NOTES AND CORRESPONDENCE

Fourier-Transform Ambiguity in Turbulence Dynamics

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ABSTRACT

Since eddies in turbulent flows are a dense collection of very short wave packets, their position and wavenumber (or lifetime and frequency) cannot be determined without ambiguity. The spectral width of eddies is about an octave, but the effects of resolution limitations on the formulation of dynamically consistent turbulence models are largely unknown. Fourier transforms are inefficient at high wavenumbers; more efficient decomposition schemes sacrifice the capability to describe the microstructure of turbulent flows.

1. Introduction

Kraichnan's study of eddy viscosity in two and three dimensions [Kraichnan (1976) this issue] uses the test-field model, a descendant of the direct-interaction approximation, to analyze spectral energy transfer in two- and three-dimensional turbulence. The theory is formulated in terms of truncated interactions among Fourier modes in wavenumber space. Back-transformed to physical space, it incorporates certain nonlocal diffusion effects. The direct-interaction approximation is invariant to Fourier transformation of the velocity field, but in the present context, which emphasizes issues related to spectral energy transfer, it is convenient to discuss it primarily in terms of Fourier coefficients of the velocity field.

Fourier transforms are integral transforms; the excitation at each wavenumber is the spatial average of the contributions of all eddies that have appreciable energy at that wavenumber. Fourier coefficients are local in wavenumber space, but very nonlocal in physical space. Conversely, the equations of motion used in fluid mechanics are usually written in such a way that they appear to be perfectly local in physical space, but infinitely nonlocal in wavenumber space. By its very nature, the Fourier-transform formalism is symmetric.

A Fourier coefficient cannot reveal at what locations the eddies contributing to its amplitude are found. Spatial reconstruction requires information on the amplitude and phase of many Fourier coefficients. It is obvious that Fourier modes do not have a one-to-one correspondence with eddies, but it is not always clear how that may influence the selection of dynamically consistent approximations in wavenumber space. Conversely, the velocity vector at a given point in physical space is ignorant of the wavenumber structure of the eddies contributing to the velocity at that point, and spatially local approximations of the interactions among eddies may be just as inappropriate as spectrally localized ones.

2. The ambiguity of wave packets

How do we decide on the spatial and spectral degrees of localness needed for a sensible and enlightening description of turbulence dynamics? In what way do eddies differ from Fourier coefficients and what is the spatial extent of their interaction domain? In an attempt to focus on this problem, I propose to define eddies as very short, nonpropagating wave packets.[1] Because the central wavenumber $\mathbf{k}$ of an eddy and its position $\mathbf{x}$ cannot be determined simultaneously with arbitrary accuracy, we have

$$\Delta\mathbf{x}\cdot\Delta\mathbf{k}\sim 2\pi. \tag{1}$$

This expression is an adaptation of the first step in Bohr's explanation of Heisenberg's uncertainty principle [a convenient reference is Pauli (1958)]. Since we are not concerned with quantum mechanics but with Fourier transforms in turbulent flows, I propose to call (1) the Fourier-transform ambiguity principle. Expressions similar to (1) occur in the theory of Doppler ambiguity; one example is the transit-time ambiguity in laser-Doppler velocity measurements (George and Lumley, 1973). Eq. (1) states that the spatial fuzziness of an eddy is complementary to its spectral width. One cannot have good spatial resolution

[1] According to J. L. Lumley, this idea was suggested first by R. W. Stewart.

and good spectral resolution simultaneously. This concept is well-known in the analysis of random data (Bendat and Piersol, 1971); the point here is that it also applies to individual eddies (Woods, 1974, 1975; Tennekes, 1976).

Fourier-transform ambiguity is symmetric in the two transform spaces involved. It suggests that the dynamics of eddy ensembles is neither local in wavenumber space nor in physical space and that pattern recognition of eddy structures is an exceedingly difficult task (we may *never* be able to take a close look at the complicated, intermittent microstructure of turbulence). On the other hand, it should be kept in mind that (1) deals with resolution problems, not with the dynamics of eddies. Without dynamical behavior and dynamical interactions based firmly on the Navier-Stokes equations, the wave packets postulated here have no structure or character.

If an eddy is a short wave packet, its spatial extent Δx is comparable to its central wavelength (a sketch is given in Tennekes and Lumley, 1972, p. 259). Therefore, $\Delta x \sim 2\pi/k$, so that (1) becomes

$$\Delta k/k \sim 1. \tag{2}$$

The spectral width of eddies thus is about an octave. This idea is used, explicitly or otherwise, in most of the spectral theory of turbulence. The turbulent energy cascade is commonly perceived to proceed in octave steps. If the energy spectrum is $E(k)$, the kinetic energy of eddies centered at wavenumber k is written as $kE(k)$; this can be done only if $\Delta k/k$ does not exhibit pathological behavior. In the same way, the enstrophy (vorticity variance) of eddies centered at k is written as $k^3E(k)$; if, as in two-dimensional or geostrophic turbulence, this happens to be a conservative quantity, one finds the well-known k^{-3} inertial range.

Relations similar to (1) and (2) hold for the time-frequency structure of eddies. If Δt is the lifetime of an eddy and ω its angular frequency measured in a Lagrangian frame of reference, then

$$\Delta t \Delta\omega \sim 2\pi, \quad \Delta\omega/\omega \sim 1. \tag{3}$$

This states that the lifetime of an eddy is proportional to the reciprocal of its vorticity and that the characteristic interaction time at wavenumber k may be estimated as $[k^3E(k)]^{-\frac{1}{2}}$. In a three-dimensional inertial range this gives the Onsager time $\tau(k) \sim \epsilon^{-\frac{1}{3}}k^{-\frac{2}{3}}$; in a two-dimensional enstrophy-conserving range this becomes $\tau(k) \sim X^{-\frac{1}{3}}$ (ϵ and X are the spectral fluxes of kinetic energy and of enstrophy, respectively). The interaction time in a k^{-3} range is independent of wavenumber; this is the principal reason why the interactions among Fourier modes in a k^{-3} range are quite nonlocal in wavenumber space.

3. Fourier transforms

Spectral cascades proceed in octave steps, but the spectral spacing of Fourier coefficients is determined by the size of the integration domain. If the wavenumber of the first Fourier mode is k_0, then successive Fourier coefficients are separated by wavenumber intervals equal to k_0. Since k_0 is small we have $k_0/k \to 0$ as $k \to \infty$ and, because of (2), $k_0/\Delta k \to 0$ as $k \to \infty$. Large eddies thus are represented by just a few Fourier modes, while small eddies are represented by many modes. This is a major drawback of Fourier transforms: most of the modes are wasted on eddies that contain very little kinetic energy, while the spectral resolution of eddies containing most of the kinetic energy is relatively poor. Fourier transforms are seen to be inefficient at the high-wavenumber end. More efficient decomposition methods have been proposed by Lorenz (1959, 1972) and Lumley (1967, 1970). Those methods are attempts to deal in an optimal fashion with the resolution issue discussed here; they do not make the task of formulating a dynamically consistent system of equations any easier. Efficient methods, such as the proper orthogonal decomposition, have to use a very crude parameterization of the intricate microstructure of turbulence. This is probably adequate for studies of the large-scale structure of turbulent flows, but not for the analysis of the many details of spectral energy transfer at small scales.

In an ensemble of small eddies, many Fourier coefficients live under the same roof. The interactions among modes within the same octave centered at some high wavenumber are thus likely to be different from those among more distant modes. The behavior of wavenumber triads has to be watched closely if the triangles involved are very elongated. If two large, neighboring wavenumbers interact with a small one, then the two large ones are not distinct as far as the interaction with the small one is concerned, because both belong to the same wave packet (Monin and Yaglom 1975, p. 308). The effect of the interaction on the small wavenumber then can be represented in terms of an eddy viscosity (Kraichnan, 1976). However, the small wavenumber affects the energy transfer between the two high-wavenumber neighbors; this leads to the coherent straining mechanism that generates the cusps in the figures presented by Kraichnan.

4. Spectral energy transfer

In this context it is worthwhile to illustrate, with the aid of a spectral eddy-viscosity model, how the original version of the direct-interaction approximation (Kraichnan, 1959) fails to reproduce the Kolmogorov spectrum. We write the energy transfer across wavenumber k, which is taken to lie within the inertial subrange, as

$$\epsilon = \nu(k)S^2(k), \tag{4}$$

where $\nu(k)$ is the eddy viscosity at k and $S(k)$ the representative strain rate. This is a spectrally local estimate, and we ignore the fact that the typical jump is about an octave in wavenumber because that does not affect the order-of-magnitude estimates that follow. Eq. (4) is a localized version of Heisenberg's (1948) spectral transfer hypothesis. Now the eddy viscosity at k is proportional to the kinetic energy at k [estimated as $kE(k)$] and to the time scale (decorrelation time) at k. Since $S^2(k) \sim k^3 E$, we can write

$$\epsilon \sim kE(k) \cdot \tau(k) \cdot k^3 E(k). \tag{5}$$

In the original version of the direct-interaction approximation (which is not invariant under Galilean transformations) the decorrelation time $\tau(k)$ is proportional to the time it takes to advect an eddy of wavenumber k past a fixed point. This time is of order $(qk)^{-1}$, where q is the characteristic velocity of the most energetic eddies. Substituting this estimate into (5), we obtain (Tennekes, 1976)

$$\epsilon \sim k^4 E^2(k) \cdot (qk)^{-1},$$

whence (Kraichnan, 1959)

$$E \sim \epsilon^{\frac{1}{2}} q^{\frac{1}{2}} k^{-\frac{3}{2}}. \tag{6}$$

The problem here is that the advection of Fourier modes at large wavenumbers by Fourier modes at small wavenumbers is permitted to dictate the rate at which phase coherence is destroyed. These "adiabatic interactions" (Kadomtsev, 1965) contribute mainly to *spatial* energy transfer, not to *spectral* transfer. We emphasize that this problem is caused by the truncations inherent in the original version of the direct-interaction approximation, not by the Fourier-transform formalism.

In subsequent versions of the direct-interaction approximation, the truncation scheme has been modified in such a way that adiabatic interactions, while still involved in spatial mixing, do not affect the spectral energy flux. When that is done, the decorrelation time becomes proportional to the Onsager time scale $\tau(k) \sim \epsilon^{-\frac{1}{3}} k^{-\frac{2}{3}}$, and Kolmogorov's form of the spectrum in the inertial subrange is retrieved. We note in passing that adiabatic interactions play a dominant role in the theory of the Eulerian frequency spectrum of turbulence (Tennekes, 1975).

5. The effects of coherent straining

In his 1976 paper Kraichnan uses the test-field model, which does not suffer from spurious advection effects. However, the cusps in his figures are caused by interactions in triads that involve coherent straining of a pair of large-wavenumber neighbors by a small wavenumber. These cusps are much narrower than an octave and one is forced to wonder if the implied spectral resolution is an artifact of the test-field model. In triads with $k = k_m - k$ small compared to k_m, the time scale associated with the coherent straining mechanism is

$$\tau(\Delta k) \sim \epsilon^{-\frac{1}{3}} (\Delta k)^{-\frac{2}{3}} \sim \epsilon^{-\frac{1}{3}} (k_m - k)^{-\frac{2}{3}}. \tag{7}$$

Presumably not by coincidence, this behavior is similar to the singularity in Kraichnan's "input contribution" to the eddy viscosity in the limit as $\Delta k \rightarrow 0$. That singularity, however, may not be consistent with the effect octave bandwidth of eddy ensembles. From a different perspective, the question is: Do eddies at wavenumbers near k_m retain their phase coherence as long as suggested by (7)? The straining probably tends to increase their effective lifetime, but would that lead to singular behavior? Is it not possible that a small eddy can escape coherent straining for extended periods by allowing some dynamical instability to destroy phase coherence? This issue does not arise in the analysis of coherent straining of small blobs of passive contaminant (Kraichnan 1976); it is nevertheless characteristic of our ignorance of the dynamical intricacy of turbulent flows.

Kraichnan (personal communication) states that the cusps in his figures are caused by the sharp truncation of wavenumber space used to determine the eddy viscosity. They are artifacts of this particular truncation, not of the dynamical approximations embodied in the test-field model. Nevertheless, it is not clear to me how resolution limitations affect the formulation of a dynamically consistent and reasonably efficient solution of the closure problem in turbulence theory.

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Trapeze Instability Modified by a Mean Shear Flow

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ABSTRACT

The influence of weak mean vertical wind shear upon the trapeze instability of Orlanski (1973) is investigated. It is found that the shear limits the growth of unstable waves unless they are propagating at nearly right angles to the mean wind vector, or in other words, the equi-phase lines are parallel to the mean wind direction.

1. Introduction

Orlanski (1973) suggested that the diurnal variations of the Brunt–Väisälä frequency N may parametrically excite internal gravity waves. In mid-latitudes the low-frequency cutoff for these unstable waves is the inertial frequency. Since there are many other mechanisms which excite internal gravity waves in the same frequency band, positive identification of the unstable waves discussed by Orlanski will be difficult. However, in equatorial regions the low-frequency cutoff is half that of the diurnal frequency. It is felt that the trapeze instability will be manifested most clearly in the equatorial region because of the absence of other clear-cut physical mechanisms which produce waves with 2-day periods. Therefore, in this note attention will be confined to the equatorial region. The equatorial region is here defined as the area which lies within 1000 km of the equator. The averaged value of the Coriolis parameter f (over this region) is 1.21×10^{-5} s^{-1}. The effects of the earth's rotation are negligible when $(f^2/N^2)(\lambda_H{}^2/\lambda_V{}^2)\ll1$. The Brunt–Väisälä frequency N is typically 0.01 s^{-1}. For internal gravity waves of 2-day periods the ratio of horizontal scale to vertical scale $\lambda_H/\lambda_V\sim300$. Using these numbers, one finds that $(f^2/N^2)(\lambda_H{}^2/\lambda_V{}^2)=0.13$ which is small compared to unity. Hence, internal gravity waves of horizontal scale perhaps as large as 200 km in an equatorially centered canal of width 2000 km may propagate as though the earth were not rotating.

In general, the vertical shear of the mean horizontal wind influences the propagation and stability properties of internal gravity waves. Near the equator the vertical shear of the mean longitudinal wind is quite weak. A typical value computed from the tables of Oort and Rasmussen (1971) is approximately 5 m s^{-1} (10 km)$^{-1}$. Although this value is small, it will be shown that the mean wind shear can greatly influence the trapeze instability.

Phillips (1966) solved the problem of determining the behavior of internal gravity waves propagating in a mean shear flow. He assumed the mean flow to depend linearly on the vertical coordinate and adopted a coordinate system which moves with the mean flow. Having N^2 as a function of time still allows this procedure to be taken. The present analysis differs with that of Phillips at the point when the final approximate solutions are required due to the time variability of N^2 in the present case.

A synthesis of the respective analyses of Orlanski (1973) and Phillips (1966) reveals that the unstable waves must propagate at nearly right angles to the mean wind vector if they are to avoid being absorbed by the mean wind before they have time to grow appreciably.

2. Analysis

The flow considered is assumed to be inviscid and adiabatic. The perturbations to the basic state are such that the Boussinesq approximation is valid. The governing equation for small perturbations to a stratified shear flow under the above conditions is then

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\nabla^2w-U_{zz}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial x}+N^2\nabla_H{}^2w=0,\quad(1)$$