The Attenuation of Vertically Propagating Internal Gravity Waves by a Randomly Varying Wind/Current Shear

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(Manuscript received 19 May 1975, in revised form 24 September 1975)

ABSTRACT

We consider the effect of a randomly varying horizontal advective flow on internal gravity waves propagating vertically in a nonrotating, stratified and unbounded fluid. Attention is focused on the alteration in the vertical growth rate for the waves when the background flow U is of the form $U=U_o+\mu$, where μ is a centered stationary random function of height (z) and U_o=constant. We show that for both long and short correlation lengths the randomness in the wind (or current) leads to a dissipation of wave energy such that the normal upward amplification of the waves can be significantly reduced. Similarly, the normal downward attenuation can be significantly enhanced. The alteration in phase speed produced by the random flow is also considered although it is not discussed in detail.

1. Introduction

With their ability to redistribute momentum and energy, internal gravity waves are an important part of the kinematics and dynamics of the atmosphere and ocean. Hines (1960), for example, first showed that the irregularities in the winds and density distributions in the upper atmosphere could be related to internal-wave-induced motions. Work being conducted in the ocean also verifies the importance of these oscillating perturbations to flow variability in stratified fluids. The generation of turbulence in the atmosphere and ocean is linked to the breaking of these waves, as well as to their absorption at critical layers and to their mutual interaction. Moreover, the momentum flux released during the breakdown of the internal gravity wave systems is capable of forcing a mean flow through gradients in the Reynold's stress. Recent investigations by Hines (1972) and by Jones and Houghton (1971, 1972) have elucidated this transfer mechanism between the waves and the mean flow for the upper atmosphere. For the ocean, Hogg (1971) and Thomson (1975) have considered the generation of longshore currents by the breaking of internal gravity waves over a shallowing bottom.

In atmospheric investigations involving the properties of internal gravity waves much use is made of the fact that decreasing fluid density leads to an amplification of the velocity oscillations associated with waves whose energy is being propagated upward. Although much less dramatic, a similar effect takes place in the ocean when internal waves originating at the bottom propagate toward the surface. In both cases, however, the kinetic energy and the energy flux in the direction of the group velocity remain conserved in the absence of any other mechanisms. Nevertheless, dissipative processes such as viscous damping and thermal conduction do exist and can lead to a significant loss of wave energy (Pitteway and Hines, 1963). This results in a corresponding reduction in the vertical amplification of the velocity oscillations of upward propagated waves or, alternatively, to an enhancement in the attenuation of downward propagated waves. Background turbulence can be another important dissipating mechanism, although analytical and numerical investigations into its overall importance suffer somewhat from the need for eddy diffusion coefficients (e.g., Hines *et al.*, 1974, pp. 421–428; LeBlond, 1966).

The purpose of this paper is to show that random vertical variations in the background horizontal wind or current profile provide a mechanism for dissipating the energy of vertically propagating internal gravity waves. As with other diffusive processes, this can lead to a significant modification of the upward amplification or downward attenuation normally associated with these waves. An advantage of the analysis used, however, is that it yields the effectiveness of the diffusive process explicitly in terms of the ensemble-averaged properties of the background flow field. (The randomness in the background flow may be considered as resulting from a superposition of other motions whose periods exceed that of the wave itself or, alternatively, as resulting from turbulence-generated fluctuations which are confined to horizontal planes by the vertical stratification.) Singular regions are avoided in the analysis since we require that the speed of the basic flow always be somewhat less than the horizontal phase speed of the internal gravity waves and that it remain stable in the presence of these waves. Under these assumptions the dispersion relation yields an expression for the vertical wavenumber which can then be used to derive the effect of the random basic flow on the wave characteristics.

2. Formulation

We consider a nonrotating, inviscid and incompressible fluid in which x is the horizontal coordinate and z the vertical coordinate measured upward parallel to the acceleration of gravity $(0, -g)$. The fluid is stratified with mean density $\rho_0(z)$ such that the Brunt-Väisälä frequency $N\{=[(-g/\rho_0)(\partial\rho_0/\partial z)]^{\frac{1}{2}}\}$ is a constant. Motions associated with the basic state are orientated in the x direction and have the form

$$U(z)=U_0+\mu(z), \tag{2.1}$$

where U_0 is a constant and $\mu(z)$ is a stationary random process with zero ensemble average, viz.,

$$\langle\mu\rangle=0.$$

This configuration is assumed to satisfy the equations of motion exactly and to be dynamically stable whereby the local Richardson number $\mathrm{Ri}=N^2/(\partial U/\partial z)^2$ is always greater than $\frac{1}{4}$ according to the stability criterion of Miles (1961) and Howard (1961).

Plane wave perturbations are then superimposed upon the basic state, in which the velocity fluctuations (u^*, w^*), the pressure perturbations p^*, and the density perturbations ρ^* have the generalized form

$$\chi^*(x,z,t)=\chi(z)\exp[i(kx-\sigma t)],$$

where k is the horizontal wavenumber and σ the radian frequency satisfying the relation $0<\sigma<N$ corresponding to the usual passband for internal waves in an incompressible fluid without inertial effects. [The admission of compressibility alters the situation somewhat in that there are now two internal wave passbands (Hines, 1960): the lower frequency internal gravity wave band $0<\sigma<\sigma_g$ and the higher frequency internal acoustic wave band $\sigma_a<\sigma<\infty$, where $\sigma_a>\sigma_g$. Attention in this paper is confined to internal waves in the former category. Moreover, the atmosphere is assumed to be isothermal so that $\sigma_g=N$.] Upon defining the Doppler-shifted frequency ω by

$$\begin{aligned}\omega&=\sigma-kU\\&=k(c-U),\end{aligned} \tag{2.2}$$

where $c=\sigma/k$ is the horizontal phase speed, the linearized equations of momentum, incompressibility and continuity become

$$\rho_0(-i\omega u+wU_z)=-ikp,$$

$$\rho_0(-i\omega w)=-p_z-g\rho \tag{2.3}$$

$$-i\omega\rho+w\rho_{0z}=0, \tag{2.4}$$

$$iku+w_z=0, \tag{2.5}$$

in which subscripts express partial differentiation. The system (2.3)-(2.5) can be solved for the pressure perturbations to yield

$$\{(\omega^2-N)[k^2(\omega^2-N)-\omega^2(\partial_z^2+N^2/g\partial_z)] -2\omega^3kU_z\partial_z\}p=0, \tag{2.6}$$

in which the Boussinesq approximation has been made.

Since ω and U are random variables, (2.6) may be expressed as

$$(L+M)p=0, \tag{2.7}$$

where L is a linear deterministic operator and M a linear random differential operator with non-zero mean. We now wish to determine the effect of the operator M, corresponding to a random advective field, on the propagation of internal gravity waves. To do this we confine our discussion to those physical situations in which the horizontal phase speed is appreciably greater than the local advective speed U, that is $|U/c|\sim O(\epsilon)$, where $0<\epsilon\ll 1$. This condition permits us to treat the operator M as a small perturbation to the deterministic operator L and hence to obtain expressions for the dispersion relation in terms of the expansion parameter ϵ. In particular, we assume that the exact wave field p can be partitioned such that $p=\langle p\rangle+p'$, where $\langle\ \rangle$ is an ensemble average over many realizations of the process and p' is the deviation from that average. The dispersion relation for the coherent or mean field $\langle p\rangle$ is then determined to second order in ϵ by the method of Keller (1967), which states that

$$\exp(-ilz)\langle L^{-1}\rangle^{-1}\exp(ilz)=0, \tag{2.8}$$

where

$$\langle L^{-1}\rangle^{-1}=L+\langle M\rangle+\langle M\rangle L^{-1}\langle M\rangle -\langle ML^{-1}M\rangle+O[(L^{-1}M)^3], \tag{2.9}$$

provided that $\|(L^{-1}M)p\|/\|p\|<1$. In this problem it follows from (2.6) and (2.7) that

$$L=(k^2/\sigma^2)(N^2-\sigma^2)^2+(N^2-\sigma^2)(\partial_z^2+N^2/g\partial_z), \tag{2.10}$$

and that to second order

$$\left.\begin{aligned}M=\epsilon[&4k^2(N^2-\sigma^2)\tilde{U}\\&+2(2\sigma^2-N^2)\tilde{U}(\partial_z^2+N^2/g\partial_z)\\&-2\sigma^2\tilde{U}_z\partial_z]+\epsilon^2[2k^2(3\sigma^2-N^2)\tilde{U}^2\\&-(6\sigma^2-N^2)\tilde{U}^2(\partial_z^2+N^2/g\partial_z)+6\sigma^2\tilde{U}\tilde{U}_z\partial_z]\end{aligned}\right\}. \tag{2.11}$$

Here the horizontal velocity U has been nondimen-

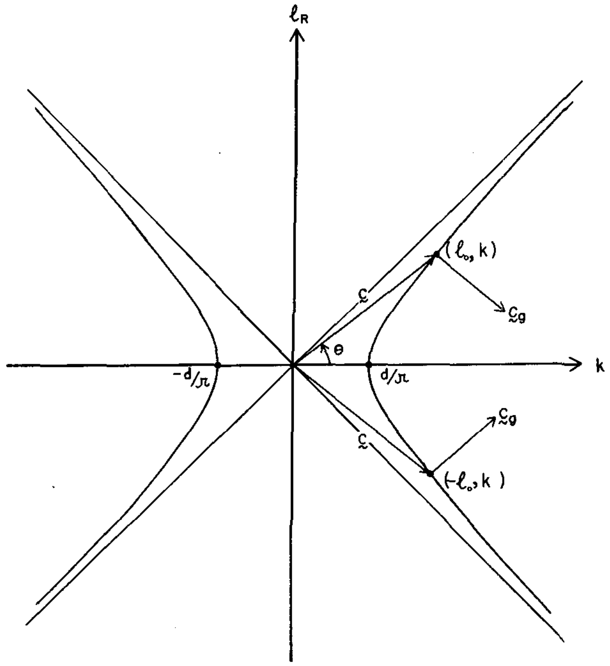

FIG. 1. The zeroth order or deterministic dispersion relation (3.3) plotted for a fixed frequency $\sigma>0$. The vector **c** gives the direction of the phase velocity while $\mathbf{c}_g$ gives the direction of the group velocity for the two cases $l_R>0$ and $l_R<0$; $\Omega\equiv(N^2-\sigma^2)/\sigma^2$.

sionalized with c via the relation

$$\begin{aligned}\epsilon\tilde{U}&=\epsilon(\tilde{U}_0+\tilde{\mu})\\&=(U_0+\mu)/c,\end{aligned}\tag{2.12}$$

in which $\tilde{U}_0$ and $\tilde{\mu}$ are of order unity. Although the vertically uniform component U_0 is not essential to the present analysis, it is nevertheless retained for the sake of generality and to aid in interpretating the results.

3. Determination of the vertical wavenumber l

Eqs. (2.8) and (2.9) give the dispersion relation correct to $O(\epsilon^2)$ and therefore lend themselves to an iterative solution based upon successive approximations to the vertical wavenumber. To derive the zeroth-order dispersion relation in the absence of advective effects we set $\epsilon=0$ (i.e., $M\equiv 0$) in (2.9) and solve the deterministic form of (2.8) given by

$$\exp(-ilz)L\exp(ilz)=0.$$

With $l=l_R+il_I$, where l_R and l_I are real, this yields the usual real dispersion relation for internal gravity waves, namely,

$$l_R^2+l_I^2=(N^2-\sigma^2)k^2/\sigma^2,\tag{3.1}$$

in which

$$l_I=N^2/2g.\tag{3.2}$$

For notational convenience we shall define $l_0=l_R$ and $d=l_I$ to be the zeroth-order approximations to the real and imaginary parts of the vertical wavenumber. Eq. (3.1) therefore becomes

$$l_0^2+d^2=(N^2-\sigma^2)k^2/\sigma^2,\tag{3.3}$$

which for fixed frequency σ consists of rectangular hyperbolas opening out at large $|k|$ (Fig. 1). The phase velocity **c** for given wavenumbers l_0, k and frequencies σ, N is directed parallel to a line drawn from the origin toward a point on the hyperbola. Since the hyperbola closes toward the k axis for increasing σ, the group velocity $\mathbf{c}_g$ is directed toward the k axis at right angles to the local tangent at that point. For upward propagating phase ($l_0>0$) the group velocity is therefore directed downward, while for downward propagating phase ($l_0<0$) it is directed upward. This feature of the wave motions will be used when we come to determine the second-order corrections to the dispersion relation.

To zeroth order in ϵ the plane wave solutions clearly have the form $p\sim\exp[-dz+i(kx+l_0z-\sigma t)]$ which implies amplification of the pressure signal for downward propagating waves and attenuation of the signal for upward propagating waves. On the other hand, the velocity fluctuations associated with the waves, as determined from p through (2.3) and (2.5) $[U_z=0]$, are amplified for upward propagating waves and attenuated for downward propagating waves. Both these features, of course, follow directly from the fact that the mean density ρ_0 of the fluid decreases exponentially in the positive z direction for constant N; i.e., $\rho_0\sim\exp(-2dz)$. However, the kinetic energy of the wave motions, $\rho_0|\mathbf{u}^2|$, remains constant with height, as does the averaged zeroth-order energy flux $\overline{p\mathbf{u}}$.

a. First-order corrections

The $O(\epsilon)$ corrections to the dispersion relation may now be determined by substituting $L+\langle M\rangle$ into (2.8), where to first order

$$\langle M\rangle=\epsilon\tilde{U}_0[4k^2(N^2-\sigma^2)+2(2\sigma^2-N^2)(\partial_z^2+2d\partial_z)].\tag{3.4}$$

Solving (2.8) iteratively by putting the zeroth-order values l_0 and d into the $O(\epsilon)$ expressions then yields

$$l_R^2+d^2=(k^2/\sigma^2)[N^2(1+2\epsilon\tilde{U}_0)-\sigma^2]+O(\epsilon^2)\tag{3.5}$$

for the real form of the dispersion relation, and

$$l_I=d+O(\epsilon^2)$$

for the amplitude modulation factor. The only effect of the advection to this order then is to shift the natural oscillation frequency N to $N(1+2\epsilon\tilde{U}_0)^{\frac{1}{2}}$ in the dispersion relation (3.5). Modifications to l_I do not arise since nonlinear terms in U do not appear until $O(\epsilon^2)$.

b. Second-order corrections

It is to this order that the effects of the random velocity fluctuations first become important, producing the amplification of the pressure and velocity perturbations along the direction of the phase regardless of their direction of propagation. To see this we first define $M^{(i)}$ as the ith order component of the random differential operator M and let

$$L^* = L + \langle M^{(1)} \rangle. \tag{3.6}$$

The dispersion relation (2.8) then becomes, to $O(\epsilon^2)$,

$$\exp(-ilz)L^* \exp(+ilz) + \exp(ilz)[\langle M^{(2)}\rangle + \langle M^{(1)}\rangle L^{-1}\langle M^{(1)}\rangle - \langle M^{(1)}L^{-1}M^{(1)}\rangle] \exp(ilz) = 0, \tag{3.7}$$

in which L^{-1} is the integral operator defined by

$$L^{-1}q(z) = \int_{-\infty}^{\infty} G(z-z')q(z')dz', \tag{3.8}$$

and $G(z)$ is the Green's function

$$LG(z) = \delta(z). \tag{3.9}$$

To determine G in (3.8) we take the Fourier transform of (3.9). This yields the transformed Green's function $\hat{G}(l)$ which is seen to have two singularities in the upper half of the complex l-plane (Fig. 2). Provided that we close the inversion path above the real l axis for $z>0$ and below the real l axis for $z<0$, and indent above the pole having Re $l>0$, we obtain the correct form for $G(z)$. Although this procedure selects waves with inward directed phase velocities it ensures that the group velocity, and hence the direction of energy propagation, is directed away from the source at $z=0$ in accordance with the Sommerfield radiation condition. The Green's function which determines L^{-1} through (3.8) is then

$$G(z) = i[2B(N^2-\sigma^2)]^{-1} \exp[-(dz+iB|z|)], \tag{3.10}$$

where

$$B = \left(\frac{N^2-\sigma^2}{\sigma^2}k^2 - d^2\right)^{\frac{1}{2}} > 0. \tag{3.11}$$

Using (3.10) in (3.8), together with the forms for the operators L and M obtained from (2.10) and (2.11), we may now calculate the dispersion relation to second order. The iterative solution to the resulting equation is then found by substituting the zeroth-order and first-order values of l_R and l_I into the $O(\epsilon^2)$ and $O(\epsilon)$ expressions, respectively. As shown in the Appendix,

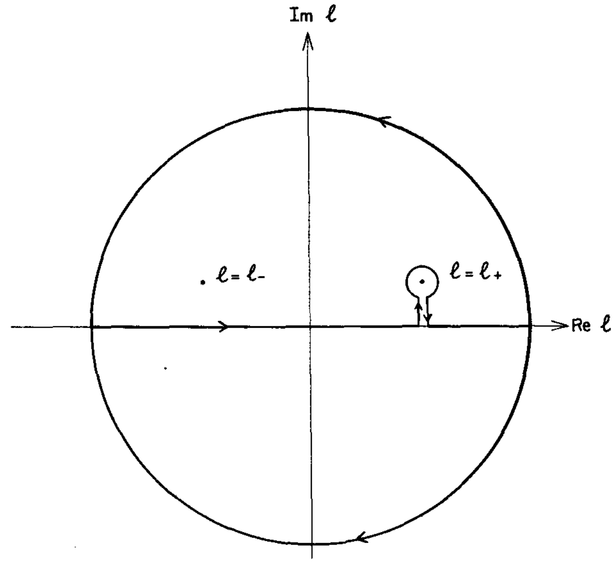

FIG. 2. Inversion contour for the transformed Green's function $G(l)$. The contour is closed above for $z>0$ and below for $z<0$; $l_\pm \equiv \pm B + id/2$, where B is defined by (3.11).

the result is

$$(N^{*2}-\sigma^2)(k^2/\sigma^2) - l^2 + 2ild + \epsilon^2\left\{\frac{k^2N^2}{\sigma^2}\left[3\tilde{U}_0{}^2 + \left(\frac{4\sigma^4}{N^2(N^2-\sigma^2)} - 1\right)\Gamma(0)\right] + 8k^2l_0\kappa^2\int_0^\infty \sin(2l_0z)\Gamma(z)dz - 8ik^2|l_0| \times\left[\frac{1}{2}\phi(0) - 2\kappa^2\int_0^\infty \cos^2(l_0z)\Gamma(z)dz\right]\right\} = 0, \tag{3.12}$$

in which

$$\left.\begin{aligned} N^{*2} &= N^2(1+2\epsilon\tilde{U}_0) \\ \kappa^2 &= 1-\left(\frac{1}{2}\frac{N^2}{\sigma^2}\frac{k}{l_0}\right)^2 \end{aligned}\right\}; \tag{3.13}$$

$$\Gamma(z) = \langle\tilde{\mu}(z')\tilde{\mu}(z'-z)\rangle \tag{3.14}$$

is the autocorrelation function (an even function); and

$$\phi(l) = \int_{-\infty}^{\infty} \exp(ilz)\Gamma(z)dz \tag{3.15}$$

is the power spectrum, with $\phi(0)>0$.

Substituting $l = l_R + il_I$ into (3.12) and separating into real and imaginary parts gives two coupled equations in l_R and l_I. These decouple in a straightforward manner however if the imaginary part is determined first, and we find

$$l_I=d-\epsilon^2 4k^2\,\mathrm{sgn}l_0\times\left[\frac{1}{2}\phi(0)-2\kappa^2\int_0^\infty \cos^2(l_0z)\Gamma(z)dz\right],\quad(3.16)$$

where as previously defined $d=N^2/2g$, while

$$\mathrm{sgn}l_0=\begin{cases}1 & \text{if}\quad l_0>0\\ -1 & \text{if}\quad l_0<0.\end{cases}$$

If the bracketed expression in (3.16) is positive, the effect of the randomness in the basic advective field is to produce an apparent second-order amplification of the pressure fluctuations irrespective of the direction of phase propagation. On the other hand, there will be an apparent attenuation if the bracketed expression is negative. In the direction of the group velocity, however, the roles are reversed. Thus a positive value of the bracket in (3.16) actually corresponds to amplitude attenuation in the direction of energy propagation while a negative value corresponds to amplifications in that direction.

Upon substituting (3.16) into the real part of (3.12) and applying the binomial expansion, we obtain the real component of the wavenumber which is expressable as

$$\frac{l_R}{l_0}-1=\epsilon\left(\frac{k}{l_0}\frac{N}{\sigma}\right)^2\hat{U}_0+\epsilon^2\frac{k^2}{l_0^2}\times\left\{\frac{1}{2}\frac{N^2}{\sigma^2}\hat{U}_0^2\left[3-\left(\frac{k}{l_0}\frac{N}{\sigma}\right)^2\right]+\Gamma(0)\left(\frac{2\sigma^2}{N^2-\sigma^2}-\frac{N^2}{2\sigma^2}\right)+4\kappa^2 l_0\int_0^\infty \sin(2l_0z)\Gamma(z)dz\right\}.\quad(3.17)$$

If the right side of (3.17) is positive, the effect of the advection is to require larger vertical wavenumbers for given σ, k and N than in the absence of such advection. In Fig. 1 this corresponds to an increase in the magnitude of the slopes of the asymptotes and therefore to an expansion of the hyperbolas away from the k axis. Associated with this is a change in direction of both the phase velocity and group velocity, and a corresponding change in magnitude of their respective vertical components. Since the vertical component ν of the phase velocity satisfies $\nu=\nu_0 l_0/l_R$, its magnitude will be decreased if (3.17) is positive, as will that of the vertical component of the group velocity. The opposite effects will of course occur if (3.17) should be negative. Clearly if $\hat{U}_0\neq 0$ the sign and magnitude of (3.17) will mostly be determined by the $O(\epsilon)$ term. Thus, $l_R>l_0$ for waves propagating in the direction of the mean flow $(U_0/c>0)$ and $l_R<l_0$ for waves propagating opposite to the direction of mean flow $(U_0/c<0)$. The effect of the second-order term is not so straightforward and requires certain approximations to be made before it can be evaluated. However, as the purpose of this paper is to describe the spatial amplification of internal gravity waves rather than changes in their phase speed we shall henceforth limit our discussion to determining approximate expressions for l_I only.

c. Short and long correlation lengths

The general expression (3.16) can reveal little more about the spatial variations in amplitude unless we say something about the structure of the advective velocity $U(z)$. To do this let S be the correlation length of the process $\tilde{\mu}(z)$ such that $\Gamma(z)\approx 0$ for $|z|>S$. If we then define the wavelength λ by $\lambda=2\pi/l_0$ we have the two limiting cases: $\lambda\gg S$ corresponding to short correlation lengths and $\lambda\ll S$ corresponding to long correlation lengths. The first case states that the vertically propagating waves are being modified by a random velocity field which is incoherent over a wavelength while the second states that the waves are being modified by a field which is somewhat coherent over a wavelength. Both limits will presumably have applications to atmospheric and oceanic conditions depending upon the process producing the mean and perturbed motions.

The leading terms in (3.16) for each limit are now obtained via the substitutions $z=Sz'$ and $l_0S=\gamma$. We further assume that the integrals

$$\int_0^\infty \Gamma(Sz')(z')^n dz'\approx\int_0^1 \Gamma(Sz')(z')^n dz',\quad n=0,1,2\cdots,$$

are of order unity. In the limit of short correlation lengths, retention of the lowest order terms in (3.16) in an expansion in terms of $\gamma(\ll 1)$ yields

$$l_I=d-\epsilon^2\frac{k^2}{l_0^2}\,\mathrm{sgn}l_0\left[2d^2+k^2\frac{N^2}{\sigma^2}\left(\frac{N^2}{\sigma^2}-2+\frac{\sigma^2}{N^2}\right)\right]\phi(0),\quad S/\lambda\ll 1.\quad(3.18)$$

To obtain the corresponding relation for long correlation lengths in (3.16), we first integrate the integral by parts and then retain lowest order terms in $\gamma^{-1}(\ll 1)$. This yields

$$l_I=d-\epsilon^2\frac{k^2}{l_0^2}\,\mathrm{sgn}l_0\frac{1}{2}\left(k\frac{N^2}{\sigma^2}\right)^2\phi(0),\quad S/\lambda\gg 1.\quad(3.19)$$

Since $N^2\geqslant\sigma^2$ and $\phi(0)>0$ it is clear that the second-order contributions to l_I in both (3.18) and (3.19) are positive or negative according to whether $l_0<0$ or $l_0>0$, respectively. Thus, in either limit the pressure perturbations $\langle p\rangle\sim\exp(-l_Iz)$ are always being attenuated to order ϵ^2 regardless of the direction of energy propagation. The same is true of the averaged streamfunction $\langle\psi\rangle$ since

$$(N^2-\sigma^2)\langle\psi\rangle\approx-(c/\rho_0)\langle p\rangle_z,$$

where $\langle\mathbf{u}\rangle=(\langle\psi\rangle_{z'}-\langle\psi\rangle_x)$ and $\rho_0\sim\exp(-2dz)$. The corresponding expressions for l_I for the velocity perturbations $\langle\mathbf{u}\rangle$ associated with the waves are in fact found simply by substituting $d=-d$ into (3.18) and (3.19). Consequently, energy is being extracted from the coherent wave field and transferred to the random motions. As to be expected the transfer in this case involves only the random part of the basic velocity field, suggesting that wave scattering is the basic mechanism producing the loss of energy from the coherent internal wave field.

d. Numerical values

Eqs. (3.18) and (3.19) may be rearranged into the normalized forms

$$\frac{(l_I/d)-1}{\epsilon^2\phi(0)d}=F,$$

whereupon defining $\alpha\equiv N^2/\sigma^2$ and $\beta\equiv d^2/l_0^2$ we have, for short correlation lengths (S),

$$F_S\equiv-\mathrm{sgn}l_0\frac{1+\beta}{\alpha-1}\left[2+\frac{1}{\beta}\frac{1+\beta}{\alpha-1}\alpha(\alpha-2+1/\alpha)\right],\quad(3.20)$$

and for long correlation lengths (L),

$$F_L\equiv-\mathrm{sgn}l_0\frac{1+\beta}{\alpha-1}\frac{\alpha^2}{2\beta}\frac{1+\beta}{\alpha-1}.\quad(3.21)$$

The magnitudes of the above functions for various values of α and β are listed in Table 1 together with the corresponding angle of phase propagation $\theta=\tan^{-1}(k/l_0)$ where from (3.3), $\tan\theta=[\alpha-1/(1+\beta)]^{\frac{1}{2}}$. Provided that $|\epsilon^2\phi(0)dF|<1$ these may be used to determine the attenuation of the pressure (or velocity) signal with height. Two obvious conclusions from the table are that 1) the smaller the horizontal or vertical wavelengths (small β) the more effective the damping for a given frequency and angle θ; and 2) the larger the angle θ the less effective the damping for fixed frequency and vertical wavenumber. The former is a consequence of the greater discrepancy between the wavelength and the scales of the "scatterers" while the latter is due to increasing alignment of the wave motions in the direction of the background shear.

4. Discussion

Referring to the solutions (3.18) and (3.19) for the two limiting cases of short and long correlation lengths, it can be seen that the validity of the analysis is dependent upon the $O(\epsilon^2)$ terms being somewhat smaller than the inverse scale height d $(=N^2/2g)$. If we use the fact that $N^2>\sigma^2$ and assume that $l_0^2\geqslant d^2$ in (3.3), this roughly translates to the requirement that

$$\epsilon^2 g/c^2\phi(0)\ll1,\quad(4.1)$$

which holds if ϵ^2 is sufficiently small. In the atmosphere, for example, where $c\approx10^2$ m s^{-1} is a reasonable value for the horizontal phase speed, we would require only that $U/c<1$ for winds having an autocorrelation function $\Gamma(z)\sim\exp(-|z|/H)$ with $H\approx1$ km.

In interpreting the results of Section 3 it has been emphasized that the second-order amplification of the pressure and velocity perturbations in the direction of phase propagation actually corresponds to amplitude attenuation in the direction of energy propagation. As the latter is the more physically relevant direction to consider, it is clear that the effect of random winds and currents will be to decrease the energy associated with the coherent part of an internal gravity wave field in the limit of short or long correlation lengths. The flux of energy ($p\mathbf{u}$) in the direction of the group velocity is therefore not conserved to second order, but rather has the form

$$|p\mathbf{u}|\sim\exp(-\epsilon^2\,\mathrm{sgn}\hat{l}_0 2Dz),$$

where $\hat{l}_0>0$ (<0) for upward (downward) propagating energy, and where D is given by the magnitude of the expressions multiplying the $O(\epsilon^2)$ terms in (3.18) and (3.19). Hence the random advective flow serves as a dissipative mechanism which scatters wave energy to the incoherent part of the wave field and/or to the background flow itself. Depending upon the magnitude of $[g\phi(0)]^{\frac{1}{2}}$ relative to the horizontal phase speed c, this "dissipation" can be of sufficient magnitude to measurably reduce the usual vertical amplification of the velocity perturbations associated with upward propagating internal wave energy. Similarly, velocity perturbations associated with downward propagating wave energy can be attenuated at an

TABLE 1. Magnitudes of the vertical attenuation factors F_S and F_L as defined by (3.20) and (3.21) for various values of the parameters $\alpha\equiv N^2/\sigma^2$ and $\beta\equiv d^2/l_0^2$. Also listed are the corresponding angles of phase propagation $\theta=\tan^{-1}[(\alpha-1)/(1+\beta)]^{\frac{1}{2}}$.

α	β	θ (rad)	$\lvert F_S\rvert$	$\lvert F_L\rvert$
2	10^{-1}	0.78	10004.0	20004.0
5	10^{-4}	1.11	10002.5	7814.1
10	10^{-4}	1.25	10002.2	6174.1
100	10^{-4}	1.47	10002.0	5102.5
2	10^{-2}	0.78	104.0	204.0
5	10^{-2}	1.11	102.5	79.7
10	10^{-2}	1.25	102.2	63.0
100	10^{-2}	1.47	102.0	52.0
2	1	0.62	8.0	8.0
5	1	0.96	5.0	3.1
10	1	1.13	4.4	2.5
100	1	1.43	4.0	2.0

appreciably greater rate than occur in the absence of the random advective field.

Finally we note that the full expression for l_I [Eq. (3.16)] indicates that energy can be extracted from the background flow and transferred to the coherent wave for intermediate ranges of S/λ, provided that the second term is positive and of greater magnitude than $\frac{1}{2}\phi(0)$. In general, however, it is expected that the dissipating effect of the scattering as measured by $\phi(0)$ will be of greater importance than the coherent exchange of energy between the waves and the random flow as measured by the integral term in (3.16).

APPENDIX

Calculation of the Dispersion Relation

To determine the dispersion relation (3.7) which is correct to second order in ϵ, we first substitute from (2.11) the expressions for the operators $M^{(1)}$ and $M^{(2)}$, defined as the $O(\epsilon)$ and $O(\epsilon^2)$ contributions to the random operator M, respectively. The operators $L^{-1}\langle M^{(1)}\rangle$ and $L^{-1}M^{(1)}$ which appear in (3.7) are then obtained in integral form from (3.8) whereby the dispersion relation becomes

$$(N^2-\sigma^2)[(N^2-\sigma^2)(k^2/\sigma^2)+r]+\epsilon\tilde{U}_0[4k^2(N^2-\sigma^2)+2(2\sigma^2-N^2)r]+\epsilon^2\Big\{\langle\tilde{U}^2\rangle[2k^2(3\sigma^2-N^2)-(6\sigma^2-N^2)r]$$
$$+\tilde{U}_0^2\exp(-ilz)[4k^2(N^2-\sigma^2)+2(2\sigma^2-N^2)R]\int_{-\infty}^{\infty}G(z-z')[4k^2(N^2-\sigma^2)+2(2\sigma^2-N^2)r]\exp(ilz')dz'$$
$$-\exp(-ilz)\langle[4k^2(N^2-\sigma^2)\tilde{U}(z)+2(2\sigma^2-N^2)\tilde{U}(z)R-2\sigma^2\tilde{U}_z(z)\partial_z]\int_{-\infty}^{\infty}G(z-z')[4k(N^2-\sigma^2)\tilde{U}(z')$$
$$+2(2\sigma^2-N^2)\tilde{U}(z')r-2\sigma^2\tilde{U}_{z'}(z')il]\exp(ilz')dz'\rangle\Big\}=0=I^{(0)}+\epsilon I^{(1)}+\epsilon^2 I^{(2)}, \quad \text{(A1)}$$

where for convenience

$$R\equiv\partial_z^2+(N^2/g)\partial_z,$$
$$r\equiv\exp(-ilz)R\exp(ilz)$$
$$=-l^2+ilN^2/g.$$

To obtain the integrals in the $I^{(2)}$ part of (A1) we first apply the operators ∂_z and R to $G(z-z')$ where it can be shown that

$$RG(z)=-(N^4/4g^2+B^2)G(z),$$

in which G is defined by (3.10) and B by (3.11). We then make the change of variable $\hat{z}=z-z'$ in the integrals and perform the multiplication of the various terms to derive integral expressions involving terms of the form $\partial^n/\partial\hat{z}^n\Gamma(\hat{z})$ for the autocorrelation function $\Gamma(\hat{z})=\langle\mu(z)\mu(z-\hat{z})\rangle$, where $n=0, 1, 2$. Upon integrating by parts in order to remove derivatives of $\Gamma(\hat{z})$ from the integrals, we then proceed to solve the generalized expressions by iteration, using the zeroth-order solution $l=l_0+iN^2/2g$ in $I^{(2)}$ and the first-order solution $l=(k^2/\sigma^2)[N^2(1+2\epsilon\tilde{U}_0)-\sigma^2]+iN^2/2g$ in $I^{(1)}$. In $I^{(2)}$ use is also made of the zeroth-order dispersion relation (3.3) and the fact that $B=|l_0|$ to zeroth order. Further simplification results from the fact that $\partial/\partial z\Gamma(0)=0$. The final form for the dispersion relation [Eq. (3.12)] follows after some rearrangement and division by $N^2-\sigma^2$.

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