{F}-1\right)\left[\frac{2-a^2/F}{(a^2/F)+(2\kappa/rF)}\right]^{\frac{1}{2}}\right\}}{\left\{\gamma_i(1+a^2/F)+\frac{\kappa}{rF}[1-\gamma_r(1+a^2/F)]\left[\frac{2-a^2/F}{(a^2/F)+(2\kappa/rF)}\right]^{\frac{1}{2}}\right\}}, \tag{4.22b}$$

$$\gamma_r=\frac{\gamma+\gamma_*}{2}=\frac{\frac{a^2}{F}\left(\frac{a^2}{F}-1\right)+\frac{\kappa}{rF}\left[2\left(\frac{a^2}{F}-1\right)+\frac{a^2}{F}(2-a^2/F)\right]+\frac{\kappa^2}{r^2F^2}(2-a^2/F)}{\frac{a^2}{F}+\frac{2\kappa}{rF}+\frac{\kappa^2}{r^2F^2}(2-a^2/F)}, \tag{4.22c}$$

$$\gamma_i=\frac{\gamma-\gamma_*}{2i}=-\left[\left(\frac{a^2}{F}+\frac{2\kappa}{rF}\right)(2-a^2/F)\right]^{\frac{1}{2}}\frac{\left[\frac{a^2}{F}+\frac{\kappa}{rF}(2-a^2/F)\right]}{\left[\frac{a^2}{F}+\frac{2\kappa}{rF}+\frac{\kappa^2}{r^2F^2}(2-a^2/F)\right]}. \tag{4.22d}$$

$$n_1=-\frac{i\kappa\lambda}{kca_0^2}\left\{\frac{FU_c\gamma_i+2\kappa/k(\gamma_r-1)}{FU_c(a^2+F)\gamma_i+\frac{2\kappa}{k}[F-\gamma_r(a^2+F)]}\right\}+\frac{\kappa}{kca_0^2}(\alpha_1a^2U_c)\frac{[FU_c(1+\gamma)+2i\kappa/k(1-\gamma)]}{\left\{FU_c(a^2+F)\gamma_i+\frac{2\kappa}{k}[F-(a^2+F)\gamma_r]\right\}}. \tag{4.22e}$$

Note that in the unstable range ($a^2<2F$) $\gamma_i<0$ (the wave in the upper layer lags behind the wave in the lower layer).

The coupled set (4.21a, b, c) describes, within the framework of the model, the interaction between the cyclone wave of amplitude A, the variation of the large-scale, zonal, atmospheric temperature 2Φ and its associated thermal wind $-2\partial\Phi/\partial y$, and the large-scale sea-surface temperature θ_2. Associated with those elements is the wavelike sea-surface temperature perturbation

$$\theta_3=\frac{i}{kc}A\left(\frac{\lambda}{a_0^2}(1-\gamma)-\frac{\alpha_1}{a_0^2}a^2\gamma U_c\right)e^{ik(x-ct)}\sin m\pi y+{}^*, \tag{4.23}$$

which passively follows the evolution of the cyclone wave.

In the next sections, the nature of the dynamics implied by (4.21a, b, c) is examined. All variables except A are strictly real and in fact by writing

$$A=|A|\exp(n_{1i}t), \tag{4.24}$$

(4.21a) may be replaced by

$$\frac{1}{2}\frac{d|A|^2}{dT}-n_{1r}|A|^2+|A|^2\int_0^1\left(M\frac{d^2\Phi}{dy^2}-N\Phi\right)\times\sin 2m\pi y dy=0, \tag{4.25}$$

where

$$n_1=n_{1r}+in_{1i}.$$

5. The decoupled finite-amplitude problem

Before examining the full problem set by (4.21a, b, c) it is useful to note some of the features of the decoupled finite-amplitude problem for the sake of comparison. The decoupled problem corresponds formally to the parameter limit

$$\lambda \to 0, \quad \alpha_1 \to 0. \tag{5.1}$$

This is also an interesting limit in the coupled problem, since with λ/a_0^2 and α_1/a_0^2 both small, this limit yields the short-time response on the time scale $T=O(1)$.

With $\lambda=\alpha_1=0$, it follows that

$$\frac{\partial\theta_2}{\partial T}=0, \quad \text{or} \quad \theta_2=\theta_2(y), \tag{5.2}$$

and hence in (4.21b) θ_2 represents a fixed, steady forcing of the zonal flow. Through the use of a Green's function representation, the solutions of (4.21b) corresponding to general θ_2 can be given. It is sufficient for the argument here to simply consider the case

$$\theta_2 = -\Theta(y-\tfrac{1}{2}), \tag{5.3}$$

i.e., a small enhancement of the basic north-south temperature gradient.

Since θ_2 is independent of T, a solution of (4.21b) which satisfies (4.19) is

$$\Phi=\frac{2km\pi\gamma_i|A(T)|^2F}{r(4m^2\pi^2+2\kappa/r)}\left[\sin 2m\pi y - \frac{2m\pi}{(2\kappa/r)^{\frac{1}{2}}}\frac{\sinh(2\kappa/r)^{\frac{1}{2}}(y-\frac{1}{2})}{\cosh\frac{1}{2}(2\kappa/r)^{\frac{1}{2}}}\right]-\tfrac{1}{2}\Theta(y-\tfrac{1}{2}). \tag{5.4}$$

When (5.4) is substituted into (4.21a), we obtain (note that $n_1=0$ when $\lambda=\alpha_1=0$)

$$\frac{d}{dT}|A|^2-\frac{N\Theta}{2m\pi}|A|^2+\Gamma|A|^4=0, \tag{5.5}$$

where

$$\Gamma=\frac{4km\pi(-\gamma_i)F}{r(4m^2\pi^2+2\kappa/r)}\left\{\frac{M4m^2\pi^2+N}{2}+\frac{[N-(2\kappa/r)M]4m^2\pi^2}{(4m^2\pi^2+2\kappa/r)}\frac{\tanh\frac{1}{2}(2\kappa/r)^{\frac{1}{2}}}{\frac{1}{2}(2\kappa/r)^{\frac{1}{2}}}\right\}, \tag{5.6}$$

whose solution is

$$|A(T)|^2=|A(0)|^2 \times\left\{\frac{\exp(N\Theta T/2m\pi)}{1+\dfrac{\Gamma|A(0)|^2}{N\Theta/2m\pi}[\exp(N\Theta T/2m\pi)-1]}\right\}. \tag{5.7}$$

For large T, $|A|^2$ approaches the limiting amplitude

$$|A(\infty)|^2=\frac{N\Theta}{2m\pi\Gamma}. \tag{5.8}$$

If $N\Theta>0$ (linear instability) and $\Gamma>0$ (nonlinear stability), the wave amplitude grows monotonically to the final steady state (5.8). In the two interesting limits of small and large κ/r it follows that, in both limits, $N>0$ and $\Gamma>0$. Thus the *decoupled* problem involves a growth to finite amplitude of A (for $\Theta>0$) along with an adjustment of the zonal flow Φ in a time T, which is O(1). It is important to note that the equilibrated amplitude given by (5.8) is realized because the adjustment of Φ due to the presence of the rectified transport of potential vorticity [the $|A|^2$ term in (5.4)] *reduces* the vertical shear of the mean flow as the wave grows. That is, in the decoupled problem, the mean flow is driven toward a stable state by the finite-amplitude baroclinic wave. If Θ is <0, i.e. if the alteration of the thermal field is in the direction of stability and reduces the vertical shear below its critical value, the wave amplitude exponentially decays to zero.

6. The coupled finite-amplitude problem—steady state

If λ and α_1 are not zero, the sea-surface temperature θ_2 will evolve in response to changes in the zonal flow Φ, which in turn is produced, in part, by the nonlinear effects of the cyclone wave. Before confronting the full time-dependent coupled problem, it is useful to examine the nature of the asymptotic steady state (assuming one exists).

In the steady state, (4.21b, c) and (4.25) may be written

$$r\frac{d^2\Phi}{dy^2}-2\kappa\Phi=-\kappa\theta_2-2km\pi\gamma_iF|A|^2\sin 2m\pi y, \tag{6.1a}$$

$$0=\frac{\lambda}{a_0^2}h(y)+\frac{\lambda}{a_0^2}(2\Phi-\theta_2)-\frac{\alpha_1U_c}{a_0^2}\frac{d^2\Phi}{dy^2}, \tag{6.1b}$$

$$\int_0^1\left(M\frac{d^2\Phi}{dy^2}-N\Phi\right)\sin 2m\pi y\,dy=n_{1r}. \tag{6.1c}$$

Eliminating θ_2 between (6.1a, b) yields

$$\frac{d^2\Phi}{dy^2}\left(1-\frac{\alpha_1U_c}{\lambda}\frac{\kappa}{r}\right)=-\frac{\kappa}{r}h-\frac{2km\pi\gamma_iF|A|^2}{r}\sin 2m\pi y. \tag{6.2}$$

It follows from (6.2) and (4.19) that for a steady state to be possible

$$\int_0^1 h(y)dy=0,$$

otherwise the heating leads to a secular growth of θ_2.

The solution of (6.2) subject to (4.19) is

$$\Phi=\frac{2km\pi\gamma_i F|A|^2}{r\left(1-\dfrac{\alpha_1 U_c}{\lambda}\dfrac{\kappa}{r}\right)}\left[\sin 2m\pi y-2m\pi(y-\tfrac{1}{2})\right]$$
$$-\frac{\kappa}{r}\left[\int_0^y (y-y')h(y')dy'\right]\left(1-\frac{\alpha_1 U_c}{\lambda}\frac{\kappa}{r}\right)^{-1}$$
$$-U_0(y-\tfrac{1}{2}).\quad (6.3)$$

In (6.3)

$$\frac{\partial\Phi}{\partial y}=U_0\quad\text{on}\quad y=0,\,1,\quad (6.4)$$

so that U_0 can be interpreted as a small *initial* increase of the shear above critical.

It is important to observe that the portion of Φ which depends on the rectified wave flux of potential vorticity [the $O(|A|^2)$ term in (6.3)] has a sign which depends on the sign of

$$1-\frac{\alpha_1 U_c}{\lambda}\frac{\kappa}{r}.$$

If this factor is *positive*, an increase of $|A|^2$ leads to a *decrease* of the large-scale, zonal thermal wind shear and then describes a nonlinear stabilizing effect. On the other hand, if $(\alpha_1 U_c/\lambda)(\kappa/r)>1$, the factor is *negative* and the growth of the wave *enhances* the thermal wind shear and is a nonlinear *destabilizing* mechanism. If (6.3) is substituted into (6.1c) we obtain an equation for $|A|^2$ in the purported steady state, namely,

$$|A|^2=\frac{\dfrac{U_0 N}{2m\pi}\left(1-\dfrac{\alpha_1 U_c}{\lambda}\dfrac{\kappa}{r}\right)}{(2m^2\pi^2 M+\frac{3}{2}N)}+\frac{\kappa}{r}\int_0^1 dy\,\sin 2m\pi y$$
$$\times\left[Mh(y)-N\int_0^y (y-y')h(y')dy'\right]+n_{1r}.\quad (6.5)$$

It is interesting to consider the case where $h(y)$ is zero, i.e., where the sea-surface temperature changes only in response to interactions with the atmosphere. In this case

$$\theta_2=-2U_0(y-\tfrac{1}{2})+\frac{2k\gamma_i m\pi F|A|^2}{r\left(1-\dfrac{\alpha_1 U_c\kappa}{\lambda r}\right)}\left(2+\frac{\alpha_1 U_c}{\lambda}\frac{\kappa}{r}-4m^2\pi^2\right)$$
$$\times\sin 2m\pi y+4m\pi(y-\tfrac{1}{2})\quad (6.6)$$

and therefore the U_0 term in (6.5) and (6.6) corresponds to an added temperature gradient (above critical) that exists in the absence of cyclone wave activity. Since n_{1r} is $O(\alpha_1 U_c/a_0^2)$, and hence negligible, it follows that if U_0 is greater than zero, which corresponds to a linearly unstable zonal state, a finite-amplitude *equilibrium state is achieved only if*

$$1-\frac{\alpha_1 U_c\kappa}{\lambda r}>0,\quad (6.7)$$

for otherwise $|A|^2$ by (6.5) would be negative. This implies that while an equilibrium super-critical steady state is likely in the absence of air-sea coupling, the addition of feedback coupling by heat exchange and wind-driven advection of surface temperature can lead to the elimination of steady states or equivalently (as shown below) to a slow, non-equilibrating growth of anomalies in the system if (6.7) is not met. In terms of the original physical parameters

$$\frac{\alpha_1 U_c\kappa}{\lambda r}=\frac{c_D U_{c*}}{\beta_* L^2}\frac{c_{pw}}{c_p}\frac{h}{h_T}\frac{U f_0 L^2\theta_s}{g(\partial\theta_s/\partial z_*)D^2\delta}.\quad (6.8)$$

It is difficult to estimate the size of several of the parameters in (6.8) but taking

$$\left.\begin{array}{ll} c_D=10^{-3}, & c_{pw}/c_p=4\\ U=U_{c*}=30\text{ m s}^{-1}, & h/h_T=10^{-1}\\ L=10^3\text{ km}, & D=10\text{ km}\\ & \delta=1\text{ km}\end{array}\right\}$$

I obtain a value of $(\alpha_1 U_c\kappa/\lambda r)$ of 2.7. Obviously, the numerical value is clearly disputable. The general tendency, however, is that increased heat coupling of air to sea (large κ) and large back coupling by the wind-driven advection (large $\alpha_1 U_c$) favors the loss of equilibrium, while frictional dissipation in the atmosphere (large r) and decreased heat storage capacity of the oceanic surface (large λ) tends to restore equilibrium.

The mechanism leading to a loss of equilibrium can be deduced from (6.1a, b, c). For small viscous dissipation (6.1a) reveals that the rectified southward (northward) advection of potential vorticity (down-gradient) in the upper (lower) layer by the cyclone wave must be balanced by vortex tube stretching accomplished by heat flux from the sea, i.e., $-(2\Phi-\theta_2)\sim|A|^2\sin 2m\pi y$. This requires, for a steady state in the absence of heat sources, a Sverdrup advection of surface-temperature:

$$v_s\frac{\partial\bar\theta}{\partial y}\sim(2\Phi-\theta_2)\sim-|A|^2\sin 2m\pi y.\quad (6.9)$$

Since

$$v_s\sim\frac{\partial^2\Phi_2}{\partial y^2}=-\frac{\partial^2\Phi}{\partial y^2},\quad (6.10)$$

this implies that

$$\frac{\partial^2\Phi}{\partial y^2}\sim|A|^2\sin 2m\pi y\bigg/\frac{\partial\bar\theta}{\partial y},\quad (6.11)$$

and therefore that

$$-\frac{\partial\Phi}{\partial y}\sim|A|^2\cos 2m\pi y\Big/\frac{\partial\bar{\theta}}{\partial y}. \tag{6.12}$$

Since $\partial\bar{\theta}/\partial y<0$, this implies that the vertical shear in the atmosphere $(-\partial\Phi/\partial y)$ is *enhanced* where the cyclone eigenfunction has its maximum amplitudes, *i.e.*, the mean field is then altered in a direction opposite to stabilization.

Naturally, such a loss of equilibrium which is parameter-dependent (6.7) may not always be realized but the advantage of a criterion such as (6.7) is that it sketches out circumstances favorable to such a change of behavior of the system.

7. The coupled finite-amplitude problem: Time dependent

It remains now to show that the inference drawn in Section 6, i.e., that the loss of the equilibrium state when (6.7) is not satisfied does indeed imply a growing disturbance anomaly.

Consider, therefore, the following initial value problem. At $T=0$ a small "seed" anomaly in the surface-temperature

$$\theta_2=-\Theta(y-\tfrac{1}{2}) \tag{7.1}$$

is placed on the system. For λ/a_0^2 and α_1/a_0^2 small, the initial evolution of the system is described for $T=O(1)$ by the decoupled problem described in Section 5, leading to the quasi-equilibrium state (5.8) for A. On this time scale θ_2 remains as given by (7.1). The subsequent evolution of the fields occurs on the λ time scale; therefore, it is natural to introduce the new "very long" time variable

$$\tau=\lambda T/a_0^2, \tag{7.2}$$

and consider all variables to become, after the quasi-equilibrated period leading to (5.8), functions of τ alone. This implies that (4.21b, c) and (4.25) can be rewritten

$$r\frac{d^2\Phi}{dy^2}-2\kappa\Phi=-\kappa\theta_2-2km\pi\gamma_iF|A|^2\sin 2m\pi y, \tag{7.3a}$$

$$\frac{\partial\theta_2}{\partial\tau}=(2\Phi-\theta_2)-\frac{\alpha_1U_c}{\lambda}\frac{d^2\Phi}{dy^2}, \tag{7.3b}$$

$$\frac{1}{2}\frac{\lambda}{a_0^2}\frac{d|A|^2}{d\tau}-n_{1r}|A|^2+|A|^2\int_0^1\left(M\frac{\partial^2\Phi}{\partial y^2}-N\Phi\right)\times\sin 2m\pi y dy. \tag{7.3c}$$

The external heating function $h(y,T)$ has been deleted from the system in order to study the evolution of the "free" coupled system. Since $\lambda/a_0^2\ll 1$, the first term in (7.3c) can be neglected. This is a singular perturbation appropriate for the period of time subsequent to achieving the quasi-steady state (5.8). The importance of noting this fact is that, for $|A|^2\neq 0$, it then follows that (7.3c) may be rewritten as

$$\int_0^1\left(M\frac{\partial^2\Phi}{\partial y^2}-N\Phi\right)\sin 2m\pi y dy=n_{1r}, \tag{7.4}$$

which is a statement that the cyclone wave field remains in a quasi-equilibrium state during the long-time evolution of the thermal field. Furthermore, (7.4) is *linear*. This allows the effective use of the Laplace transform to study the transient, long-term behavior of (7.3a, b) and (7.4). If the Laplace transform of any variable, say, $\Phi(y,\tau)$ is denoted by an overbar, *viz*:

$$\bar{\Phi}(y,s)=\int_0^\infty d\tau e^{-s\tau}\Phi(y,\tau), \tag{7.5}$$

it follows that

$$\bar{\theta}_2(s+1)=-\Theta(y-\tfrac{1}{2})+2\bar{\Phi}-\frac{\alpha_1U_c}{\lambda}\frac{\partial^2\bar{\Phi}}{\partial y^2}, \tag{7.6}$$

$$r\frac{\partial^2\bar{\Phi}}{\partial y^2}-2\kappa\bar{\Phi}=-\kappa\bar{\theta}_2-2km\pi\gamma_iF\overline{|A|^2}\sin 2m\pi y, \tag{7.7}$$

$$\int_0^1\left(M\frac{\partial^2\bar{\Phi}}{\partial y^2}-N\bar{\Phi}\right)dy\sin 2m\pi y=n_{1r}/s, \tag{7.8}$$

subject to

$$\frac{\partial\bar{\Theta}}{\partial y}=-\frac{\Theta}{2s}\quad\text{on}\quad y=0,1. \tag{7.9}$$

Elimination of $\bar{\theta}_2$ leads to an equation relating $\bar{\Phi}$, $|A|^2$ and Θ:

$$\frac{d^2\bar{\Phi}}{dy^2}\left[r(s+\tfrac{1}{2})-\frac{\alpha_1\kappa U_c}{\lambda}\right]-2\kappa s\bar{\Phi}=+\kappa\Theta(y-\tfrac{1}{2})-2km\pi\gamma_iF|A|^2(s+1)\sin 2m\pi y, \tag{7.10}$$

the solution of which is

$$\bar{\Phi}=\frac{2km\pi\gamma_iF|A|^2(s+1)}{\{2\kappa s+4m^2\pi^2[(s+1)r-(\alpha_1\kappa U_c/\lambda)]\}}\left[\sin 2m\pi y-\frac{2m\pi}{\eta}\frac{\sinh\eta(y-\frac{1}{2})}{\cosh\eta/2}\right]-\frac{\Theta}{s}(y-\tfrac{1}{2}), \tag{7.11}$$

where

$$\eta=\left[\frac{2\kappa s}{(s+1)r-(\alpha_1\kappa U_c/\lambda)}\right]^{\frac{1}{2}}. \tag{7.12}$$

Insertion of (7.11) into (7.8) then yields

$$(-2km\pi\gamma_i F)\overline{|A|^2}=\frac{N\Theta(\eta^2+4m^2\pi^2)\{2\kappa s+4m^2\pi^2[(s+1)r-(\alpha_1\kappa U_c/\lambda)]\}}{2m\pi\{[(M4m^2\pi^2+N)/2](\eta^2+4m^2\pi^2)-(8m^2\pi^2/\eta)\tanh\eta/2(M\eta^2-N)\}(s+1)s},$$
$$=I(s), \tag{7.13}$$

where the n_{1r} term has been ignored as small $[O(\alpha_1 U_c/a_0^2)]$.

The behavior of $|A(\tau)|^2$ can be inferred by examining the singularities of (7.13). A time-independent contribution from the pole at $s=0$ corresponds to the steady-state solution given by (6.5) with $u_0=+\Theta/2$. This contribution will be a proper steady state solution (i.e., will have $|A|^2>0$) only if (6.7) is satisfied. Additional poles occur at the points

$$\eta=i2\mu,$$

where μ satisfies

$$2(M4m^2\pi^2+N)(m^2\pi^2-\mu^2)=-4m^2\pi^2\frac{\tan\mu}{\mu}(M\mu^2+N). \tag{7.14}$$

There are an infinite number of such solutions for μ which for large μ asymptote to $(2k+1/2)\pi$. In terms of these μ poles, the corresponding *exponential growth rates* of these contributions is

$$S\mu=-\mu^2\frac{(1-\alpha_1\kappa U_c/\lambda r}{(\mu^2+2\kappa/r)}. \tag{7.15}$$

The pole of $I(s)$ corresponding to each μ is

$$\lim_{s\to s\mu}(s-s\mu)I(s)$$
$$=+\frac{4N\Theta}{4\mu^2(s\mu+1)}\frac{(m^2\pi^2-\mu^2)^2(\kappa/m\pi)e^{s\mu\tau}s\mu}{[(4\mu^2/\mu)+(r\mu^4/8\mu\kappa)]}$$
$$\times\left[\frac{M4m^2\pi^2+N}{2}+\frac{4m^2\pi^2}{\cos^2\mu}\left(M+\frac{N}{\mu^2}\right)+\frac{4m^2\pi^2\ \mathrm{tau}\mu}{\mu}(M-N/\mu^2)^2\right]^{-1}, \tag{7.16}$$

and hence, decreases with increasing μ.

Thus if (6.7) is satisfied, the steady solution (6.5) is finally achieved on the long (λ) time scale and the contributions of (7.16) to the rectified wave flux of heat and potential vorticity will slowly decay. On the other hand, if (6.7) is not satisfied, $S\mu>0$ by (7.15) which implies a slow *exponential growth* of the anomalous temperature field and accompanying cyclone wave field on the advective time[1] scale $(\alpha_1 U_c)^{-1}$. In this case the poles given by (7.16) are eventually dominating (and they are positive) in the long-term representation of the anomaly fields.

[1] In dimensional units the dimensional time scale can be reconstructed, with the aid of the definition of α_1, to be $t_*=L/U$ $(\alpha_1 U_c)^{-1}=\beta_* h_T L/\mathrm{curl}\ \tau_*=L/v_{s*}$, where v_{s*} is the *dimensional* Sverdrup velocity.

8. Conclusions

In order to consider a tractable analytical model, vast simplifications in the representation of the dynamics have been required. For example, the rectified effects of transient cyclone waves have been represented by the effect of a single finite-amplitude wave in an atmospheric flow which is marginally stable and the advective transport of temperature in the ocean has been parameterized in a very crude way. In compensation for this departure from accuracy, this simple, schematic model has revealed a mechanism whereby the air-sea system, coupled both by heat and momentum exchange, can enhance initially small sea-surface temperature anomalies and accompanying changes in the thermal field of the atmosphere and its cyclone wave activity.

The sea-surface temperature anomalies that are produced have a two-fold structure. The most significant thermal field is the x independent field θ_2, whose evolution contributes to altering the mean field in which the cyclone wave is embedded and from which it draws energy. At the same time an atmospherically impressed thermal anomaly θ_3 exists which though smaller than θ_2 is especially significant because it is *wave-like* and passively follows the long-period evolution of the atmospheric wave. Both θ_2 and θ_3 grow along with A.

In this model the large-scale temperature fields in the ocean and atmosphere are independent of longitude and grow only temporally. In fact, a natural though technically complex extension of the present theory would allow θ_2 and Φ to be functions of longitude on length scales which are long compared to the cyclone scale. In such a case the basic instability mechanisms described above would still operate but there would probably be at least two additional features of interest. It is natural to suppose that the temporal growth would be at least partially altered to a west-to-east spatial growth as the θ_2 anomaly, introduced on an oceanic western boundary, slowly advects eastward in mid-latitude gyres. This would lead to an eastward bias in the anomaly strength, a fact that has been noted observationally. This would also lead to the possibility of longitudinal modulation of the anomalies or the production of more realistic temperature "pools" in the sea-surface temperature.

The second feature is that since the growth rate in the case of anomaly enhancement is on the advective time scale, the instability process described above is not catastrophic. For within an *e*-folding time scale the anomaly will have moved across the ocean and leave the region where the instability process acts either by interaction with the eastern boundary or meridional advection, out of the region of cyclone activity. These are clearly speculative statements which require further analysis for their confirmation, but I think they are clearly suggested by the results already derived.

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