The Effects of Fluctuations in Liquid Water Content on the Evolution of Large Drops by Coalescence

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ABSTRACT

Because of the nonlinearity of the coagulation equation the evolution of large drops by coalescence is likely to be very sensitive to liquid water content. This implies that a cloud containing pockets of high liquid water content can produce rain much more quickly than an identical cloud in which the liquid water content is uniform.

1. Introduction

A considerable number of papers have been and are being devoted to the stochastic process of hydrometeor growth by droplet coagulation, while serious attempts are being made to parameterize the process so that hydrometeor growth can be handled more realistically in numerical circulation studies. The purpose of this note is to point out and exemplify the sensitivity of the process to the liquid water content of the cloud in which the coagulation is occurring. A consequence of this sensitivity is the inadequacy of any mean value of liquid water for describing the coagulation process in a cloud with spatially fluctuating liquid water content.

2. Liquid water content as a parameter in the coagulation equation

The coagulation equation, written in terms of droplet volume v and time t, is

$$\frac{d}{dt}n(v) = -n(v)\int_0^\infty K(v,u)n(u)du + \frac{1}{2}\int_0^v K(v, v-u)n(u)n(v-u)du, \quad (1)$$

where $n(v)\Delta v$ represents the concentration of drops in the category $v \rightarrow v+\Delta v$ and $K(v,u)$, the coagulation kernel, is $\pi(r+\rho)^2|V-U|E(v,u)$ if r and ρ are the radii and V and U the falling speeds of drops of volume v and u respectively; $E(v,u)$ is the collection efficiency for drops of those sizes.

The derivation of (1) and a similar slightly more complicated equation in terms of drop radius has been given many times and can be found in standard textbooks. It will not be repeated here, but it is worth noting that v and u are merely identifiers of drop size; the numbers $n(u)\Delta u$ in the category $u \rightarrow u+\Delta u$ interact with the number $n(v)\Delta v$ in the category $v \rightarrow v+\Delta v$; provided $K(v,u)n(v)n(u)\Delta v\Delta u$ gives the rate of interaction per unit volume, Eq. (1) or a very similar equation follows. For Eq. (1) to apply unchanged the only additional requirement is the additivity of v and u upon coagulation, i.e., that coagulation of a v drop and a u drop give a $(v+u)$ drop.

Liquid water content w does not appear explicitly in (1) but is contained implicitly through the relationship

$$\rho\int_0^\infty un(u)du = w$$

which applies when u is the drop volume. There are many ways in which $n(u)$ can vary in response to a change in w; an increase in w can be realized by increasing the number or the size of the drops, or by allowing both to change simultaneously. Suppose that Eq. (1) holds for $w=1$ and that changes in w are produced only through changes in n (Case I). Then for $w\neq 1$ one has to put $wn(u)\Delta u$ and $wn(v)dv$ for the number of drops per unit volume in the category $u \rightarrow u+\Delta u$ and $v \rightarrow v+\Delta v$ respectively; this leads to the equation (in which water density ρ has been omitted for simplicity, this quantity being numerically unity)

$$\frac{dn(v)}{d(wt)} = -n(v)\int_0^\infty K(v,u)n(u)du + \frac{1}{2}\int_0^v K(v, v-u)n(u)n(v-u)du, \quad (2)$$

so that variation of w is equivalent to a proportional change in the time scale—doubling the liquid water content causes the coagulative evolution of $n(u)$ to proceed twice as fast. If, for example, a given proportion of large drops is produced in 1000 s at $w=1$, it will be produced in 500 s at $w=2$ and 2000 s at $w=\frac{1}{2}$.

A second possible dependence of w is that given by holding the drop concentration and the shape of the distribution to be independent of w but allowing drop volume to change in proportion to w: that is, if $n(u)\Delta u$ gives the number of drops with volume $u \rightarrow u+\Delta u$ when $w=1$, then for $w\neq 1$ there are $n(u)\Delta u$ drops with volume $wu \rightarrow wu+\Delta(wu)$. In this case (Case II) Eq. (1) holds unchanged, but K now must refer to the coagulation of drops of volume wu with drops of volume wv. In general, the kernel increases with increasing overall size because cross section, falling speed and collection efficiency are all increased for a constant ratio of drop sizes when the size is increased. To keep the discussion as simple as possible it will be assumed that $K(wu,wv)=w^\alpha K(u,v)$.

This assumption would hold strictly in the Stokes law region if the collection efficiency $E(v,u)$ were constant (for then $K\propto$ area$\times$fall speed, i.e., $K\propto r^4$ and $w^{\frac{4}{3}}$) and an inspection of plotted kernels suggests that it is not too unreasonable an approximation in general. Furthermore, Manton (1974) has given theoretical arguments to obtain for drops of radius r and ρ ($r>\rho$) an expression of the form $r^4 f(\rho/r,r)$ for the collection kernel, with f initially constant (for fixed values of the ratio ρ/r), then increasing rapidly (as r^6) to a maximum and falling off slowly as r^{-1} beyond the maximum. Thus in different regions one finds r^4, r^{10} and r^3 proportionalities in the kernel K. Manton also discussed the effects of shear flow and concluded that the shear-induced kernel would show an r^6 dependence where without shear an r^4 dependence was indicated.

It seems appropriate therefore to look at the effect of a kernel which varies as r^3, r^4 and r^6 or, in terms of volume, as w^1, $w^{\frac{4}{3}}$ and w^2. When K scales with w, Eq. (2) again applies, the effect being exactly the same as that obtained by making drop numbers n proportional to w. For powers of w other than 1, we simply write the kernel in Eq. (1) as $w^\alpha K(u,v)$, and get

$$\frac{d}{d(w^\alpha t)}n(v)=-n(v)\int_0^\infty K(v,u)n(u)du + \frac{1}{2}\int_0^v K(v,\,v-u)n(v-u)n(u)du, \quad (3)$$

so again all that happens when w is changed is a rescaling of the time scale. For $w^{\frac{4}{3}}$ and w^2 dependence the scaling factor increases more rapidly; doubling the water content, for example, compresses the time scale by 2.5 and 4 respectively. If $n_1(u,t)$ represents a solution of Eq. (1), however obtained, for $w=1$, the same solution can be applied for another liquid water content simply by scaling the time appropriately, i.e.,

$$n_w(u,t)=\begin{cases} n_1(u,wt) & \text{for Case I} \\ & \quad (\text{concentrations}\propto w \text{ or } K\propto w) \\ n_1(u,w^\alpha t) & \text{for Case II} \\ & \quad [K(wu,wv)\propto w^\alpha K(u,v)]. \end{cases}$$

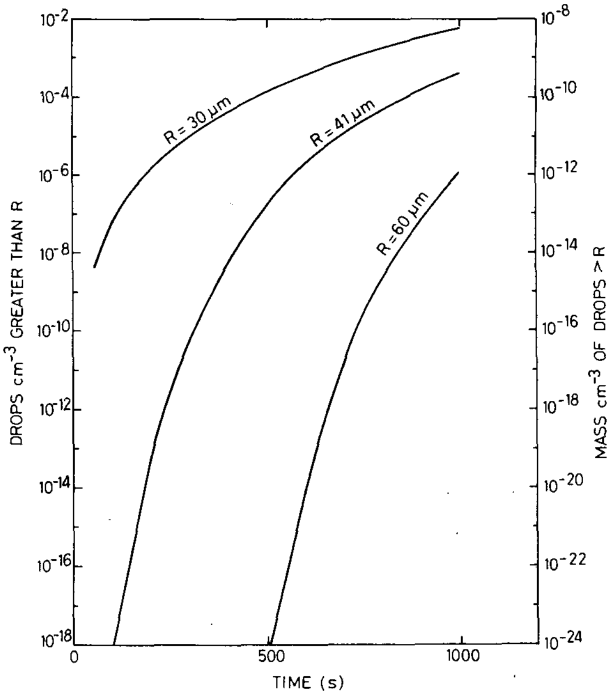

FIG. 1. Data computed by Ryan (1974) for the evolution of a cloud droplet population (initial concentration 200 cm^{-3}, liquid water content 1 g m^{-3}; largest droplets initially present 24 μm). Klett and Davis (1973) kernels were used.

If a cloud or portion of a cloud contains a spatially varying liquid water content, the probability of a water content between w and $w+\Delta w$ being $p(w)\Delta w$, the overall evolution process within the cloud is evidently given by averaging the solutions for different liquid water content using $p(w)$ as a weighting function. The result is a kind of time convolution of the solution for constant liquid water:

$$\bar{n}(u,t)=\int_0^{w_a} p(w)n_1(u,wt)dw \quad \text{for Case I,} \quad (4a)$$

$$\bar{n}(u,t)=\int_0^{w_a} p(w)n_1(u,w^\alpha t)dw \quad \text{for Case II.} \quad (4b)$$

The same holds true for any linear functional of the size distribution, such as the mean radius, cumulative number larger than a prescribed size, etc. The adiabatic liquid water content w_a represents the maximum possible value for w under normal circumstances.

The tail of a droplet distribution evolves with time in a highly nonlinear way, as exemplified by Fig. 1; the data for this figure were computed by Ryan (1974). It follows that when the liquid water fluctuates statistically the average evolution process proceeds quite a lot faster than it would if the liquid water were constant.

3. Application

If the distribution of liquid water content were known it would be a simple matter to take it into account using Eqs. (4). Unfortunately such data are not available. We do know, however, that the liquid water content fluctuates greatly within a cloud and that it is almost everywhere considerably less than the adiabatic value which would be given by conservative ascent of an air parcel from saturation at cloud base. Mixing with drier air is generally accepted as the reason for this, but details of the mixing process are as yet poorly understood. Cotton (1975), discussing numerical cloud models, suggests that the mixing "constant" used in such models even seems to vary with geographic location.

When the convolution-type integral in Eqs. (4a) and (4b) is examined, we notice that $n_w(u,t)$ must increase rapidly with w; because of the time-scaling it parallels the time-dependence of $n_1(u,t)$ illustrated in Fig. 1, and when we are dealing with the very few privileged drops in the extreme tail of the distribution (and it is these which become hydrometeors) the rate of increase is especially fast. Notice for example the curve labelled $R=41$ μm, which gives the concentration of drops >41 μm radius; around 600 s it attains an appreciable (i.e., comparable to hydrometeor) concentration at a rate of about a decade every 100 s. This gives a value of more than 10 for $\Delta(\ln N)/\Delta(\ln t)$, roughly corresponding to a variation of number with (time)^{10} during that interval of growth for Case I, or to an even higher power for Case II. Thus when one comes to evaluate

$$\int_0^{w_a} p(w)n_1(u,w^\alpha t)dw$$

or other linearly related quantities, the integrand is found to increase rapidly with w, and unless $p(w)$ decreases with w faster than n_1 increases, the integral is dominated by the behavior of the integrand at the largest values of w, near the adiabatic value which presumably represents the maximum possible value of w.

It is easy to show that the effect of water content fluctuations is profound when much higher values than the average occur—even if they occur infrequently. Consider for example a hypothetical situation where the water content is 0.33 g m^{-3} in 99% of the cloud volume but an adiabatic value of (say) 3 g m^{-3} exists in the remaining 1% of the cloud volume. The latter contributes little to the average liquid water content, which becomes 0.35 g m^{-3} rather than 0.33 g m^{-3}, but droplets greater than 40 μm are produced in concentrations several orders of magnitude higher in the regions of high liquid water, and those regions are dominant from the point of view of large drop growth. Fig. 2 shows for such a mixture the averaged evolution

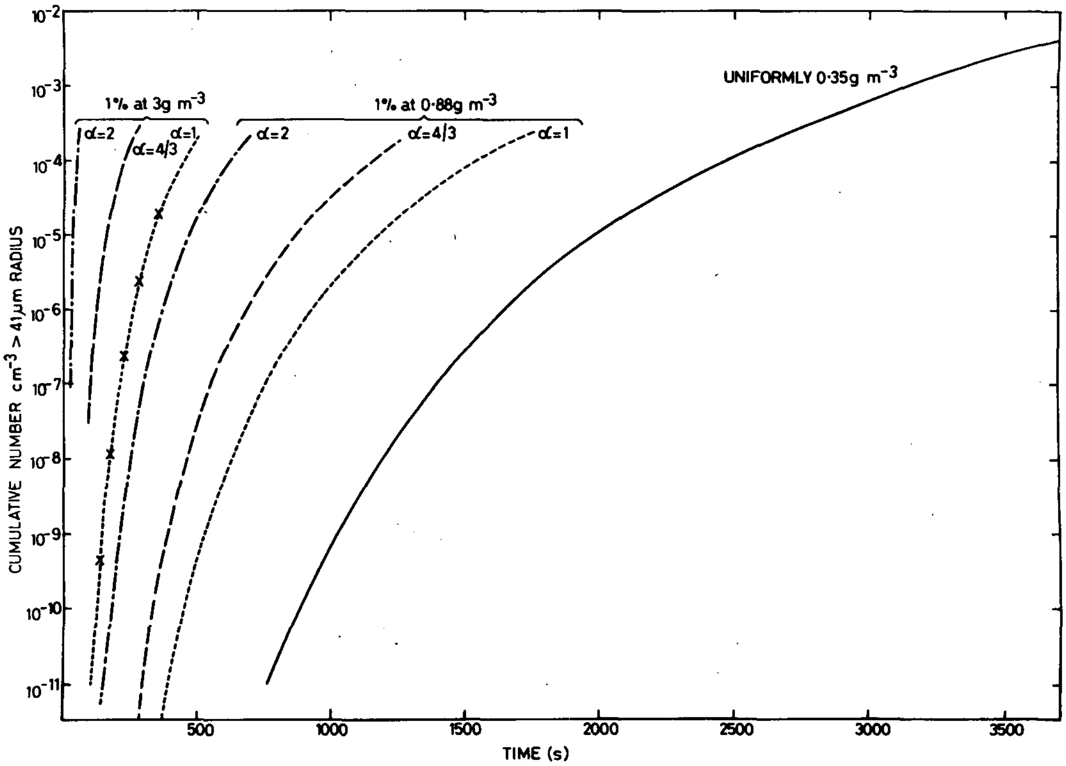

FIG. 2. The effect of liquid water fluctuations on the evolution of a distribution similar to that of Fig. 1 but containing 70 droplets cm^{-3} initially and an average liquid water content of 0.35 g m^{-3}.

of drops >41 μm for $\alpha=1$ (dotted curve), $\alpha=\frac{4}{3}$ (broken curve) and $\alpha=2$ (dash-dotted curve). These curves were obtained from the solid curve $n(t)$ (which relates to a constant liquid water content of 0.35 g m^{-3}) simply by calculating

$$0.99n\left(\frac{0.33}{0.35}t\right)+0.01n\left(\frac{3.0}{0.35}t\right) \quad \text{for} \quad \text{Case I } (\alpha=1)$$

and

$$0.99n\left(\left|\frac{0.33}{0.35}\right|^{\alpha}t\right)+0.01n\left(\left|\frac{3.0}{0.35}\right|^{\alpha}t\right) \quad \text{for} \quad \text{Case II.}$$

The numbers are not important—the important point is that the average behavior is dominated by the regions of high liquid water content.

Fig. 3 was provided by Warner (1975, private communication) from measured data for 400 samples each of volume 10 cm^{-3} taken in cumulus clouds in 1967 in maritime air near the coast of northern New South Wales. It shows the distribution of the ratio of measured liquid water content on individual samples to average value at the same altitude. Evidently the distribution is approximately log-normal and indicates (at levels of 1% and less) that liquid water contents two or three times the average occur on 0.5% of occasions; a possible, although hardly justifiable, extrapolation suggests five or so times the average liquid water content could occur in 0.1% of the cloud volume. If Warner's data are used to produce a possible $p(w)$ one finds the integral to be dominated by the behavior of $p(w)$ at very small values of p; indeed the magnitude of the integral is grossly different depending on whether the observed distribution is truncated at the largest observed w/w_{mean} or extrapolated, and different extrapolations give quite different results.

To illustrate the effect of a distribution such as that found by Warner (1975, private communication), we note that values of 2.5 or more times the average occurred in about 0.8% of the slides; the effects of including 0.8% of regions with $w=0.88$ g m^{-3} are shown in Fig. 2. It is evident that again the average number of large drops during the first few minutes is several decades higher than would be produced with a uniform liquid water content. The differences are greatest at low concentrations.

4. Conclusions

Since the average liquid water content in clouds is consistently well below the adiabatic value, regions of higher than average liquid water content probably exist because of the vagaries of the mixing processes. These can have an effect on the overall production of large drops which is quite out of proportion to the

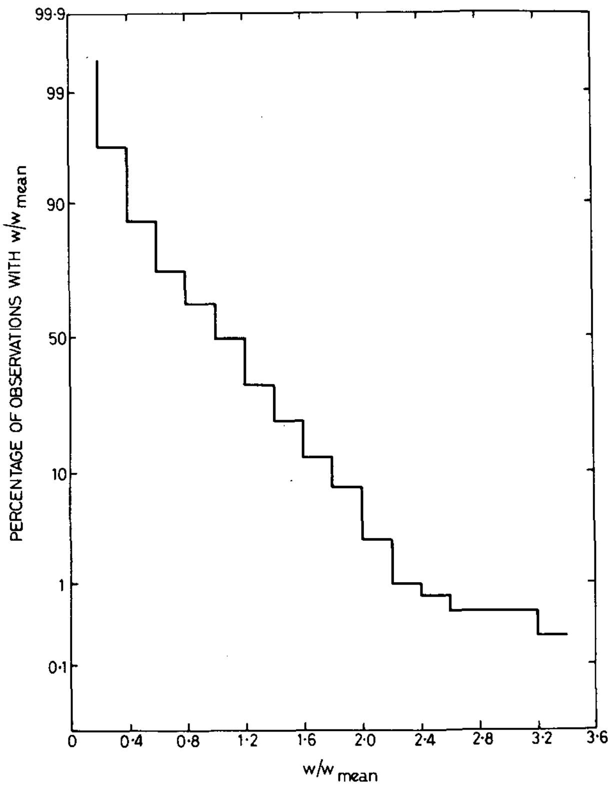

Fig. 3. Data from Warner (1975, private communication) for the frequency of occurrence of various values of liquid water content. The measurements relate to a sample volume of 10 cm^3. (w_{mean} is the average liquid water content at a given height.)

fraction of cloud volume occupied by them. If a few regions exist which contain (say) 3 g m^{-3} liquid water and if they maintain their integrity for a few minutes, calculations show that in those few minutes those regions produce large drops so quickly that even if they comprise only 1% of the cloud they can give rise to significant (say 10^{-6} cm^{-3}) concentrations of "large" drops when averaged over the entire cloud volume, whereas in the absence of regions of high liquid water content (but with the same average liquid water) "large" drop concentrations are negligible (10^{-15} cm^{-3} or less).

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