Applications of a Fast Computational Method for Telford's Theory of Cloud Droplet Coalescence Growth[1]

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ABSTRACT

A fast computational method for evaluating Telford's simplified stochastic model of the coalescence process of cloud droplet growth is shown to be useful for approximating the spectrum development in the early stages of precipitation, particularly for cases of narrow droplet size spectra. The method uses a saddle-point integration of the Laplace transform of the growth probability function, and turns out to be at least two orders of magnitude faster than the usual kinetic equation computations. Simple, rough approximations for large and small time intervals are developed.

Instead of setting the collection efficiency $E=1$ as in Telford (1955), realistic formulas for E taken from the Scott-Chen (1970) formula, the Neiburger-Lee-Lobl-Rodriguez (Lee, 1975) formula, or a Lagrange interpolation to fit the Davis-Sartor and Schafrir-Neiburger theories are used, yielding considerably different conclusions from those of Telford. Davies' and Beard and Pruppacher's terminal velocity formulas were both used, with little difference found between them.

A simple modification of our method was developed to treat the continuous collection model for a single particle falling in a uniform cloud of droplets. The results show that the continuous and discrete models have nearly the same growth rate when the growing droplets are larger than 50 μm in radius; the probability curves of the two models have nearly the same behavior. Consequently, the growth rate calculated by the continuous model is used for droplets from 50 to 500 μm. A typical result shows that 0.01% of the droplets with an initial size of 30 μm can grow to a size of 400 μm within 30.77 min.

Three examples which consider a single large drop size and either a single or narrow discrete spectrum of smaller droplet sizes are presented for the comparison of Telford's model with a kinetic equation calculation, arranged to treat volume categories which are integral multiples of a smallest size. An estimate of the range of validity of Telford's model as an approximation to the calculation of the kinetic equation is discussed. Under the same initial conditions, the calculation of the kinetic equation in FORTRAN needs about 50 s of computer time but the modified Telford method needs only 0.2 s in BASIC.

1. Introduction

The early theoretical work on the growth-by-collection process for warm rain assumed that larger drops falling through smaller ones collected the latter as if they were uniformly spaced. The resulting steady rate of growth by average sweep-out is often called "continuous growth," as opposed to "stochastic growth" for the case of randomly spaced droplets. The continuous model made all drops grow equally, whereas, as Telford (1955) showed in his pioneer paper on stochastic growth, a probabilistic model will allow a few drops to grow faster than the rest and become the initiators of precipitation. Since then the kinetic equation of aerosol theory has been used and developed for the calculation of stochastic growth (Golovin, 1963; Scott, 1968; Berry, 1967; Berry and Reinhardt, 1974; Gillespie, 1972). The main problem with the kinetic equation method is its need for large amounts of computer time and memory. As we shall show, since there is an extremely easy and rapid way of computing the Telford result, it is of interest to see how far a Telford type of calculation can be used to stimulate or model the kinetic equation approach. The early stages of a shower are of particular interest in this regard, but we also have useful results for longer times. Our fast calculational method is also of use in assessing the effects of varying collection efficiency and terminal-velocity formulas on the early stages of growth. Furthermore, we are able to use our method to make explicit and quantitative the transition from stochastic to continuous collection as the collector drop grows larger.

For comparison purposes, we apply the kinetic equation to the hypothetical case in which the volumes of all drops are multiples of a smallest size. Some interesting features of the early stages of stochastic collection are revealed by this model.

[1] The work reported in this paper forms part of the thesis submitted by one of us (C. Y. Chen) for the M.S. degree at the University of Nevada, Reno.

2. Telford's model

Telford's model is not only a stochastic one, but assumes that the small droplets are all the same size V_0 and that their spectrum remains constant rather than shrinking. It considers the growth of each large drop independently, or equivalently treats a group of large ones all of the same size without interactions among themselves. Telford (1955) derived the probability that a drop of volume mV_0 at $t=0$ reaches the volume $(m+k)V_0$ in a time between t and $t+dt$, and then made a calculation showing that the stochastic process of droplet collection produced the required small number of large precipitation drops much more rapidly than could have been generated on the average by the continuous model. In his paper, Telford assumed the collection efficiency to be independent of drop size, i.e., $E=1$, and determined the terminal velocity by Stokes' law. Under typical conditions, he found that drops 23 μm in radius could form in the order of 5 min in a cloud of 10 μm droplets from larger droplets initially 12.6 μm in size. Under the same conditions, the time for growing by the continuous growth model required the order of 33 min.

In the present paper we use two methods of obtaining results with Telford's formula for the calculation of the probability of coalescence $I_k(T)$ which describes the number of the growing drops that will grow to a certain size k or larger in a given dimensionless time T, where k is the integer ratio of the volume of the growing drop to that of one of the small droplets. We also make use of $Q_k(T)$, the probability for growth to exactly size k. The first method is a direct calculation from the integrand of Telford's result, with an algebraic modification useful for calculation when the dimensionless time T is small. The second method uses a saddle-point calculation involving a Laplace transform. After comparing these methods, we find that the saddle-point method is the easiest and fastest as well as having quite satisfactory accuracy, so we use this as our numerical method in the rest of this paper. We compute the collection efficiency as a function of the radius of the collecting droplets in most cases by the Scott-Chen (1970)[2] approximate formula which is fitted to the Davis and Sartor (1967) and Shafrir and Neiburger (1963) collision efficiency theories. In a few cases, we use the Neiburger-Lee-Lobl-Rodriguez formula (Lee, 1975) which is fitted to the Shafrir-Neiburger efficiencies and in other cases a direct Lagrange interpolation method is used with Davis and Sartor (1967), Shafrir and Neiburger (1963) and Davis (1972) efficiencies. The terminal velocities of the collecting droplets are determined by two almost equivalent methods, Davies' formula (1945) and that of Beard and Pruppacher (1969). We calculate the growth rate of droplets up to 100 μm; for sizes beyond, we investigate the difference of the growth rate between the discrete model and the continuous model, always under the assumption that the larger drops do not collect each other and the small droplet spectrum remains constant rather than shrinking. We find that when the size of the growing droplets is larger than 50 μm, both models have nearly the same rate of growth. Because the continuous model is still easier in calculation than the discrete model, we combine these two models, using the discrete model for droplets smaller than 100 μm and the continuous model for larger droplets up to 500 μm. The values of the atmospheric variables we consider are: radius of smallest collected cloud droplets $R_0=10$ μm; water content $w=1$ g m^{-3} (except where noted); viscosity of air $\eta=0.1708\times10^{-3}$ poise; density of air $\rho_a=0.1148\times10^{-2}$ g m^{-3}; pressure $=900$ mb; temperature $=0$°C; gravity $=981$ cm s^{-2}.

a. *Telford's theory developed by use of a Laplace transform*

We begin with Telford's (1955) expression for the probability dP_j of a drop of volume jV_0 collecting a cloud droplet of volume V_0 in time dt. This probability can be calculated in a well known way, as shown for instance by Gillespie (1975). The collision and collection process is treated as deterministic once the position of the smaller (collected) droplet is located within the effective volume swept out by the relative motion of the two droplets. The infinitesimal geometrical volume swept out in dt is $\pi R^2(U-U_0)dt$, where R is the radius of the large drop, U and U_0 the terminal velocities of collecting drops and collected droplets; to get the effective volume we multiply by the collection efficiency E. The required probability becomes the probability that a droplet is within this effective volume, which is just the ratio of this volume to the mean volume per droplet in the cloud. In terms of the cloud water content ω of droplets, the density of liquid water ρ and the radius R_0 of the droplets, the mean number of droplets per unit volume is $3\omega/4\pi R_0^3\rho$. With the relation $R=R_0 j^{\frac{1}{3}}$, we find

$$dP_j=\frac{3E\omega U_0}{4\rho R_0}j^{\frac{2}{3}}\left(\frac{U}{U_0}-1\right)dt, \tag{2}$$

where j is an integer and R_0 the radius of the cloud droplet. We define a dimensionless time T given by

$$T=\frac{3\omega U_0}{4\rho R_0}t \tag{2a}$$

and A_j by

$$A_j=Ej^{\frac{2}{3}}(U/U_0-1), \tag{2b}$$

so that we have

$$dP_j=A_j dT. \tag{3}$$

[2] There is a misprint in Eq. (9) of this paper. The corrected formula is

$$B=\{1.587a_L+32.73+344(20/a_L)^{1.56}\times\exp[-(a_L-10)/15]\sin[\pi(a_L-10)/63]\}/a_L^2.$$

From (3), the probability that no collision occurs in a time interval T, namely $\exp(-A_jT)$ for a drop of size jV_0, can be derived in the well-known way.

If the probability that a drop whose size at $T=0$ is mV_0 undergoes its kth collision between T and $T+dT$ is expressed by $q_k(T)dT$, we can derive a relation between q_k and q_{k-1}. Choose a time interval from 0 to T and let $0\leqslant T'\leqslant T$. Then the probability that the $(k-1)$th collision occurs between T' and $T'+dT'$ is $q_{k-1}(T')dT'$.

For a drop of volume $(m+k-1)V_0$, the probability that no collision occurs after T' until T is $\exp[-A_{m+k-1}\times(T-T')]$ and the probability that it collects a droplet between T and $T+dT$ is $A_{m+k-1}dT$.

Therefore, the probability that a drop with volume mV_0 at $T=0$ undergoes its kth collision between T and $T+dT$ is

$$q_k(T)dT=A_{m+k-1}dT\int_0^T dT'q_{k-1}(T')\times\exp[-A_{m+k-1}(T-T')]=A_{m+k-1}dTQ_{k-1}(T), \quad (4)$$

where $Q_{k-1}(T)$ is the probability that exactly $k-1$ collisions occur by time T, i.e.,

$$Q_{k-1}(T)=\frac{q_k(T)}{A_{m+k-1}}. \quad (5)$$

Eq. (4) is a generalization of Gillespie's (1975) Eq. 23b. By using a Laplace transform

$$\mathfrak{L}\{q_k(T)\}=\tilde{q}_k(s)=\int_0^\infty e^{-sT}q_k(T)dT \quad (6)$$

together with the fact that $q_1(T)dT=A_me^{-A_mT}dT$, we obtain

$$\left.\begin{aligned}\tilde{q}_1(s)&=\frac{A_m}{s+A_m}\\ \tilde{q}_k(s)&=\prod_{j=m}^{m+k-1}\frac{A_j}{s+A_j}\end{aligned}\right\}. \quad (7)$$

And by evaluating the usual Laplace inversion formula at the poles $s_m=-A_m$, we find Telford's formula

$$q_k(T)=\sum_{l=m}^{m+k-1}\exp[-A_lT]\prod_{j=m}^{m+k-1}A_j/\prod_{\substack{i=m\\ i\neq l}}^{m+k-1}(A_i-A_l), \quad (8)$$

where Π' signifies a product with the vanishing factor omitted.

Gillespie's (1975a) result that when all the A_l are equal $Q_k=(AT)^k\exp(-AT)/k!$ may be obtained by evaluating the Laplace inversion of (7) at the multiple pole at $S=A$, or for successive values of k by a suitable limiting process on (8), such as writing $A_{m+j}=A+j\epsilon$, where ϵ is allowed to approach zero after expanding the exponentials to the $k-1$ power of ϵ.

The probability $I_k(T)$ that a drop with volume mV_0 at $T=0$ has k or more collisions by the time T is then

$$I_k(T)=\int_0^T q_k(T')dT'=\sum_{l=m}^{m+k-1}[1-\exp(-A_lT)]\prod_{\substack{j=m\\ j\neq l}}^{m+k-1}{}'(1-A_l/A_j)^{-1}. \quad (9)$$

b. Alternative equation for small T

The numerical calculation of (9) involves working to a large number of significant figures when T is very small. To avoid this defect, an alternative form of (9) is derived.

Telford (1955) has shown that

$$\sum_{l=m}^{m+k-1}(-A_l)^p\prod_{\substack{j=m\\ j\neq l}}^{m+k-1}{}'\frac{A_j}{A_j-A_l}=0, \quad 1\leqslant p\leqslant k-1, \quad (10)$$

where p is an integer. This is an algebraic theorem valid for any set of A_l's whatever as long as no two are equal.

On expanding $1-\exp(-A_lT)$ and using (10), (9) becomes

$$I_k(T)=-\sum_{l=m}^{m+k-1}\sum_{p=k}^{\infty}\frac{(-A_lT)^p}{p!}\prod_{\substack{j=m\\ j\neq l}}^{m+k-1}{}'\frac{A_j}{A_j-A_l}. \quad (11)$$

Obviously (11) is advisable for small T only. When T is larger, too many terms of p are needed for convergence.

c. Saddle-point calculations

We use the saddle-point method to calculate both $q_k(T)$ and $I_k(T)$. It is convenient to illustrate the method with the latter function. By the standard rules, we can find the Laplace transform of $I_k(T)$:

$$\tilde{I}_k(s)=\mathfrak{L}\left\{\int_0^T q_k(T')dT'\right\}=(1/s)\tilde{q}_k(s).$$

Letting γ be a suitable real positive number, the usual inversion formula gives

$$I_k(T)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{sT}}{s}\prod_{j=m}^{m+k-1}\left(\frac{A_j}{s+A_j}\right)ds=(1/2\pi i)\prod_{j=m}^{m+k-1}A_j\int_{\gamma-i\infty}^{\gamma+i\infty}e^{g(s,T)}ds, \quad (12)$$

where

$$g(s,T)=sT-\ln s-\sum_{j=m}^{m+k-1}\ln(s+A_j).$$

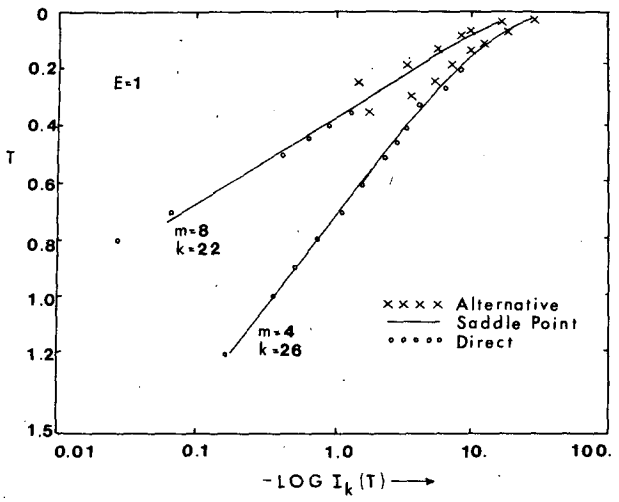

FIG. 1. Comparison of the results of the direct calculation, the alternative calculation and the saddle-point calculation for $I_k(T)$. For each of the values of k the saddle-point line joins with the alternative calculation for small values of the dimensionless time T and with the direct calculation for large values of T. The collection efficiency E equals 1.0 and Davies' terminal velocities are used.

The path of integration is parallel to the axis of imaginaries and to the right of all singularities. When s approaches either 0 or ∞ on the real axis $g(s,T)$ approaches infinity positively. Therefore, $g(s,T)$ has a minimum for some value of s on the positive real axis, which point is therefore a saddle point.

If we designate the saddle point by s_0, we have

$$\left.\frac{\partial g(s,T)}{\partial s}\right|_{s=s_0} = 0 = T - s_0^{-1} - \sum_{j=m}^{m+k-1} (s_0 + A_j)^{-1}. \quad (13)$$

For easier computation, we choose a series of values of s_0 and calculate for each the resulting T, which we write T_0. Then we have

$$T_0 = s_0^{-1} + \sum_{j=m}^{m+k-1} (s_0 + A_j)^{-1}. \quad (14)$$

If in (12) we integrate along a path passing through s_0, almost the entire contribution of the integrand to the integral will come from the region in which the magnitude of $g(s,T)$ takes on a positive minimum value along the real axis, and a maximum value along the integration path, i.e., the integration of (12) is dominated by the saddle point. Expanding $g(s,T)$ about the point s_0, we have

$$g(s,T_0) \approx g(s_0,T_0) + \tfrac{1}{2}(s-s_0)^2 g''(s_0,T_0),$$

where

$$g''(s_0,T_0) = \left.\frac{\partial^2 g(s,T_0)}{\partial s^2}\right|_{s=s_0} = s_0^{-2} + \sum_{j=m}^{m+k-1} (s_0 + A_j)^{-2}, \quad (15)$$

and the first derivative term vanishes because of $g'(s_0,T_0)=0$.

Writing $s - s_0 = iy$ and using the approximation for $g(s,T_0)$ we have

$$I_k(T) \approx \sum_{j=m}^{m+k-1} A_j \exp[g(s_0,T_0)] \times \int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}y^2 g''(s_0,T_0)]dy \approx \prod_{j=m}^{m+k-1} A_j \exp[g(s_0,T_0)][2\pi g''(s_0,T_0)]^{-\frac{1}{2}}. \quad (16)$$

The computation for $q_k(T)$ differs from the above only in the absence of the term s_0^{-1} in (13) and the term s_0^{-2} in (15), and in the replacement of 0 by $-A_m$ as the right-hand most finite value of s for which $g(s,T)$ approaches infinity.

For improving the accuracy of the saddle-point calculation, we can use the correction term which has been used by Scott (1968). A test of accuracy was made by comparing results with and without the correcting factor for cases with k from 6 to 62 and $I_k(T)$ from 10^{-1} to 10^{-10}. On the logarithmic scale of our plots there was no visible difference. So for the results reported below, the correcting factor was not used.

d. *Comparison of the methods*

Under typical atmospheric conditions as illustrated above, three methods, i.e., direct calculation from Telford's formula, use of the alternate equation for small T, and use of the saddle-point method (SPM) are used to calculate the probability of $I_k(T)$. Fig. 1 shows the results. We find that in the large T region, the curve of the saddle-point method joins with the curve calculated by direct calculation from Telford's method and in the smaller T region it joins with the curve calculated by the alternate equation. In other words, Fig. 1 shows that in the region from $I_k(T)=10^{-10}$ to $I_k(T)=0.8$ for the calculation of $I_k(T)$ we can use the saddle point method only instead of using either a direct calculation from Telford's method or the alternative equation. The quantity $Q_k(T)$ can be found from

$$Q_k(T) = I_k(T) - I_{k+1}(T) = q_{k+1}(T)/A_{m+k}, \quad (17)$$

which follows directly from the definitions of $Q_k(T)$ and $I_k(T)$, as well as by a direct saddle-point calculation. We have used both, and found the results to agree. By comparing the correction terms for $Q_k(T)$ and $I_k(T)$, we have confidence that the results of the SPM for $Q_k(T)$ are at least as accurate as that for $I_k(T)$.

3. Effect of collection efficiency formulas

Telford (1955) used 100% efficiency ($E=1$) for illustrative purposes. To show the substantial difference made by using realistic values for E and the small

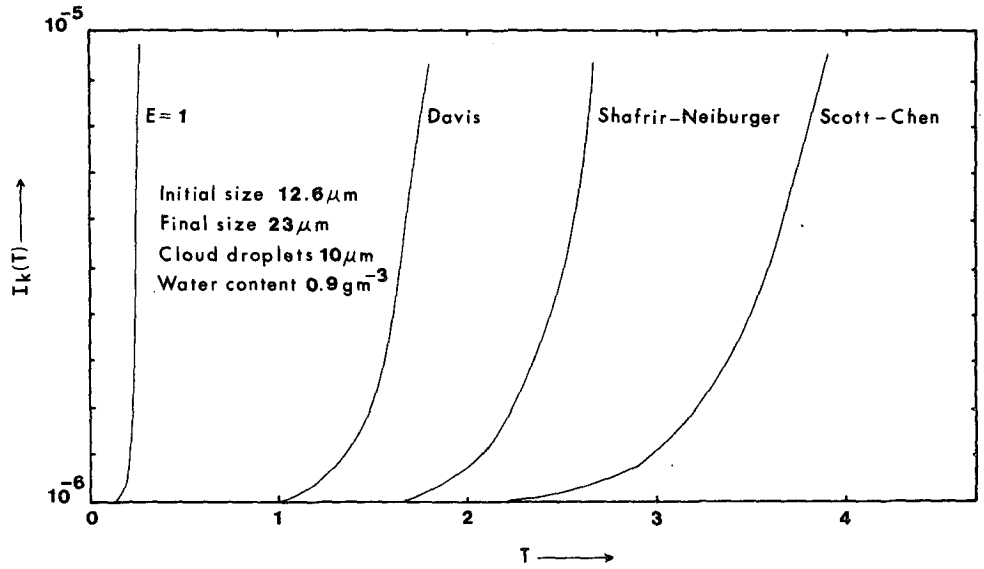

FIG. 2. Comparison with Telford's (1955) results for the fraction of 12.6 μm ($m=2$) radius drops which grow to greater than 22.9 μm ($k=10$) radius after time T. Both the Davis (1972) and Shafrir and Neiburger (1964) curves are calculated by using Lagrange interpolation to determine E when $\omega=0.9$ g m^{-3}, $a=10$ μm, $\rho=1.0$ g cm^{-3}, $u_0=1.273$ cm s^{-1}. Here real time $t=1163.7\times T$ [s].

effect of a simple reduction of E from 1.0 to 0.8, we used these two values and the Scott-Chen formula for the cases where the large droplet starts at twice and at 27 times the small droplet volume.

In order to compare with Telford's (1955) original results a special range of $I_k(T)$ plotted as a function of T on a linear scale is shown in Figs. 2 and 3 under the same atmospheric conditions as used in Telford's original paper. In addition to the Scott-Chen efficiencies, we used a simple linear Lagrange interpolation in two dimensions, i.e., the standard four-point formula applied to data points from the curves given by Shafrir and Neiburger (1963) and Davis (1972). Note especially that the introduction of realistic collision efficiencies drastically increases the growth time over the Telford estimate and that the different efficiency formulas give significantly different results. The need for improved precision in these formulas is clearly evident. For example, when the initial large-droplet size is 12.6 μm, the final size is 23 μm, water content $\omega=0.9$ g m^{-3}, we find at $I_k(T)=8.4\times10^{-6}$:

For $E=1$: $T=0.257$ or $t=4.98$ min.
For E determined by Lagrange interpolation of Davis' (1972) data: $T=1.75$ or $t=33.94$ min.
For E determined by Lagrange interpolation of Shafrir and Neiburger (1963) data: $T=2.125$ or $t=41.2$ min.
For E determined by Scott-Chen formula with Davis-Sartor-Shafrir-Neiburger data: $T=3.8$ or $t=83.7$ min.

In Telford's (1955) paper, he obtained a real time $t=5.11$ min or $T=0.257$. The difference from our results for $E=1$ comes from the different value of terminal velocity we used, $U_0=1.273$ cm s^{-1} for 10 μm of cloud drops, as compared with Telford's value 1.248 cm s^{-1}.

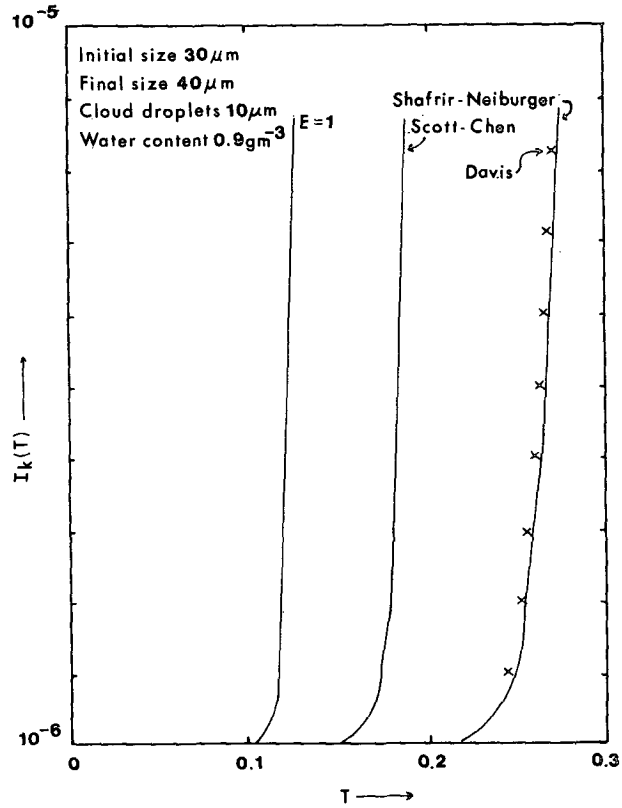

FIG. 3. As in Fig. 2 except for the fraction of 30 μm ($m=27$) radius drops ≥ 40 μm ($k=37$) radius after time T.

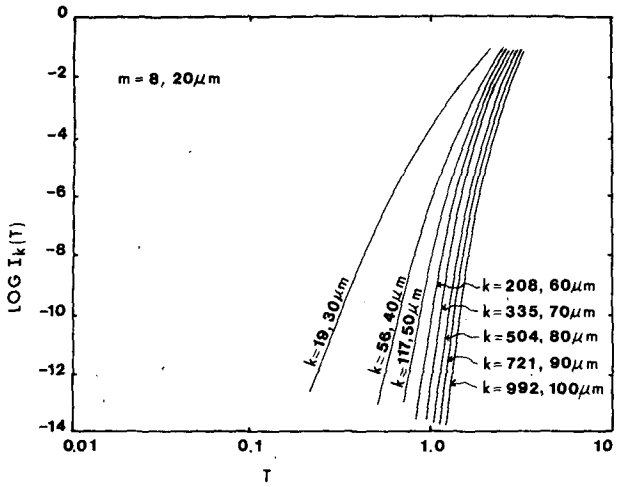

FIG. 4. Plots of $\log I_k(T)$ as a function of $\log T$ with the Beard-Pruppacher terminal velocity and Scott-Chen collision efficiency, $m=8$ (20 μm).

Examples for the growth of drops to successively larger sizes are shown by means of $I_k(T)$ in Figs. 4 and 5. In these figures the relation between values of m, k, j, T and real physical values are

Real time $=1047T$ [s].
Radius of cloud droplets $=10$ μm.
Radius of initial collecting drops $=10\times m^{\frac{1}{3}}$ μm.
Radius of final collecting drops $=10\times(m+k)^{\frac{1}{3}}$ μm $=10\times j^{\frac{1}{3}}$ μm.

For the calculation of terminal velocity, we used Davies' and Beard-Pruppacher's formulas. For the radius of droplets smaller than 30 μm, the two formulas give nearly the same result ($<1\%$ difference), the difference becoming larger as the size increases beyond 30 μm. However, the difference of terminal velocity between these two methods does not make much difference on the calculation of the probability of coalescence. Davies' values were used for Fig. 1 and Beard and Pruppacher's for the others.

4. Limiting behavior of $I_k(T)$ for large and small T

The shapes of the $I_k(T)$ curves suggest the possibility of finding approximate analytical fits that might for instance be of use to modelers. From the graphs of $I_k(T)$, with coordinate $\log T$ plotted against $\log I_k(T)$, we find that when T is very small $\log T$ approaches a straight line and the slope approaches k, as can be seen from Figs. 4 and 5. This result can be obtained theoretically from (9). When T is very small, the first term with $p=k$ dominates the results. After taking the logarithm on both sides, we have

$$\log I_k(T)\approx k\log T+\text{constant}. \tag{18a}$$

And when T is very large

$$I_k(T)\approx 1-\exp(-aT+b), \tag{18b}$$

where a and b are constant, $a>0$. This result can be obtained by using (9) when T is so large that only a few values of j dominate.

5. Comparison of the growth rates for the continuous and stochastic models

From (3), the probability that a single collecting drop of volume $V=jV_0$ collects a cloud drop in time dT is $dP_j=A_j dT$. From the relation between δP_j and δV, the mean time for collecting one droplet of volume $\delta V=V_0$ is $\delta T=1/A_j$.

Thus the mean time during which a growing drop with initial volume $V=mV_0$ collects k cloud drops is

$$\Delta T_{mk}=\sum_{j=m}^{j=m+k}1/A_j. \tag{19}$$

Eq. (19) can be approximated by an integral

$$\Delta T_{mk}=\int_{j=m}^{m+k}\frac{dj}{A_j}=\int_{j=m}^{m+k}\frac{dj}{j^{\frac{1}{3}}E(U/U_0-1)}. \tag{20}$$

Eq. (20) is reasonably accurate only when A_j is not small. For the integration of (20) we use the Reynolds number Re as the variable instead of j. To write j and U as a function of Re, we note that Beard and Pruppacher (1969) quote their results for terminal velocity by relating the Reynolds number to the ratio of the actual drag force (the net gravitational value) to the Stokes value.

Using this integral, we have calculated the growth time for the size of growing droplets up to 500 μm, the upper limit being chosen because the Beard-Pruppacher formula is valid up to about Re$=200$, for which the corresponding radius of the growing drop is 488 μm.

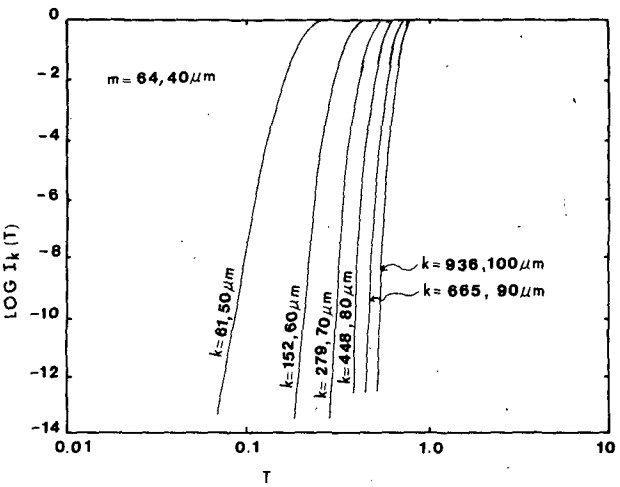

FIG. 5. As in Fig. 4 except $m=64$ (40 μm).

After comparing the growth rates between the continuous and stochastic models, we found that when the size of the growing droplets is larger than 50 μm both models give nearly the same results for the growth rate. The time increments for growth from 80 to 100 μm all agree within about 4%. Therefore, for the calculation of growth rate we can use the continuous model instead of the discrete model when the size of the growing droplet is larger than 100 μm.

From graphs of $I_k(T)$, such as those in Figs. 4 and 5, we find that when the droplets grow from 50 to 100 μm, the respective curves of $I_k(T)$ approach a straight line and are parallel to each other, so we could assume that when the growing droplets are larger than 100 μm the respective curves of $I_k(T)$ will also be nearly parallel to each other. Under this assumption, we can calculate the growth rate by the continuous model and obtain the probability $I_k(T)$ easily for the size of growing droplets that are larger than 100 μm. The results are shown in Fig. 6. We find that within 30.77 min, 0.01% $[I_k(T)=10^{-4}]$ of the growing droplets with an initial size of 30 μm can grow to a size of 400 μm.

6. A comparison of the Telford and kinetic equation calculations

We illustrate the comparison of the Telford method with the use of the kinetic equation (KE) by three examples. For the calculation with the KE, 64 mass values that are integral multiples of the smallest size were chosen so as to cover the range of sizes from radius 10 to 40 μm. Using V_0 to represent the volume of droplets with radius 10 μm, the volume of other sizes of drops can be expressed as integer multiples of V_0. This categorization assures that all the resultant drop sizes will fall on categorical points so no interpolation for splitting between categorical points is necessary, and the integrals in the kinetic equation reduce to simple summations. Whenever the radius of the growing drop becomes larger than 40 μm, the calculation is cut off automatically.

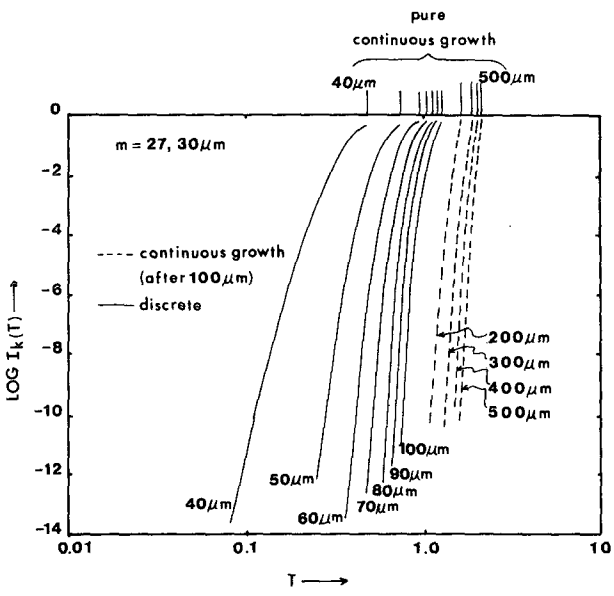

FIG. 6. As in Fig. 4 except for a combination of the continuous and discrete models with $m=27$ (30 μm). When the droplets are not larger than 100 μm we use the discrete model and for $R>100$ μm the continuous model is used. The top part of the figure shows the growth time for pure continuous growth.

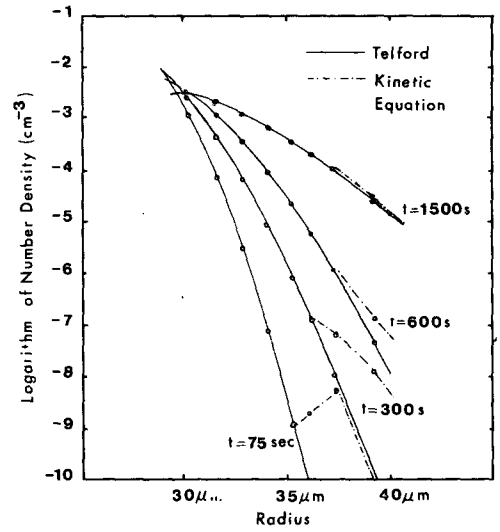

FIG. 7. A comparison of the Telford calculation with the kinetic equation calculation. The sizes and number densities of droplets and drops are 15.87 μm, 5 cm^{-3} and 28.84 μm, 0.01 cm^{-3}, respectively. When the sizes of the growing drops are less than 37.33 μm, the two results coincide graphically. The actual difference is that Telford's results are ~2% greater than the kinetic equation results. The spectra are discrete in character; the connectional lines which pass through Telford's results are for ease in viewing. Dashed lines are used for the kinetic equation results when they differ from the Telford results.

The first example involves discrete droplet and drop sizes of volumes respectively $4V_0$ and $24V_0$. That is, the radius of droplets is taken as 15.87 μm with density 5 cm^{-3}, and radius of the drops is 28.84 μm with density 0.01 cm^{-3}. For Telford's calculation, the probability of exactly k collisions $Q_k(T)$ was obtained from $I_{k+1}T-I_k(T)$.

For this case we used the Neiburger-Lee-Lobl-Rodriguez (Lee, 1975) collision efficiency formula. The results are shown in Fig. 7. In this figure, the spectra are all discrete, of course, as shown by the points. The smooth lines were drawn to aid the eye. In the range where the radius is smaller than 37.3 μm (volume $52V_0$) the results of two calculations are indistinguishable graphically. In actuality, Telford's results are about 2% greater than the KE results. Since drops of equal size have zero probability of coalescence in our case in which wake capture is not included, the smallest size that can arise from a large drop to large drop collision is that of a $28V_0$ and a $24V_0$ drop. The kinetic equation thus shows a jump where Telford's will not. However, for larger times the loss and gain terms in the kinetic

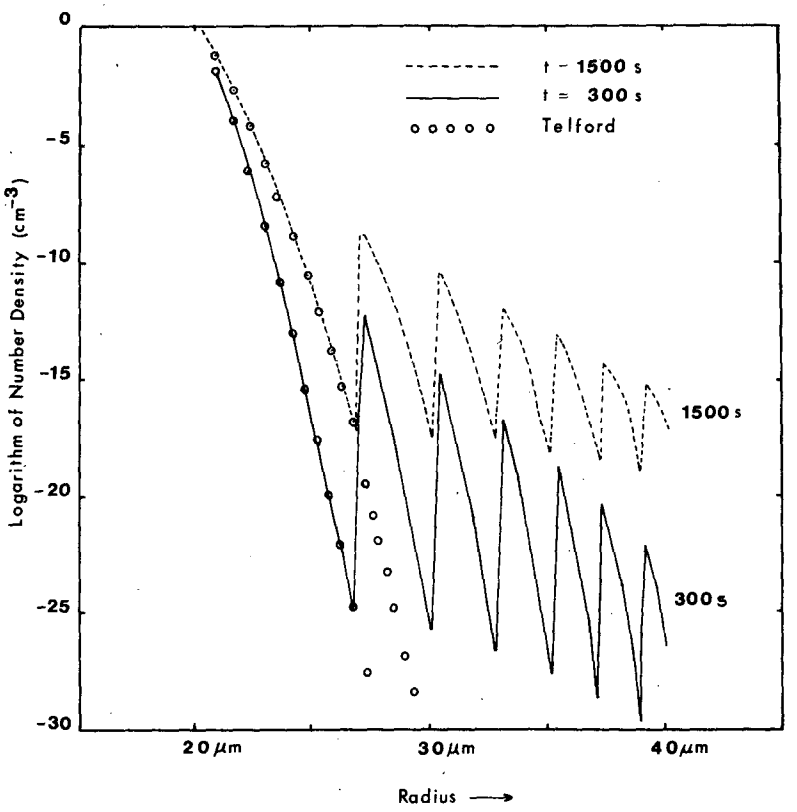

FIG. 8. The effect of multiple collisions among drops. The initial sizes and densities for droplets and drops are 10 μm, 2.0 cm^{-3} and 20 μm, 1.05 cm^{-3}, respectively. The Telford results are shown by circles which almost coincide with the continuous lines showing the kinetic equation results in the range where the collisions between large drops are not involved.

equation balance out and the two theories coincide even above 37.3 μm.

To test Telford's assumption of the constant density of small droplets, we found in this example that the decrease of the density of the droplets is 0.014, 0.06, 0.14 and 0.4% for t equal to 75, 300, 600 and 1500 s, respectively. In this case the ratio of number densities between drops and droplets is 1 to 500. It is clear that the concentration of droplets almost remains constant as Telford assumed. In order to check the effect of the 40 μm cutoff for the 1500 s calculation, we extended the category points from 64 to 125 points, i.e., a 50 μm cutoff. The results of using these two cutoffs differ by about 3%.

The second example is one which shows the effect of multiple collisions among large drops. Fig. 8 shows the result at $t=300$ and 1500 s for the case of 10 μm small droplets, number density 2.0 cm^{-3} and 20 μm large drops with density 1.05 cm^{-3}. The collisions of large drops on each other come from the nonlinear aspect of the KE, so that the relatively large density of the larger drops exaggerates the jumps that occur at volumes corresponding to the collection of large drops on each other. Because the Neiburger-Lee-Lobl-Rodriguez formula gives zero collision efficiency between a drop with a volume $8V_0$ and those of volumes $9V_0$, $10V_0$ and $11V_0$, the jumps in Fig. 8 occur at resulting volumes of $20V_0$, $28V_0$, $36V_0$, $44V_0$, $52V_0$ and $60V_0$.

The accuracy of the Telford model is less here, but the value for the early small size part of the process is still evident. Of course, if the number density of large drops decreases, the probability of collisions between large drops will reduce and then the Telford method will be more applicable. A side consequence of these results is that they show that the Berry-Reinhardt method with the assumption of smoothness implied in its interpolation and integration methods will not work for discrete or narrow initial spectra of the type used here. In this case the "bumps" persist even when the duration is as long as 1500 s.

The third example is similar to the first one. Instead of considering a single size of droplets at 15.87 μm (with volume $4V_0$, number density 5 cm^{-3}), we broadened the size of droplets to include 12.6, 14.42, 17.10 and 18.17 μm. The corresponding volumes and number densities are $2V_0$, 0.1 cm^{-3}; $3V_0$, 1 cm^{-3}; $5V_0$, 1 cm^{-3}; and $6V_0$, 0.1 cm^{-3}, respectively. For the Telford calculation we considered all the droplets to have the same size as 15.87 μm (volume $4V_0$). The results are shown in Fig. 9.

In this figure we see the obvious fact that broadening the initial spectrum of droplets produces more possible sizes of growing drops for the kinetic equation calculation. Because the water content is conserved, the density of each size of drops becomes less as compared with Telford's calculation. However, if we split the mass of each intermediate size into two nearby neighboring sizes which can be calculated by the Telford method—for example, split the mass of size $34V_0$ drops equally into size $32V_0$ and $36V_0$ and convert the mass of size $33V_0$ and $35V_0$ drops respectively into size $32V_0$ and $36V_0$—we find that the resultant spectra become very close to the results of the Telford calculation especially in the case of $t=1500$ s. This example shows

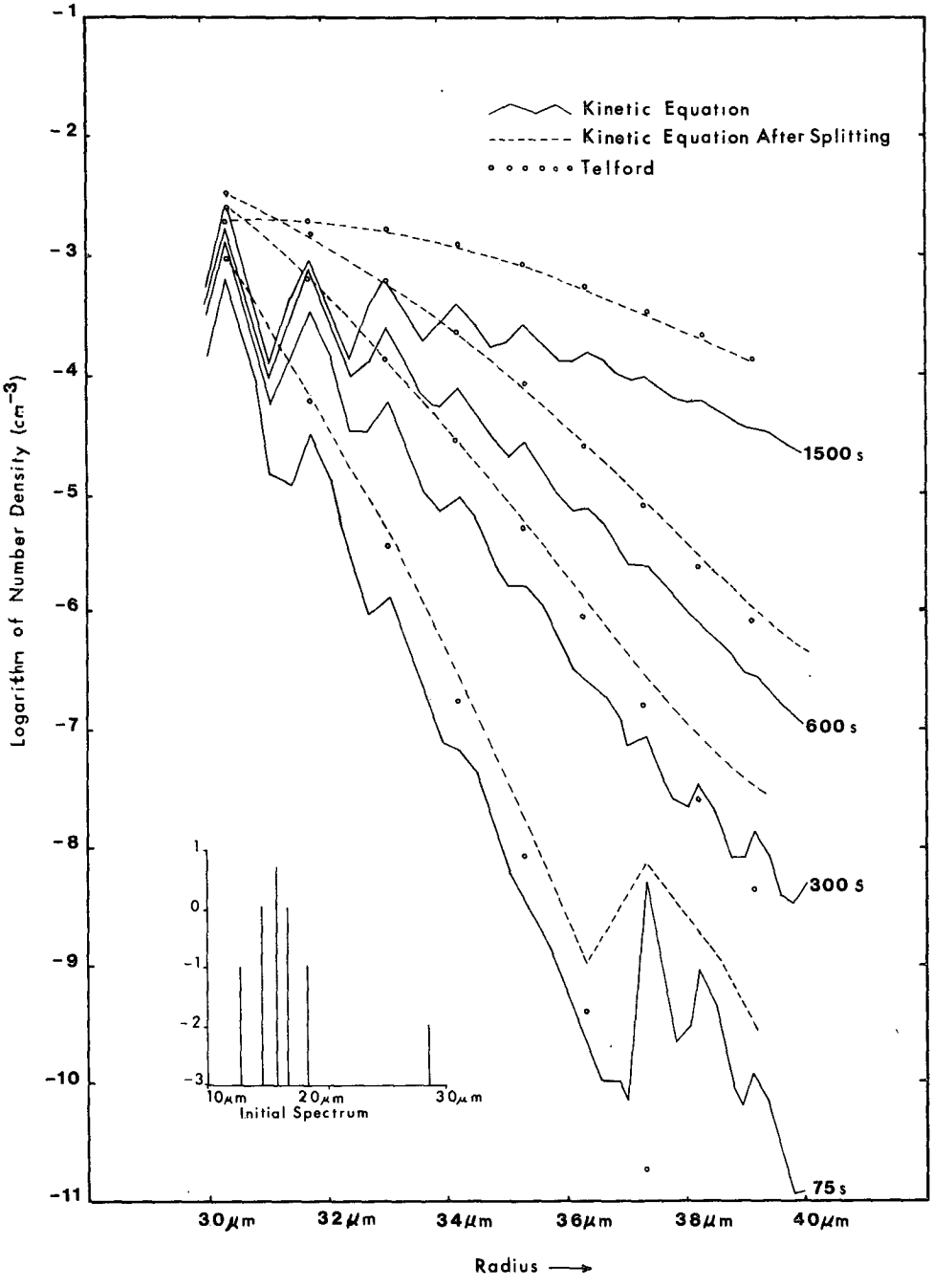

FIG. 9. Comparisons between the results of the Telford calculation and that of the kinetic equation calculation for a spread-out initial spectrum with and without splitting. The sizes and number densities of initial spectrum are 12.6 μm ($2V_0$), 0.1 cm^{-3}; 14.42 μm ($3V_0$), 1 cm^{-3}; 15.87 μm ($4V_0$), 5 cm^{-3}; 17.10 μm ($5V_0$), 1 cm^{-3}; 18.17 μm ($5V_0$), 0.1 cm^{-3}; and 28.86 μm ($24V_0$), 0.01 cm^{-3}. For the Telford calculation, all the droplets are considered as to be 15.87 μm in size.

us that we can extend the Telford calculation to the case where the initial distribution of droplets is a narrow spectrum.

The modified Telford's calculation provides us a means of impressively fast computation. For example, to obtain the results shown in Fig. 7, the kinetic equation calculation needed about 50 s of central computer time (using FORTRAN language) for each t but the Telford calculation only needs 0.2 s (using BASIC language). The speed of the Telford method convinces us that it is worthwhile to investigate further the valid range of Telford's model. These investigations are under way.

7. Conclusions

1) We have improved the calculational method for Telford's stochastic collection model by means of a Laplace transformation and use of the saddle-point method. This modified calculation of the probability $I_k(T)$ of k or more collections provides us a fast way for investigation of the effects of each factor in the coalescence growth of cloud droplets, and the way in which stochastic growth goes over into continuous growth.

We found that when the growing droplets are larger than 50 μm, the continuous and discrete models have nearly the same growth rates.

2) When the size of the growing droplets is as small as 12.6 μm, the effect of the reduction of the collection efficiency below unity is to slow down considerably the growth rate of the collecting droplets. As shown in Figs. 2 and 3 we concluded that Telford's simplified model is not as adequate as had been thought for explaining the early stage development of growing droplets, owing to the sensitivity of the effect of the collection efficiency on the growth of small droplets.

3) We compared the Telford calculation with the kinetic calculation by considering two discrete pairs of drop and droplet sizes. The radii and number densities of drops and droplets were 28.84 μm, 0.01 cm^{-3} and 15.87 μm, 5 cm^{-3}, respectively. In this case of a discrete type initial spectrum, the Berry-Reinhardt formulation is not applicable for the kinetic equation calculation.

To test Telford's assumptions, we find the conception that the density of droplets does not decrease was valid in this case. The other assumption, that large drops do not collide with each other, will decrease the growth rate of larger drops just after the early stage of development. However, in the longer duration range, the results of the two calculations will approach each other in those cases so far tried.

4) We have extended the Telford calculation to a case where the initial size of the droplets is not single-valued but a narrow discrete spectrum. The results of the Telford and kinetic equation calculations are very close to each other when the latter results are grouped into the Telford categories. Even when the duration is as long as 1500 s the difference between these two calculations is less than 4%. In all our comparisons we find that the modified Telford calculation is of the order of 250 times faster than the other.

5) In this paper we have considered only one set of atmospheric variables and a few sizes of droplets. Work is in progress to investigate a range of values of the atmospheric variables and of the sizes, distributions and times for which the Telford results model useful aspects of the kinetic equation.

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