On Chahine’s Relaxation Method for the Radiative Transfer Equation

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ABSTRACT

The iteration scheme proposed by Chahine for the solution of the radiative transfer equation is discussed in the context of the inverse problem for the thermal structure of the atmosphere. Sufficient conditions which insure the convergence of the iteration are given.

1. Introduction

The present paper is devoted to an investigation of the method proposed by Chahine (1968, 1970, 1972, 1974) for the solution of various inverse problems associated with atmospheric sounding by means of satellites. The method has already been used for the determination of the temperature profile, the extent and height of cloud layers, and the concentration of absorbing gases in the lower part of the atmosphere. We shall present our analysis within the framework of the inverse problem for the thermal sounding both because of its relative simplicity and because of its intrinsic importance.

To that effect, a brief review of the problem is required. The basic equation of radiative transfer provides an expression for the outgoing radiance in terms of the temperature profile, namely

\[ I(\nu) = B[\nu, T_s] \tau(\nu, \rho_s) + \int_{\rho_s}^{\rho} B[\nu, T(\rho)] d\rho. \]  (1)

In the above, \( I(\nu) \) is the radiance at frequency \( \nu \), \( T \) is the temperature, and \( \rho \) the pressure. A subscript \( s \) and an overbar indicate respectively that a quantity is evaluated at the ground surface or at the altitude of the satellite. Finally,

\[ B[\nu, T] = \frac{a \nu^3}{\exp(b \nu/T) - 1} \]  (2)

is the classical Planck function and \( \tau(\nu, \rho) \) the transmisson function between levels \( \rho \) and \( \rho' \). \( \tau(\nu, \rho) \) can be thought of as the probability that a photon with energy \( h\nu \) emitted at level \( \rho \) will reach level \( \rho' \) without being absorbed.

The above version of the radiative transfer equation contains several assumptions and oversimplifications. In particular, it assumes that (i) the atmosphere is in thermodynamic equilibrium (i.e., that the emission is purely thermal), (ii) there is no scattering (i.e., extinction is solely due to absorption), and (iii) the transmission function is independent of temperature. We refer the interested reader to the excellent article by Wark and Fleming (1966) for a justification of these assumptions as well as for a more discursive introduction to the basic problem.

Because CO\(_2\) is uniformly mixed in the lower part of the atmosphere, its absorption bands at 4.3 and 15 \( \mu \)m are usually chosen as the frequency bands in which to measure the radiance.

A fairly typical formulation of the inverse problem would then be: given a series of measured radiances \( \{I(\nu_j)\}_N \) at frequencies \( \nu_1, \nu_2, \ldots, \nu_N \), what can be inferred about the temperature profile \( T(\rho) \)? There are several questions which arise in connection with the formulation of the problem. For example, how should the frequencies \( \nu_1, \nu_2, \ldots, \nu_N \) be chosen in order to minimize the effects of noise? Or, to what extent can \( T(\rho) \) be determined from a finite number of measurements? Without trying to minimize the importance of such questions or to give the mistaken impression that they have been answered, I would like to bypass them and to consider the iteration scheme proposed by Chahine to solve (1).

It is both helpful and customary to use the logarithm of the pressure as a vertical coordinate, \( v \)

\[ x = - \ln \frac{\rho}{\rho_s} \]  (3)

and to denote by

\[ H = - \ln \frac{\rho}{\rho_s} \]  (4)

the “altitude” of the satellite. Then the basic equation (1) can be written

\[ I(\nu) = B_s(\nu) \tau_s(\nu) + \int_0^H B[\nu, T(x)] K(\nu,x) dx, \]  (5)
where the kernel of the integral equation is
\[ K(\nu, x) = \frac{\partial \tau(\nu, x)}{dx}. \] (6)

Graphs of \( \partial \tau/\partial x \) for various values of \( \nu \) have appeared in many papers (e.g., Wark and Fleming, 1966; Conrath, 1968). Looked upon as a function of \( x \), \( K(\nu, x) \) has a single peak and is positive for most of the frequencies which have been used. We shall make use of these features of \( K(\nu, x) \) in the sequel.

In closing this rapid introduction to the inverse problem for the thermal structure, it should be mentioned that very little is known about the mathematical properties of Eq. (5) and in particular, about the existence and uniqueness of a physically meaningful solution. In all likelihood, (5) has a positive, real solution only if the function \( I(\nu) \) belongs to a particular class of functions. Unfortunately, this class of functions is not easy to characterize. We shall encounter this same difficulty under a slightly different guise at a later stage.

2. Chahine's relaxation method

Iterative techniques have been extensively used for the solution of nonlinear problems. Therefore, it is natural to try to obtain \( T(x) \) by means of successive approximations. However, in order to implement this program, one needs a means to update the initial approximation. Since the unknown \( T(x) \) enters solely under the integral sign (i.e., the integral equation is of the first kind), there is no standard, obvious way of “improving” the first approximation. In order to use a method of successive approximations, an ad hoc relation between two successive approximations must be added to (5). The relationship introduced by Chahine is not only very simple, but has several important advantages.

In order to motivate this relationship, let us assume that we have obtained the \( n \)th approximation to the temperature profile, say \( T^{(n)}(x) \). Then, we can define
\[
I^{(n)}(\nu) = B^{(n)}(\nu)T^{(n)}(x) + \int_0^H K(\nu, x)B[\nu, T^{(n)}(x)]dx. \] (7)

If \( I^{(n)}(\nu) \) differs from the measured \( I(\nu) \), we can rewrite (7) as
\[
I(\nu) = \frac{I(\nu)}{I^{(n)}(\nu)}B^{(n)}(\nu)T^{(n)}(x) + \int_0^H \frac{I(\nu)}{I^{(n)}(\nu)}B[\nu, T^{(n)}(x)]K(\nu, x)x. \] (8)

The above expression suggests that we should equate \( B[\nu, T^{(n)}(x)]I(\nu)/I^{(n)}(\nu) \) with \( B[\nu, T^{(n+1)}(x)] \) in order to get a natural up-dating relation. The difficulty with this suggestion stems from the fact that both sides of such an equality would be functions of \( x \) and \( \nu \), and that the functional dependence might be incompatible. We can overcome this objection by making a slight modification to the original suggestion. The proposed modification stems from the following observation. For a given value of \( \nu \), most of the measured outgoing radiation originates from an atmospheric layer centered around a level \( x \) located roughly at the peak of \( K(\nu, x) \). Thus, there exists a pairing between frequency and “sampled” heights. This pairing is known in the astrophysical literature as the Barbier-Eddington approximation [see Kourganoff (1963) and Pecker (1965)]. Chahine does not give an explicit formula for this pairing, but for the sake of exposition we can assume that the relationship between \( \nu \) and \( x \) is obtained by solving the equation
\[
\frac{\partial}{\partial x}K(\nu, x) = 0 \] (9)

for either one of the variables, namely
\[
\nu = f(x), \] (10a)
\[
x = g(\nu). \] (10b)

Inserting (10) in the contemplated equality, we get the up-dating relation proposed by Chahine [Eq. (9), 1970, Eq. (6), 1972]:
\[
B[\nu, T^{(n+1)}(x)] = \frac{I(\nu)}{I^{(n)}(\nu)}B[\nu, T^{(n)}(x)]. \] (11)

Eqs. (7) and (11) constitute the basic two steps of the iterative procedure used by Chahine to solve the inverse problem.

Actually, a discrete version of (7) and (11) is used both because of computational reasons and because of our limited knowledge of \( I(\nu) \) due to measurement noise. Therefore, the following version of the iteration scheme is more representative:
\[
I^{(n)} = \sum_{j=1}^{N} K_{ij}\beta_{j}^{(n)}, \] (12)
\[
\beta_{i}^{(n+1)} = \frac{I_{i}}{I_{i}^{(n)}}\beta_{i}^{(n)}. \] (13)

1) Two remarks are now in order: Eq. (12) contains one additional approximation, namely the \( \nu \) dependence of \( B \) has been replaced by the function \( f(x) \) defined in (10a). The resulting function,
\[
\beta(x) = B[f(x), T(x)], \] (14)
appears in (12) and (13), where it is evaluated for values of \( x \) obtained from (10a). As a result
\[
\beta(x) = B[\nu, T(x)], \] (15)
Alternatively, $\nu$ could have been replaced by an average value $\bar{\nu}$. The justification for either approximation lies in the narrowness of the frequency band and in the relative weak dependence of $B(\nu, T)$ on frequency.

\[2\) The matrix $K$, whose elements are proportional to $K(\nu, x_i)$ is a square matrix. This may seem surprising in view of the fact that the number of pressure levels used in the numerical integration is usually larger than the number of sampling frequencies. Nevertheless, the squareness of $K$ is an intrinsic feature of Chahine’s method and is a direct consequence of the updating relation (13) which implies that $I$ and $\beta$ have the same number of entries. Thus, having selected the $N$ frequencies $\nu_1, \ldots, \nu_N$ optimally, Chahine’s method aims at computing the temperature solely at the heights $g(\nu_1), \ldots, g(\nu_N)$. At other intermediary levels, the temperature is obtained via interpolation.

We shall investigate the convergence of the iteration scheme as written in (12) and (13). If the iteration converges, then it is clear that

\[ I = K \beta . \tag{16} \]

A very legitimate question arises at this stage: why can’t we evaluate $\beta$ by finding the inverse of $K$? There are at least two reasons why this is not an acceptable procedure. The first has to do with the fact that the matrices $K$ of interest are ill-conditioned. This unfortunate state of affairs is a direct consequence of the fact that integral equations of the first kind are associated with ill-posed problems. Although Chahine’s method cannot cure this ill (after all the condition number is an intrinsic parameter), it avoids the computation of $K^{-1}$ which can be a very delicate operation. For the problem at hand, computing $K^{-1}$ stems from the possibility that round-off errors coupled with noise in $I$ could lead to a solution $\beta$ with negative elements, i.e., to a complex temperature! This leads us to the second advantage of Chahine’s method which is related to the positivity of $\beta$. Indeed, if we focus our attention upon the physically relevant case in which $K$ is a non-negative matrix, i.e.,

\[ K_{ij} \geq 0, \quad i, j = 1, 2, \ldots, N, \quad (17) \]

as well as upon positive “radiance” vectors $I$, i.e.,

\[ I_i > 0, \quad i = 1, 2, \ldots, N, \quad (18) \]

then, provided that

\[ I^{(0)} > 0, \quad (19) \]

all the subsequent approximations generated by (12)–(13) will be positive. Thus, with the provisos (17), (18) and (19), Chahine’s method, if it converges, will always produce a real positive temperature.

It should be noted that even if (17) and (18) are in force, there is no a priori reason to expect the solution of (16) to be positive. For this to be the case, further restrictions have to be placed upon the vector $I$. We have alluded to this problem already in the context of the existence of a solution of the integral equation (5). Once again, the class of admissible vectors $I$ is not easy to characterize. Chahine’s method avoids, so to speak, this problem altogether. Indeed, if in spite of (17) and (18) the solution of (16) (which exists provided that $\det K \neq 0$) were to be a nonpositive vector $\beta$, then in view of our previous remark the iteration scheme started off by a $\beta^{(0)}$ which satisfies (19) would simply not converge. This case could arise if the radiance measurements were very badly contaminated by noise.

3. A sufficient condition for convergence

We previously remarked that $K(\nu, x)$ is a positive kernel with a peak, and that the location of this peak is used in order to pair heights and frequencies. Translated in terms of matrices, this suggests that the class of matrices of interest is the class of positive matrices with dominant diagonals. For one such class of matrices, we can actually prove the convergence of Chahine’s scheme.

**Theorem:** Given an $N \times N$ matrix $K = (K_{ij})$ such that

\[ K_{ii} > 0, \quad i, j = 1, 2, \ldots, N, \quad (20a) \]

\[ K_{ij} > \sum_{j \neq i} K_{ij}, \quad i = 1, 2, \ldots, N, \quad (20b) \]

an $N$-column vector $I$ such that

\[ I > 0, \quad (21a) \]

\[ K^{-1} I > 0, \quad (21b) \]

and an initial approximation $\beta^{(0)}$ such that

\[ \beta^{(0)} > 0, \quad (22) \]

then the iteration scheme (12)–(13) converges to the solution of (16).

**Proof:** Note that condition (20b) implies that $K^{-1}$ exists (Taussky, 1949). Thus (21b) is meaningful as it stands. As a result, the matrix equation (16) has a solution $\beta$ and because of (21b) it is positive, viz.

\[ \beta_i > 0, \quad i = 1, 2, \ldots, N. \quad (23) \]

To establish the convergence of the iteration, we start by writing (13) in the form

\[ \beta^{(n+1)}_i - \beta_i = \beta^{(0)}_i - \beta_i - \frac{\beta^{(0)}_i}{I^{(0)}_i} [I^{(0)}_i - I_i]. \tag{24} \]

But

\[ I^{(0)}_i - I_i = \sum_{j=1}^N K_{ij} [\beta^{(0)}_j - \beta_j]. \tag{25} \]
Therefore (24) can be written as
\[ \beta^{(n+1)} - \beta = (U - Q^{(n)}) (\beta^{(n)} - \beta), \]
where \( U \) stands for the \( N \times N \) unit matrix and
\[ Q^{(n)} = \left( \frac{\beta^{(n)}}{I^{(n)}} K_{ij} \right). \]
(27)

Making use of “infinity norms” (see, e.g., Wilkinson 1965, p. 56), we deduce that
\[ \| \beta^{(n+1)} - \beta \|_\infty \leq \| U - Q^{(n)} \|_\infty \| \beta^{(n)} - \beta \|, \]
(28)
where
\[ \| U - Q^{(n)} \|_\infty = \max_i \sum_j \left| \delta_{ij} - \frac{\beta^{(n)}}{I^{(n)}} K_{ij} \right|, \]
(29)
and \( \delta_{ij} \) is the Kronecker delta. As previously mentioned, conditions (20a), (21a) and (22) imply that
\[ \begin{align*}
\beta^{(n)}_{i} &> 0 \\
I^{(n)}_{i} &> 0
\end{align*} \quad \text{for } i = 1, 2, \ldots, N \text{ and all } n \text{ 's}. \]
(30)

Consequently
\[ I^{(n)}_{i} \geq K_{ii} \beta^{(n)}_{i}. \]
(31)

Because of (30) and (31), we can rewrite (29) as
\[ \| U - Q^{(n)} \|_\infty = \max_i \left\{ 1 - \frac{\beta^{(n)}_{i}}{I^{(n)}_{i}} K_{ii} + \sum_{j \neq i} \frac{\beta^{(n)}_{j}}{I^{(n)}_{j}} K_{ij} \right\}. \]
(32)

In view of the diagonal dominance condition (20b), we conclude that
\[ \| U - Q^{(n)} \|_\infty < 1, \]
(33)
and hence
\[ \| \beta^{(n+1)} - \beta \| < \| \beta^{(n)} - \beta \|. \]
(34)

The above inequality implies the convergence of the iteration.

4. Toward necessary conditions

Conditions (20a), (20b), (21a), (21b) and (22) are sufficient to insure that the iteration (12)–(13) converges toward a positive solution. Since these are not necessary conditions, the iteration may converge even if some of these conditions were violated. It is therefore natural to inquire how these conditions should be modified in order to obtain necessary ones.

Conditions (20a) and (21a) are related to the positivity of \( K \) and \( I \). They are suggested by the physics of the problem and a relaxation of either one is not required.

Condition (22) is related to the positivity of the initial approximation. It should be viewed as a device by means of which one arrives at the solution of (16) only if the latter is positive. This is a useful constraint which should be retained.

Condition (21b), viz. \( K^{-1}I > 0 \), is essential from the physical point of view. As we previously remarked, if (22) is in force and (21b) is violated, then the iteration cannot converge. Some of the numerical experiments carried out by Chahine (1970) provide an illustration of this type of failure of the iteration scheme. I am referring to the cases in which \( I \) was respectively an exact synthetic radiance vector and a radiance vector contaminated by random errors. The iteration converged in one case but not in the other. Since the matrix \( K \) was the same in both runs, the failure of the iteration must be due to the breakdown of (21b) coupled with (22).

This brings us to the diagonal dominance condition (20b). As far as I can tell, the matrices \( K \) used by Chahine do not satisfy this condition. It should be noted that for a given kernel \( K(x,x) \), the finer the discretization the less likely it is that condition (20b) will be satisfied. Connath and Revah (1972) observed that Chahine’s method was sometimes numerically unstable. In particular, they state that this instability “disappears when the number of spectral intervals is reduced and the use of 7 intervals produces better results than 16.” Conceivably, this observation might be due to a weakening of the diagonal dominance which results from an increase in the size of the matrix. Be that as it may, condition (20b) is certainly very restrictive and could be relaxed. To that effect, it is important to note a direct consequence of this condition and of Gerschgorin’s theorem (Wilkinson, 1965, p. 71). This theorem implies that the eigenvalues \( \lambda_i(K) \) of \( K \) lie in the circles \( C_i \) defined as
\[ |\lambda - K_{ii}| = \sum_{j \neq i} K_{ij}. \]
(35)

Thus, as a result of (20b), we deduce that
\[ \text{Re} \lambda_i(K) > 0, \quad i = 1, 2, \ldots, N. \]
(36)

Of course, conditions (36) may be satisfied even if \( K \) was not diagonally dominated. They are therefore weaker conditions. I suspect that analogous conditions on the real part of the eigenvalues of \( K \) (or possibly on the eigenvalues of a matrix closely related to \( K \)) would be required to insure the convergence of the iteration scheme. This conjecture is based on the following argument. Let us replace (12)–(13) by a system of differential equations in which the continuous variable \( t \) plays the role of the discrete iteration index. The simplest such system would be
\[ \begin{cases}
J(t) = K \gamma(t) \\
\frac{d \gamma(t)}{dt} + \gamma(t) = \frac{I_t}{J(t)} \gamma(t)
\end{cases} \]
(37)

The “point” \((I, \beta)\) is the only critical point of this system. In order to investigate the nature of this limit
point, we follow the usual procedure and write

\[ \mathbf{J}(t) = \mathbf{I} + \mathbf{r}(t) \]
\[ \gamma(t) = \beta + \mathbf{c}(t) \]  \hspace{1cm} (38)

Substituting (38) in (37) and linearizing, we get

\[ \mathbf{r}(t) = \mathbf{Kc}(t), \]  \hspace{1cm} (39)
\[ \frac{d\mathbf{c}_i}{dt} = -\gamma_i \mathbf{I}_i. \]  \hspace{1cm} (40)

Since \( \mathbf{c}_i(t) \) and \( \gamma_i(t) \) are exponential functions of \( t \), we look for solutions of the form

\[ \mathbf{c}(t) = \mathbf{C}e^{-\alpha t}, \]
\[ \mathbf{r}(t) = \mathbf{R}e^{-\alpha t} \]  \hspace{1cm} (41)

Then (39) and (40) imply that

\[ \mathbf{Q} \mathbf{C} = \alpha \mathbf{C}, \]  \hspace{1cm} (42)

where \( \mathbf{Q} \) is the limit of the matrix \( \mathbf{Q}^{(n)} \) defined in (27). If all the eigenvalues of \( \mathbf{Q} \) are such that

\[ \text{Re} \alpha_i(\mathbf{Q}) > 0, \quad i = 1, 2, \ldots, N, \]  \hspace{1cm} (43)

then the limit point \( \{\mathbf{I}, \beta\} \) is locally stable. This suggests that for some class of initial conditions, \( \{\mathbf{J}(t), \gamma(t)\} \) tends to \( \{\mathbf{I}, \beta\} \) as \( t \to \infty \).

The preceding argument suggests that the convergence of the iteration is related to the eigenvalues of the matrix \( \mathbf{Q} \). In some sense, this result is pleasing since \( \mathbf{Q} \) is related to both \( \mathbf{K} \) and \( \mathbf{I} \). However, the presence of the solution \( \beta \) in the definition of \( \mathbf{Q} \) detracts greatly from the usefulness of such a criterion.

In closing, I should mention that the above discussion revolves around the difficulty in making precise the statement that the kernel \( \bar{K}(\nu,x) \), and hence \( \mathbf{K} \), is "peaked." From this point of view, it is obvious that in spite of the shortcomings of the diagonal dominance condition, such a condition provides a pictorial description of this essential feature of the kernel.

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