Forcing of Stratospheric Kelvin Waves by Tropospheric Heat Sources

CHIH-PEI CHANG

Department of Meteorology, Naval Postgraduate School, Monterey, Calif. 93940

(Manuscript received 19 November 1975, in revised form 22 January 1976)

ABSTRACT

The problem of scale-selection of Kelvin waves in the stratosphere by forcing from tropospheric heating is analyzed using a simple linear model. The effect of vertical wind shear is excluded because the phase speed of the waves is fast relative to the range of the mean zonal wind in the vicinity of the tropopause at which level the upward energy flux due to forcing is evaluated. Results of this analysis modify Holton's (1973) theory in that 1) the forcing is most efficient for the longest zonal wavelength even if the heat sources are distributed randomly, and 2) the most favored vertical wavelength of the excited waves is about twice the vertical scale of heating. The calculated vertical wavelengths exceed slightly those observed and the discrepancies are discussed.

1. Introduction

There is ample evidence on the existence of atmospheric Kelvin waves in the equatorial stratosphere (Wallace and Kousky, 1968; and others). These waves propagate eastward and downward with a zonal wavenumber of 1–2, a periodicity of 10–15 days, and a vertical wavelength of 8–12 km. Their zonal wind component has a typical amplitude of 10 m s^{-1} at the equator while no appreciable fluctuation of the meridional wind component is observed. The zonal wind and pressure perturbations are both symmetric with respect to the equator and are close to geostrophic balance in the meridional direction, but the waves behave like internal gravity waves in the zonal and vertical directions. More detailed observational and theoretical descriptions of these waves are given in Wallace (1973) and Holton (1975). These waves are believed to play a very important role in the quasi-biennial oscillation of the equatorial stratosphere by supplying westerly momentum to the zonal mean flow (Lindzen and Holton, 1968; Holton and Lindzen, 1972).

The vertical structure of the Kelvin waves indicates that they are maintained by energy propagating from the troposphere. In addition, the possibility that they are forced by interaction with mid-latitude motions, as suggested by Mak (1969), is quite small because of the small meridional wind perturbations which rule out the importance of lateral energy fluxes. Dynamic instability also appears unlikely due to the very long zonal scale and the fast westerly phase speed of the waves.

Due to the presence of the large amount of deep cumulus convection in the tropical troposphere, it has been suggested that a plausible energy source for these waves is the latent heat released by cumulus towers. There are two possible mechanisms by which the latent heating in the troposphere may maintain the Kelvin waves:

1) A two-way interaction such that the cumulus convection, which supplies energy to the waves, is itself controlled by the large-scale wave motion field. This scale interaction results in an unstable situation which is generally called the conditional instability of the second kind (CISK).

2) The waves are forced by cumulus heating which is controlled by processes unrelated to the waves. Hayashi (1970) and Lindzen (1974) have investigated the first mechanism and found that the growth rates of CISK are unsatisfactory in explaining the Kelvin waves. Holton (1972, 1973) studied the second mechanism with a numerical diagnostic model in which the diabatic heating due to cumulus convection is specified. Damping in the form of Rayleigh friction and Newtonian cooling are also included. He found that the wave response in the stratosphere to a tropospheric heat source resembling the Kelvin wave mode has a structure in close agreement with observations. However, the observed periodicity of the Kelvin waves must be specified although no corresponding peak in the cloud brightness spectra can be found. On the other hand, his numerical calculations suggest that the vertical scale of the excited waves is close to that of the specified forcing. In equatorial wave theory (Holton and Lindzen, 1968) the vertical scale determines the wavenumber-frequency relationship. By postulating a red-noise spectral distribution of the tropospheric heat sources, Holton suggested that the

vertical scale and the large low-frequency variance of the forcing result in a band-pass selectivity of the Kelvin waves.

Holton's theory may indeed be valid, especially because an oscillation of the monsoon heating in South Asia, with a quasi-periodicity of about 15 days, has been observed during the northern summer, although it is not clear whether this periodicity exists in other seasons. The purpose of this paper is to point out that the atmosphere acts like a band-pass filter even if the heat sources of the troposphere are distributed randomly in the frequency domain. The relationship between the vertical scale of the response and that of heating will also be examined.

2. The forced model

The zonal mean wind in the tropical stratosphere has considerable vertical shear which may influence the upward propagation of wave energy (Lindzen, 1971). However, due to the fast phase speed of the Kelvin waves the shear is probably not very significant at the tropopause level for which our discussion will be most relevant. The zonal mean wind will therefore be neglected in our simple analysis. The effect of damping will also be excluded. The linearized zonal momentum, meridional momentum, hydrostatic, thermodynamic energy and continuity equations on an equatorial beta-plane, with a single zonal wavenumber k, a Doppler-shifted phase speed c, and no perturbation in the meridional velocity may then be written

$$-ikcu=-ik\phi, \tag{1}$$

$$\beta yu=-\frac{\partial\phi}{\partial y}, \tag{2}$$

$$\frac{\partial\phi}{\partial z}=\frac{RT}{H}, \tag{3}$$

$$-ikcT+w\Gamma=\frac{Q}{c_p}, \tag{4}$$

$$iku+e^{z/H}\frac{\partial}{\partial z}(e^{-z/H}w)=0, \tag{5}$$

where u, w, T, ϕ and Q are the perturbation zonal velocity, vertical velocity, temperature, geopotential and the diabatic heating rate, respectively; H is a constant scale height, Γ the static stability, c_p the specific heat at constant pressure, R the gas constant, β the meridional gradient of the vertical component of earth's vorticity, y the meridional coordinate, and $z=-H\ln(p/p_0)$ is the vertical coordinate with p the pressure and p_0 a reference pressure.

Eqs. (1)–(2) give the meridional structure of the Kelvin waves:

$$u,\phi\propto\exp\left(-\frac{\beta y^2}{2c}\right), \tag{6}$$

where the condition $c>0$ is required to satisfy the trapping condition on an equatorial β-plane. If a height-dependence factor, $\exp(z/2H)$, is separated from the perturbation quantities, Eqs. (1) and (3)–(5) may be combined to form a single equation in w:

$$\frac{\partial^2 w'}{\partial z^2}+\lambda^2 w'=\frac{Q'}{c^2}, \tag{7}$$

where

$$\left.\begin{aligned}\lambda^2&=\frac{S}{c^2}-\frac{1}{4H^2}\\ w'&=we^{-z/(2H)}\\ Q'&=\frac{R}{c_pH}Qe^{-z/(2H)}\\ S&=\frac{R}{H}\Gamma\end{aligned}\right\}. \tag{8}$$

The parameter λ is a measure of the vertical wavenumber. If the heating function Q' is given, (7) can be solved with suitable boundary conditions. In our problem the following boundary conditions are used:

$$w=\begin{cases}0, & \text{at}\quad z=0 \qquad (9\text{a})\\ C_1e^{i\lambda z}+C_2e^{-i\lambda z}, & \text{at}\quad z=z_t \qquad (9\text{b})\end{cases}$$

where

$$C_1=rC_2.$$

Here z_t is the height of tropopause and the condition (9b) results from the requirement that latent heating vanishes at z_t. The parameter r is a reflection coefficient which, in the absence of vertical wind shear, is given by the model specification of the static stability distribution. The solution to (7) at $z=z_t$ may now be written as

$$w(z_t)=-\frac{re^{i\lambda z_t}+e^{-i\lambda z_t}}{\lambda(1+r)}\int_{z_c}^{z_t}\sin(\lambda z)\frac{Q'}{c^2}dz, \tag{10}$$

where z_c is the height of the cloud base.

3. The energy flux at tropopause

We will consider the vertical energy flux $\overline{\phi' w'}$, where the overbar denotes the zonal average and $\phi'=\phi\exp(-z/2H)$, at the tropopause level (z_t), to be a measure of the efficiency of forcing. Despite the actual existence of the vertical shear of the zonal mean flow in the stratosphere, the response of the waves should be largely determined by this quantity.

At z_t the solution is given by (9b) and only the C_2 term needs to be considered for upward energy propagation or downward phase propagation.

Eqs. (1) and (5) may now be rewritten as

$$-ikcu' = -ik\phi',$$

$$iku' - \left(i\lambda + \frac{1}{2H}\right)w' = 0,$$

where $u' = u\exp(-z/2H)$. From these relationships it follows immediately that

$$\phi' = cu' = \frac{c}{k}\left(\lambda - \frac{i}{2H}\right)w'.$$

Since $H \approx 7$ km for the atmosphere, $\lambda \gg 1/(2H)$ if the vertical wavelength $\ll 88$ km. This is the case for all practical purposes so we may write

$$\phi' \approx \frac{c\lambda}{k}w'.$$

From the definition of λ given in Eq. (8) we may also write

$$c\lambda \approx S^{\frac{1}{2}},$$

which leads to

$$\phi' \approx \frac{S^{\frac{1}{2}}}{k}w',$$

and

$$\overline{\phi' w'} \approx \frac{S^{\frac{1}{2}}}{k}\overline{w'^2}. \tag{11}$$

Eq. (11) indicates that, if the amplitude of the vertical velocity perturbation at the tropopause level is fixed, the upward energy flux will be inversely proportional to the zonal wavenumber. Examination of (7) reveals that w' depends only on λ (c is uniquely determined by λ) as long as Q is independent of k. Thus for given vertical scale the forcing at the tropopause level becomes maximum for the longest horizontal wavelength. This result obviously remains the same if one considers the upward momentum flux $\overline{u'w'}$, because $u' = \phi'/c$.

In order to examine the selection of vertical scales, we assume a heating function

$$Q' = \begin{cases} mSq(y)\sin\pi\left(\dfrac{z - z_c}{\Delta z}\right), & z_t \geqslant z \geqslant z_c \\ 0, & z > z_t \text{ or } z < z_c \end{cases} \tag{12}$$

where m specifies the strength of the heating and $\Delta z = z_t - z_c$ is a measure of the vertical scale of the heating. The function $q(y)$ which specifies the meridional distribution of heating will be assumed to take the form of (6) for the time being. The vertical profile as given by (12) resembles closely that of the typical observations (Chang, 1976). For the simplest case, we shall first assume that the static stability parameter S has a constant tropospheric value

$$S = S_t$$

throughout the atmosphere. In this case $r = 0$ and only upward energy propagation at $z = z_t$ is allowed. The solution (10) at z_t is now found to be

$$w' = \hat{w}\exp[-i\lambda z_t - \beta y^2/(2c)], \quad z = z_t, \tag{13}$$

where

$$\hat{w} = -\frac{m\pi\Delta z}{\lambda}\frac{\lambda^2 + 1/(4H)^2}{\pi^2 - \lambda^2\Delta z^2}\sin\left[\frac{\lambda}{2}(z_t + z_c)\right]\cos\left(\frac{\lambda\Delta z}{2}\right) \tag{14}$$

denotes the amplitude of the solution in this simple model. We note here that no actual singularity exists in (14). The vertical energy flux at the equator and $z = z_t$ is thus

$$\overline{\phi' w'} = \frac{S^{\frac{1}{2}}}{2k}\overline{\hat{w}^2}.$$

From (14) if one assumes $\lambda \gg 1/(4H^2)$, it can be shown that this energy flux reaches maximum at

$$\lambda = \frac{\pi}{\Delta z},$$

where

$$\overline{\hat{w}^2} \approx \frac{m^2\pi^2}{16}\sin^2\left[\frac{\pi}{2\Delta z}(z_t + z_c)\right].$$

The tropospheric forcing is therefore most efficient at a vertical wavelength

$$L_z \equiv \frac{2\pi}{\lambda} = 2\Delta z,$$

or twice the vertical scale of the heating. This result is somewhat similar to those of Green (1965) and Lindzen (1966). A plot of the energy flux profile at z_t is given in Fig. 1 in which it is assumed that $z_t = 15\,z_c$ and $H = 7z_c$. If we measure the vertical scale of Holton's (1972, 1973) heating function the value Δz can be found to be $\sim$11 km, which predicts a maximum response of 22 km vertical wavelength in the troposphere. Since in reality the stratospheric static stability is about three times that of the troposphere, using (8) we may estimate that the stratospheric vertical wavelength corresponding to a 22 km tropospheric value would be

$$L_s \approx L_z\left(\frac{S_s}{S_t}\right)^{-\frac{1}{2}} = 22 \times 3^{-\frac{1}{2}} \approx 12.8 \text{ km},$$

where the subscript s denotes stratosphere. If we assume $\Delta z \approx 14$ km, then $L_s \approx 16.3$ km.

It may be of some interest to consider the effect of different static stability between stratosphere and troposphere on the solution (13), because the "efficiency" of reflection of wave energy at $z=z_t$ may be scale-dependent. In this case we have

$$S=S_s, \quad \text{for } z>z_t,$$
$$S=S_t, \quad \text{for } z\leqslant z_t.$$

Another boundary condition is now needed and the simplest form available is the radiation condition above z_t, i.e.,

$$w'=A\exp[-i\lambda_s z-\beta y^2/(2c)], \quad \text{for } z>z_t, \qquad (15)$$

where λ_s is the vertical wavenumber in the stratosphere. The continuity conditions of w' and ϕ' across z_t lead to

$$r=\frac{1-(S_s/S_t)^{\frac{1}{2}}}{1+(S_s/S_t)^{\frac{1}{2}}}\exp(-2i\lambda_t z_t), \quad \text{for} \quad S_s>S_t,$$

where λ_t is the vertical wavenumber in the troposphere. The solution (10) at z_t in the form of (15) is

$$A=\frac{2\lambda_t\exp[i(\lambda_s-\lambda_t)z_t]\hat{w}}{(\lambda_s+\lambda_t)-(\lambda_s-\lambda_t)\exp(-2i\lambda_t z_t)},$$

where $\hat{w}$ is given by (14) with all λ replaced by λ_t. The vertical energy flux at the equator and z_t is now

$$\overline{\phi'w'}\approx\frac{S^{\frac{1}{2}}}{2k}AA^*\approx\frac{S^{\frac{1}{2}}}{2k}E\hat{w}^2,$$

where

$$E=2[(1+S_s/S_t)+(1-S_s/S_t)\cos 2\lambda_t z_t]^{-1}$$

is the effect of different static stabilities above and below z_t and the asterisk denotes complex conjugate. For $S_s=3S_t$, E varies between a maximum of 1 when $\lambda_t=0,\ \pi/z_t,\ 2\pi/z_t,\ \ldots$, and a minimum of 0.33 when $\lambda_t=\pi/(2z_t),\ 3\pi/(2z_t),\ 5\pi/(2z_t),\ \ldots$. Assuming $z_t=15$ km and $\Delta z=14$ km, the variation of E is plotted in the upper section of Fig. 1 and the modified energy flux profile is indicated by the dashed curve in the lower section. It is seen that the most efficient forcing is slightly larger than twice the scale of heating. Hence we may conclude that the variation in static stability between troposphere and stratosphere does not cause great effect in the vertical scale of the most favored response.

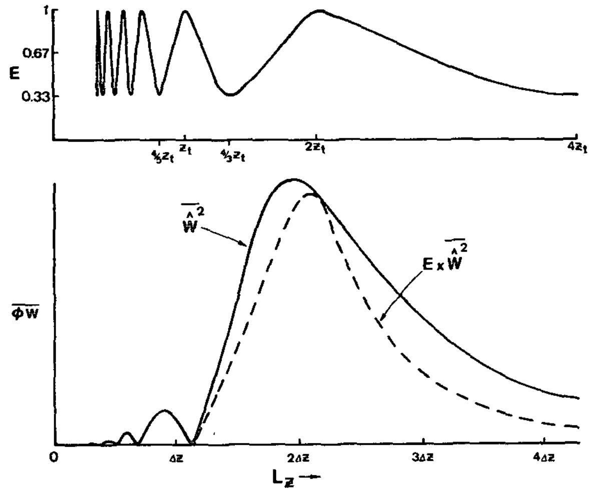

FIG. 1. Profile of vertical wave energy flux $\overline{\phi'w'}$ as a function of vertical wavelength L_z. The solid curve in the lower diagram is for the case $S_s=S_t$ and the dashed curve for the case $S_s=3S_t$. The upper diagram is the effect on E of the difference in static stability on the vertical wave energy flux profile for the case $S_s=3S_t$.

4. Concluding remarks

We have shown that for randomly distributed tropospheric heat sources the Kelvin waves of the largest zonal scale are excited most efficiently in the absence of damping. This may explain why most of the Kelvin waves observed in the lower stratosphere are of zonal wavenumber 1 and some are of wavenumber 2. The vertical wind shear has been excluded from our analysis but it should not significantly alter this conclusion.

The finding that the most favored vertical wavelength is about twice the vertical scale of heating gives a reasonable estimate for Kelvin waves in the lower stratosphere, although the calculated wavelength of 13–16 km is slightly larger than those observed. Several influences which are excluded in our model may alter this aspect. For example, Lindzen (1971) found that the focusing effect of the vertical wind shear in the stratosphere would reduce the vertical scale of the waves. A two-scale expansion technique can be similarly applied to our forced model and the resultant prevailing vertical wavelength in the stratosphere should be reduced.

Another effect which has been excluded is the actual north-south distribution of the heating function. The meridional structure equation (6) indicates that the meridional scale of the waves is coupled with the vertical scale through the phase speed c. The variation of the meridional scale with the vertical scale may be considered slow because the e-folding width of the Gaussian distribution is proportional only to the square root of c, which is approximately proportional to the vertical wavelength. In the real atmosphere the meridional heating profile $q(y)$ in (12) does not adjust to different vertical scales as we have assumed in our calculations. A proper treatment of this problem is, perhaps, to multiply the previously calculated response by the projection

$$P=\frac{\int_0^\infty q(y)\exp[-\beta y^2/(2c)]dy}{\int_0^\infty \exp[-\beta y^2/(2c)]^2 dy}. \quad (16)$$

Here we could specify $q(y)$ as we did for the vertical heating profile, although there appears to be more observational uncertainty. For the purpose of the present qualitative discussion, however, this is not strictly necessary. We know that the latent heating is largely confined to the tropics, so that the meridional extent of $q(y)$ must be quite limited. Therefore, the effect of (16) should be such that waves whose vertical scales are larger than that corresponding to the meridional scale of $q(y)$ will suffer from having a large portion of their north-south domain unsupported by heating. If we assume that the meridional scale of $q(y)$ can be represented by an e-folding width $y_d \approx 1500$ km of a Gaussian distribution about the equator, and that $S_t = 1.22\times10^{-4}$ s^{-2} which corresponds to $\Gamma = 3$ K km^{-1}, then those waves with a tropospheric vertical wavelength

$$L_z > \pi\beta S_t^{-\frac{1}{2}} y_d^2 \approx 14.2 \text{ km},$$

or a stratospheric wavelength

$$L_s > 8.2 \text{ km},$$

will have their amplitude reduced by (16). The reduction will be more severe for longer vertical wavelengths. This obviously would also shorten the previously calculated most favored vertical scales of the Kelvin waves.

Acknowledgments. The author wishes to thank Profs. J. R. Holton and R. S. Lindzen for discussion and Profs. R. T. Williams and G. J. Haltiner for reading the manuscript. This research was supported by the Atmospheric Sciences Section, National Science Foundation, under Grant DES75-10719. Parts of the material in this paper were presented at the AMS Ninth Conference on Hurricanes and Tropical Meteorology, May 1975, Miami, Fla.

REFERENCES

Chang, C.-P., 1976: Vertical structure of tropical waves maintained by internally induced cumulus heating. *J. Atmos. Sci.*, **33**, 729–739.

Green, J. S. A., 1965: Atmospheric tidal oscillations: An analysis of the mechanics. *Proc. Roy. Soc. London*, **A288**, 564–574

Hayashi, Y., 1970: A theory of large-scale equatorial waves generated by condensation heat and accelerating the zonal wind. *J. Meteor. Soc. Japan*, **48**, 140–160.

Holton, J. R., 1972: Waves in the equatorial stratosphere generated by tropospheric heat sources. *J. Atmos. Sci.*, **29**, 368–375.

——, 1973: On the frequency distribution of atmospheric Kelvin waves. *J. Atmos. Sci.*, **30**, 499–501.

——, 1975: The dynamic meteorology of the stratosphere and mesosphere. *Meteor. Monogr.*, **15**, No. 37, 240 pp.

——, and R. S. Lindzen, 1968: A note on Kelvin waves in the atmosphere. *Mon. Wea. Rev.*, **95**, 385–386.

——, and ——, 1972: An updated theory for the quasi-biennial cycle of the tropical stratosphere. *J. Atmos. Sci.*, **29**, 1076–1080.

Lindzen, R. S., 1966: On the relation of wave behavior to source strength and distribution in a propagating medium. *J. Atmos. Sci.*, **23**, 630–632.

——, 1971: Equatorial planetary waves in shear: Part I. *J. Atmos. Sci.*, **28**, 609–622.

——, 1974: Wave-CISK in the tropics. *J. Atmos. Sci.*, **31**, 156–179.

——, and J. R. Holton, 1968: A theory of the quasi-biennial oscillation. *J. Atmos. Sci.*, **25**, 1095–1107.

Mak, M.-K., 1969: Lateral driven stochastic motions in the tropics. *J. Atmos. Sci.*, **26**, 41–64.

Wallace, J. M., 1973: General circulation of the tropical lower troposphere. *Rev. Geophys. Space Phys.*, **11**, 191–222.

——, and V. E. Kousky, 1968: Observational evidence of Kelvin waves in the tropical stratosphere. *J. Atmos. Sci.*, **25**, 900–907.