A Self-Contained Model for the Pressure Terms in the Turbulent Stress Equations of the Neutral Atmospheric Boundary Layer

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ABSTRACT

In this paper we develop an abbreviated model for the pressure-gradient velocity correlation terms in the equations for the Reynolds-stress components in the neutral boundary layer. The model contains three terms: a nonlinear return-to-isotropy term, a mean strain-rate term, and a mean vorticity term. There are three free constants in the model, which are determined with the aid of experimental results on the ratios between the Reynolds-stress components in the neutral surface layer. Since three independent equations are involved, the model is self-contained. Through its mean vorticity term, the model incorporates the effects of a rotating coordinate system. The application of the model to a neutral Ekman layer gives realistic results.

1. Reynolds-stress equations

The transport equation for the Reynolds-stress tensor in a neutral boundary layer is (Wyngaard *et al.*, 1974a)

$$\frac{D\overline{u_iu_k}}{Dt}=-\overline{u_iu_j}\frac{\partial U_k}{\partial x_j}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\overline{u_iu_ku_j}+\frac{2}{3}\frac{\delta_{ik}}{\rho}\overline{pu_j}\right)$$
$$-\pi_{ik}-\tfrac{2}{3}\bar{\epsilon}\delta_{ik}+2\epsilon_{ipq}\Omega_q\overline{u_ku_p}+2\epsilon_{kpq}\overline{u_iu_p}. \quad (1)$$

Here, capital letters refer to mean quantities, and lower-case letters refer to fluctuations. The viscous dissipation term is taken to be isotropic; $\bar{\epsilon}$ is the mean dissipation rate per unit mass. The angular velocity Ω_q of the frame of reference creates the last two terms in (1); the effects of these terms on the momentum fluxes in the planetary boundary layer have been analyzed by Wyngaard *et al.* (1974a).

In (1), the part of the pressure terms that contributes to the flux divergence of Reynolds stress is placed next to the other flux-divergence term. The pressure-gradient velocity correlation tensor π_{ik} is a symmetric tensor with zero trace, so that it does not contribute to the flux divergence. It is defined by

$$\pi_{ik}=-\frac{1}{\rho}\left(\overline{u_k\frac{\partial p}{\partial x_i}}+\overline{u_i\frac{\partial p}{\partial x_k}}-\tfrac{2}{3}\delta_{ik}\frac{\partial}{\partial x_j}\overline{pu_j}\right). \quad (2)$$

In the surface layer of an adiabatic planetary boundary layer the Coriolis terms of (1), i.e., those that contain Ω_p, are negligible compared to the other terms. Also, the surface layer is a constant-flux layer in first approximation, so that the components of the Reynolds-stress tensor are independent of height. Since it is difficult to conceive of non-zero flux divergences associated with uniform stresses, we neglect the flux-divergence terms of (1) in the neutral surface layer. If the coordinate system in the surface layer is aligned with the surface wind, and if conditions are taken to be stationary and horizontally homogeneous, the component equations of (1) reduce to

$$\frac{D\overline{u_1^2}}{Dt}=0=-2\overline{u_1u_3}\frac{\partial U_1}{\partial x_3}-\pi_{11}-\tfrac{2}{3}\bar{\epsilon}, \quad (3)$$

$$\frac{D\overline{u_2^2}}{Dt}=0=-\pi_{22}-\tfrac{2}{3}\bar{\epsilon}, \quad (4)$$

$$\frac{D\overline{u_3^2}}{Dt}=0=-\pi_{33}-\tfrac{2}{3}\bar{\epsilon}, \quad (5)$$

$$\frac{D\overline{u_1u_3}}{Dt}=0=-\overline{u_3^2}\frac{\partial U_1}{\partial x_3}-\pi_{31}. \quad (6)$$

This sytem is constrained by the kinetic energy budget [the sum of (3), (4) and (5)], which reads

$$-\overline{u_1u_3}\frac{\partial U_1}{\partial x_3}=\bar{\epsilon}. \quad (7)$$

Because of this constraint (which is based on $\pi_{11}+\pi_{22}+\pi_{33}=0$), the set (3)–(7) contains only three independent equations. Therefore, a self-contained model for π_{ik} can have only three undetermined constants.

We shall develop such a self-contained model below, though we realize that this approach limits us to an abridged form of the comprehensive models that are being developed by our colleagues (Lumley and Khajeh-Nouri, 1974a, b).

2. Development of the model

In the original return-to-isotropy model of Rotta (1951) the term π_{ik} is approximated by

$$-\pi_{ik}=-\frac{C_1}{\tau}\left(\overline{u_iu_k}-\frac{\delta_{ik}}{3}q^2\right)+\alpha_0q^2\left(\frac{\partial U_i}{\partial x_k}+\frac{\partial U_k}{\partial x_i}\right). \quad (8)$$

Here, $\frac{1}{2}q^2$ is the mean kinetic energy of the turbulence, and τ is a time scale for the energy-containing eddies, while C_1 and α_0 are numerical constants. The value of α_0 was computed by Lilly (1967), Crow (1968) and Rotta (1972); they give $\alpha_0=\frac{1}{5}$. Wyngaard *et al.* (1974a) used both terms in (8) for some of their calculations, but they found that the term containing α_0 caused unphysical effects. They decided to delete the α_0 term from their model; apparently, if that term is to be retained, other terms involving the mean shear have to be included, too.

If Rotta's return-to-isotropy model is applied to the stress equations for the surface layer, two shortcomings become apparent. First, the model does not allow for differences between the component energies in the vertical and lateral directions; instead, it predicts that $\overline{u_2^2}=\overline{u_3^2}$. Second, a realistic value of the shear-stress component $(\overline{-u_1u_3})$ can be obtained only if the constant C_1 is permitted to assume a different value in the equation for the shear stress.

A review of the literature on the subject shows that the interim models for the pressure term that have been proposed by various authors cannot be applied to the atmospheric surface layer without having to adjust the constant in each component equation in order to match the observed stress components (e.g., Lewellen and Teske, 1973; Mellor, 1973). This is a serious drawback, because other phenomena, such as the dispersion of pollutants from surface sources, depend strongly on the differences among the component energies. Also, except in the comprehensive model being developed by Lumley, the effects of the mean vorticity field are generally ignored, so that it is impossible to account for the effects of the Coriolis terms on the pressure fluctuations.

Our approach to the development of a self-contained model for π_{ik} is based on the following premises:

1) All terms in (3)–(7) are known from field and laboratory experiments, except for the free constants associated with the model for π_{ik}. Since the system (3)–(7) contains three independent equations, we allow three free constants. Fewer than three constants would impose unnecessary constraints on the ratios between the stress components, more than three would make the system indeterminate.

2) In addition to the classical return-to-isotropy terms given in (8), the model has to include terms representing the interaction between the turbulence and the mean shear. These terms should represent the effects of the mean strain rate separately from those of the mean vorticity, because we want to include Coriolis effects in the term representing the influence of the mean vorticity.

3) The model has to possess correct symmetry and invariance properties (Donaldson, 1971). In particular, each of the terms in the model for π_{ik} should be trace-free (because $\pi_{ii}=0$) and symmetric (because $\pi_{ik}=\pi_{ki}$).

4) The individual contributions to the model should have comparable orders of magnitude. It is not our intent to formulate a definitive, comprehensive model, but to find a sensible compromise between rigor and simplicity.

The starting point for our analysis is the Poisson equation governing pressure fluctuations in an incompressible fluid. It is a straightforward exercise to derive the contribution to the pressure fluctuations that is associated with the interaction between the turbulence and the mean flow. That contribution, represented by the symbol p^m, is given by

$$\frac{1}{\rho}p^m(\mathbf{x})=\frac{1}{2\pi}\iiint_V\left(\frac{\partial U_i(\mathbf{x}')}{\partial x_j'}-2\epsilon_{ijk}\Omega_k\right)\times\frac{\partial u_j(\mathbf{x}')}{\partial x_i'}\frac{dx_1'dx_2'dx_3}{|\mathbf{x}-\mathbf{x}'|}.$$

Here, as in (1), Ω_k is the angular velocity vector of the frame of reference. We have neglected the contribution made by the pressure fluctuations at the solid surface.

The formal expression for the part of the pressure-gradient velocity correlation that corresponds to $p^m(\mathbf{x})$ as defined above becomes, if the mean-shear field is approximately homogeneous over a distance comparable to the integral scale of the turbulence:

$$\frac{1}{\rho}\left(\overline{u_p\frac{\partial p^m}{\partial x_q}}+\overline{u_q\frac{\partial p^m}{\partial x_p}}\right)=\frac{1}{2\pi}\left(\frac{\partial U_i}{\partial x_j}-2\epsilon_{ijk}\Omega_k\right)\times\iiint_V\left(\frac{\partial^2\overline{u_j(\mathbf{x}')u_p(\mathbf{x})}}{\partial x_i'\partial x_q'}+\frac{\partial^2\overline{u_j(\mathbf{x}')u_q(\mathbf{x})}}{\partial x_i'\partial x_p'}\right)\times\frac{dx_1dx_2dx_3}{|\mathbf{x}-\mathbf{x}'|}. \quad (9)$$

Since the term representing the mean flow gradient can be taken outside the integral, the model for π_{ik}

can be constructed in terms of products of mean-flow and turbulence properties. In the surface layer of a neutral boundary layer, the requirement of approximate vertical homogeneity of the mean shear is not satisfied. Nevertheless, the structural similarity of surface-layer turbulence permits the use of (9), with the understanding that the consequences of inhomogeneity are absorbed into the undetermined numerical coefficients that occur in the modeled equations.

The tensor in front of the volume integral in (9) can be rewritten as

$$\frac{\partial U_i}{\partial x_j}-2\epsilon_{ijk}\Omega_k=S_{ij}+R_{ij}. \quad (10)$$

The mean strain rate S_{ij} (a symmetric tensor) is defined by

$$S_{ij}=\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right). \quad (11)$$

The rotation tensor R_{ij} is skew-symmetric; it includes the contribution made by the rotation of the coordinate system:

$$R_{ij}=\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}-\frac{\partial U_j}{\partial x_i}\right)-2\epsilon_{ijk}\Omega_k. \quad (12)$$

The tensor R_{ij} is related to the "absolute" vorticity of the mean-flow field, i.e., the mean vorticity observed in an inertial coordinate system. We shall develop the vorticity-related contribution to our model for π_{ik} in terms of R_{ij}; that way, the effects of the earth's rotation will be included automatically.

Eqs. (9) and (10) suggest that a model for the pressure term π_{ik} should contain products of S_{ik} and R_{ik} with the Reynolds-stress tensor, $-\overline{u_iu_k}$. In order to obtain the required tensor properties it is convenient to distinguish between the trace $q^2=\overline{u_iu_i}$ of the stress tensor and the stress-anisotropy tensor b_{ik}; the latter is defined as

$$b_{ik}=\frac{\overline{u_iu_k}}{q^2}-\frac{\delta_{ik}}{3}. \quad (13)$$

Searching for products of S_{ik} and R_{ik} with q^2 and b_{ik}, we find that the following have the required properties:

$$q^2S_{ik}, \quad (14)$$

$$q^2A^s_{ik}=2q^2(S_{ij}b_{jk}+S_{jk}b_{ij}-\tfrac{2}{3}S_{jl}b_{lj}\delta_{ik}), \quad (15)$$

$$q^2A^r_{ik}=2q^2(R_{ij}b_{jk}-R_{jk}b_{ij}). \quad (16)$$

These terms are equivalent to those used in a recent version of Lumley's "rapid distortion" model (Lumley and Khajeh-Nouri, 1974b). All three terms are of the same order as Rotta's terms because S_{ij} and R_{ij} are of order $1/\tau$, where τ is a turbulence time scale. Note that (14) corresponds to the second term of (8). Also, all three terms are symmetric and have zero trace. If we add terms based on (14)–(16) to Rotta's first term $(C_1b_{ik}q^2/\tau)$, we obtain the following model for π_{ik}:

$$-\pi_{ik}=-\frac{C_1}{\tau}q^2b_{ik}+q^2(2\alpha_0S_{ik}+\alpha_1A^s_{ik}+\gamma_1A^r_{ik}). \quad (17)$$

Here, A^s_{ik} and A^r_{ik} are defined by (15) and (16), respectively. Because the constant $\alpha_0=\frac{1}{5}$ is known from the work of Lilly, Rotta, and Crow, this model contains three free constants: C_1/τ, α_1, and γ_1. Since τ can be expressed as

$$\tau=\beta q^2/\epsilon,$$

the dimensional coefficient C_1/τ is related to the nondimensional coefficient C_1/β. Our analysis cannot determine C_1 and β individually; C_1 itself can be found if Lumley's value of β is adopted (Lumley and Khajeh-Nouri, 1974b).

Our analysis now departs from Lumley's in one significant respect. Lumley and Khajeh-Nouri perform an expansion of the integral in (9); this leads to a relation between α_1 and γ_1 and reduces the number of free constants by one. In effect, the expansion formulated by Lumley and Khajeh-Nouri specifies the nature of the cross-talk between the strain-rate terms and the vorticity terms. There is indeed some analytical evidence that the response to deformation is modified by the presence of vorticity (e.g., Pearson, 1959), but we have decided to neglect that interaction because we feel that it is more important to represent the response to deformation as an effect physically distinct from the response to rotation. This approach is dictated partly by our desire to include Coriolis effects: even if $\partial U_i/\partial x_j$ in (9) is zero, the Ω_k term would have a distinct effect on the return to isotropy.

3. Determination of the constants

If (3)–(6) are nondimensionalized with $\bar{\epsilon}=u_*^3/kz$ (as is appropriate for a neutral surface layer) and if (7) determines the logarithmic wind profile, the following set of equations results:

$$2(1-\tfrac{1}{3}\alpha_1-\gamma_1)-b_{11}C_1/\beta-\tfrac{2}{3}=0, \quad (18)$$

$$\tfrac{4}{3}\alpha_1-b_{22}C_1/\beta-\tfrac{2}{3}=0, \quad (19)$$

$$-\tfrac{2}{3}\alpha_1+2\gamma_1-b_{33}C_1/\beta-\tfrac{2}{3}=0, \quad (20)$$

$$-\frac{\overline{u_3^2}}{u_*^2}-\gamma_1(b_{11}-b_{33})\frac{q^2}{u_*^2}+\alpha_0\frac{q^2}{u_*^2}-\alpha_1b_{22}\frac{q^2}{u_*^2}+(C_1/\beta)\frac{u_*^2}{q^2}=0. \quad (21)$$

For each set of experimentally determined values of u_1^2/q^2, u_2^2/q^2, and q^2/u_*^2, a corresponding set of con-

TABLE 1. Numerical values of model parameters.

Case	$\overline{u_1^2}/q^2$	$\overline{u_2^2}/q^2$	$\overline{u_3^2}/q^2$	u_*^2/q^2	C_1/β	α_1	γ_1
1	0.480	0.280	0.240	0.170	3.2389	0.3704	0.3057
2	0.473	0.343	0.184	0.118	3.8005	0.5276	0.2235
3	0.570	0.280	0.150	0.160	1.9734	0.4211	0.2928
4	0.530	0.260	0.210	0.130	3.4662	0.3094	0.2227
5	0.530	0.270	0.200	0.160	2.6765	0.3729	0.2792
6	0.650	0.230	0.120	0.113	1.7781	0.3622	0.2644
7	0.565	0.340	0.095	0.117	1.9357	0.5097	0.2726

References
1. Champagne *et al.* (1970): homogeneous shear flow in a wind tunnel.
2. Cramer (1967): atmospheric surface layer.
3. Klebanoff (1955): wind-tunnel boundary layer on a smooth surface.
4. Wyngaard *et al.* (1974a): atmospheric surface layer, modified data.
5. So and Mellor (1972): surface layer in wind tunnel; smooth wall.
6. Comte-Bellot (1965): turbulent flow between parallel plates.
7. Hinze (1959): turbulent pipe flow.

stants C_1/β, α_1, and γ_1 can be computed from (18)–(21). Table 1 shows the numerical values obtained from seven different data sets. The constants α_1 and γ_1 appear to be relatively insensitive to the scatter in the data; all are positive and of order 1. However, C_1/β varies by a factor of 2. This scatter may be attributed to experimental errors, variation in Reynolds number, and flow configuration. For comparison, numerical values of the corresponding constants were computed for Lumley's model. These values are $\alpha_1=0.76$, $\gamma_1=-0.06$, and $C_1'=1+7\text{II}$ ($\text{II}=b_{ij}b_{ji}$, second invariant of the tensor b_{ij}). Setting the constant $\beta=0.312$ (see Lumley and Khajeh-Nouri, 1974b), the constant C_1 in our model ranges between 0.56 and 1.2, while the values of $C_1=1+7\text{II}$ vary from 1.25 to 2.2. The average value of γ_1 is 0.26, much larger than -0.06 in Lumley's model. No correlation was found between the constant C_1 in Table 1 and $C_1'=1+7\text{II}$ (II was based on experimental data). On the contrary, C_1 was found to be inversely proportional to the second invariant II, diverging from the values of $1+7\text{II}$. These discrepancies cannot be attributed to experimental errors; we have found no satisfactory explanation so far.

4. Physical interpretation of the mean shear and rotation terms

The effect of the mean strain-rate terms $\alpha_0 S_{ij}$ and $\alpha_1 A_{ij}^s$ can be explained by the "rapid distortion" argument of Batchelor (1953): if a fluid element is distorted sufficiently rapidly by the mean flow field, so that the inertial and viscous forces have little effect, the distortion results in redistribution of turbulent vorticity and consequently of the turbulent energy as well.

Pure rotation of the fluid elements, caused by the vorticity terms $\gamma_1 A_{ij}^r$, results in fluctuating Coriolis forces. Again, the net effect is redistribution of the turbulent energy. The actual physical meaning of the rotation terms is not easily understood. To illustrate how the turbulent rate equations are affected by the vorticity part of the pressure tensor π_{ik} let us consider a plane homogenous shear flow rotating with angular velocity $\mathbf{\Omega}$ aligned with the gradient of the mean velocity $\partial U_1/\partial x_3$, i.e., $\mathbf{\Omega}=\Omega_3$. The Coriolis forces will generate a stress component $\overline{u_1u_2}$. The rate equarion for $\overline{u_1u_2}$ is then

$$\frac{\partial\overline{u_1u_2}}{\partial t}=-(1-\gamma_1)2\Omega_3(\overline{u_1^2}-\overline{u_2^2})-C_1\frac{\overline{u_1u_2}}{\tau}. \qquad (22)$$

Note that the pressure term due to the rotation occurs in the equation as a linear combination of the explicit Coriolis terms, and that the stress $\overline{u_1u_2}$ switches sign with Ω_3. If we assume that the pressure-gradient velocity correlation terms always have a tendency to reduce the anisotropy of the turbulent field, it makes sense that the constant γ_1 is positive and smaller than one [see Eq. (22)]. From Table 1, the average value of the constant γ_1 is 0.26 ± 0.04. This value is consistent with our assumptions.

Wyngaard *et al.* (1974b) suggested that some kind of negative feedback on the explicit Coriolis terms in the turbulent stress equations is desirable when a convective planetary boundary layer with baroclinic geostrophic wind is modeled. Again, if the constant γ_1 is positive, the rotation part of the pressure term π_{ik} generates negative feedback on the explicit Coriolis terms.

5. Application to a neutral, barotropic Ekman layer

Numerical computations were performed for a neutral, barotropic, horizontally homogeneous planetary boundary layer with friction velocity $u_*=0.3$ m s^{-1}, surface roughness $z_0=0.01$ m, surface wind orientation along the positive x axis, oriented west–east. A latitude of 45°N was selected, with the Coriolis

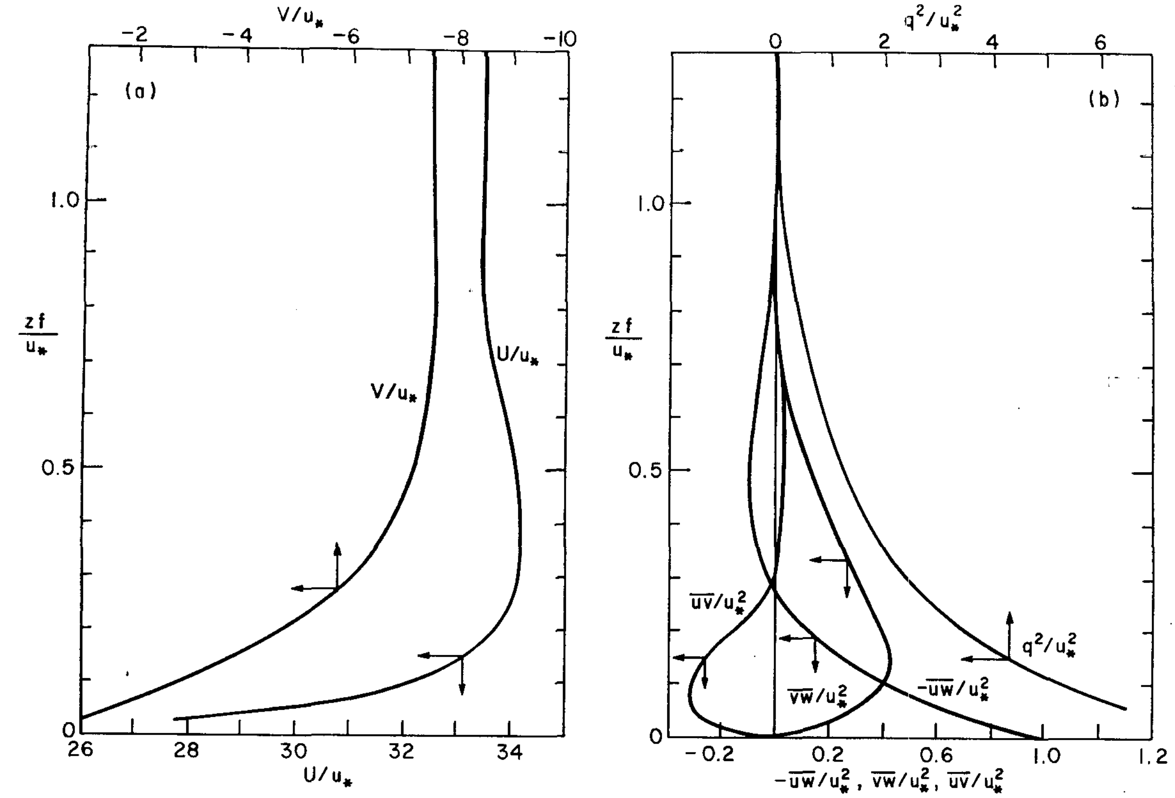

FIG. 1. Vertical distributions of mean wind components (a) and turbulent energy and stresses (b) for the neutral planetary boundary layer.

parameter $f=2\Omega\cos\phi=2\Omega\sin\phi=10^{-4}$ s^{-1}. As an example, the surface-layer data of Cramer (1967), and the corresponding set of constants (see Table 1, Case 2) were selected; they were used as lower boundary conditions at $zf/u_*=0.0016$. The upper boundary was placed at $zf/u_*=1.75$, with zero Reynolds stresses and wind gradients as boundary conditions. The dissipation $\bar{\epsilon}$ was modeled in accordance with Lumley and Khajeh-Nouri (1974b):

$$\frac{D\bar{\epsilon}}{Dt}=-3.73\frac{\bar{\epsilon}}{q^2}(\bar{\epsilon}-bP)-\frac{\partial}{\partial x_j}\overline{u_j\epsilon}. \qquad (23)$$

Here, $P=-\overline{u_iu_j}(\partial U_i/\partial x_j)$ represents the mean kinetic energy production, and the constant b was taken to be $\frac{3}{4}$, in agreement with Wyngaard *et al.* (1974a). The turbulent transport terms in Eq. 1 were approximated by a scalar gradient transport model of the form $\overline{\psi u_i}=C_\psi\overline{u_iu_j}(\partial\bar{\psi}/\partial x_j)$, where $\bar{\psi}$ is $\overline{u_ju_k}$ or $\bar{\epsilon}$. With the exception of the dissipation transport constant C_ϵ, all C_ψ's were set at unity. C_ϵ was determined exactly in the surface layer where $\bar{\epsilon}=u_*^3/kz$. Setting $(D\bar{\epsilon}/Dt)=0$ in (23), we found C_ϵ to be 0.67. The time constant $\beta=\tau/(q^2/\bar{\epsilon})$ was taken to be 0.15.

The results of these computations, given in Fig. 1, show that the nondimensional height of the PBL, defined as the height to the first zero crossing of $\overline{uw}$, is $zf/u_*=0.28$; this agrees with observations. The profiles of the wind components U/u_* and V/u_* are similar to observed profiles, but the geostrophic angle $\alpha_0=\tan^{-1}(V_g/U_g)$ is less than that observed. Application of a fully invariant transport model of the form

$$\overline{u_iu_ju_k}=\left(\overline{u_iu_e}\frac{\partial\overline{u_ju_k}}{\partial x_e}+\overline{u_ju_e}\frac{\partial\overline{u_ku_i}}{\partial x_e}+\overline{u_ku_e}\frac{\partial\overline{u_iu_j}}{\partial x_e}\right)$$

may alter results in the middle and upper part of the boundary layer; however, the changes are not expected to be appreciable.

6. Conclusions

The model of the pressure term described here is capable of predicting features of the entire boundary layer and of recovering the experimentally determined values of the turbulent stresses in the surface layer. The numerical values of the free constants α_1, γ_1 and C_1/β depend on experimental data; they are subject to error. Because of this, the model presented here cannot and does not claim universal applicability. It is, however, attractive for applications in which it is necessary to properly represent the measured turbulent stress components in the surface layer. The variation of the constant C_1 associated with the return-to-isotropy term suggests that C_1 may be dependent on some parameter representing the presence of a rigid (smooth or rough) surface. In our model we did not explicitly account for the "reflection" of pressure terms at the surface, but the contribution of such an effect could be appreciable.

We have applied similar procedures to the determination of the "rapid-deformation" part of the pressure-gradient temperature correlation terms in the heat flux equations. In that case only two free constants are involved. We are using these models in studies of the convective boundary layer; we hope to report on that research in the near future.

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