The General Circulation of Two-Dimensional Turbulent Flow on a Beta Plane[1]

H. TENNEKES[2]

Department of Aerospace Engineering, The Pennsylvania State University, University Park 16802

(Manuscript received 15 November 1976, in revised form 16 February 1977)

ABSTRACT

It is likely that several features of the mid-latitude circulation in the earth's atmosphere will also be observed in two-dimensional, nondivergent flow with buoyant forcing and surface friction. Properly scaled, buoyancy effects are surprisingly similar to baroclinic effects. A linear stability analysis shows that the growth rate of unstable disturbances depends on zonal wavenumber in much the same way as that of baroclinic waves, except for the absence of a high-wavenumber cutoff related to the Rossby radius of deformation. The energy conversion mechanisms in buoyancy-driven two-dimensional flow closely resemble those in the atmosphere: eddy kinetic energy is maintained primarily by conversion of eddy potential energy, the kinetic energy of the mean zonal flow is maintained primarily by a reverse energy cascade, and the flow owes its existence and dynamics to the mean temperature contrast between latitude circles. The equations studied in this paper include those for enstrophy and temperature variance; the spectral fluxes of these quantities are taken into account. The maintenance of the general circulation in two-dimensional flow is described in part by a system of flux-maintenance equations. These shed light on such issues as the magnitude of the poleward eddy heat flux in developing storms and the countergradient eddy momentum flux in middle latitudes.

1. Introduction

The study of two-dimensional turbulence of an incompressible fluid is motivated in part by problems that arise in the analysis of the statistical dynamics of nondivergent, barotropic models of atmospheric motion. If the fluid involved is inviscid, such motion conserves both its total kinetic energy and its total enstrophy (the enstrophy is defined as one-half of the vorticity variance). However, enstrophy is not a conservative quantity if the viscosity of the fluid is not identically equal to zero (Batchelor, 1969). This problem arises because random, two-dimensional flow fields tend to intensify vorticity gradients, thus forcing enstrophy toward smaller scales. If such an "enstrophy cascade" is permitted to proceed for a long enough time, the smallest scale containing significant enstrophy will become small enough to be affected by viscosity. The resulting irreversible viscous loss of enstrophy will gradually decrease the total enstrophy of the flow field, no matter how small the viscosity is.

Due to the "spectral blocking" tendency of two-dimensional flow (Fjørtoft, 1953), it is difficult—if not impossible—to cascade kinetic energy toward ever smaller scales. In the limit as the viscosity approaches zero, the viscous dissipation rate of kinetic energy vanishes (Batchelor, 1969). For all practical purposes the total kinetic energy of two-dimensional turbulence at very high Reynolds numbers is a conservative quantity.

Barotropic, nondivergent flow of an almost-inviscid fluid can be maintained in a statistically steady state only if there is a source of vorticity fluctuations that balances the enstrophy cascade rate. In studies of the spectral dynamics of two-dimensional turbulence one often employs random stirring at a given wavenumber. Such stirring, however, also acts as a source of eddy kinetic energy. This implies that energy has to be removed by some other process. A convenient way to do this is by introducing some kind of parameterized surface friction, which makes up for the absence of an energy cascade. In this way, the spectral dynamics of statistically steady states can be studied.

Most of the practical problems of turbulence dynamics, however, arise from the interactions between the mean flow and the velocity fluctuations. For example, what happens when two-dimensional turbulence is exposed to a mean wind shear in the plane of motion? There are reliable indications (Kraichnan, 1967) that one must expect a reverse energy cascade in two-dimensional flow: kinetic energy tends to migrate toward larger scales of motion and eventually feeds

[1] Contribution No. 410, Department of Atmospheric Sciences, University of Washington.

[2] Permanent affiliation (after 1 August 1977): Royal Netherlands Meteorological Institute, de Bilt.

into the mean flow. The mid-latitude westerlies, for example, are maintained by converting eddy kinetic energy into mean-flow kinetic energy. In other words, two-dimensional flows tend to be barotropically stable. Because of this tendency, it is unlikely that a statistically steady general circulation with a nontrivial mean velocity distribution can be maintained without providing some source for eddy kinetic energy and a corresponding sink for mean-flow kinetic energy. If surface friction is a suitable sink, what would be a realistic source of eddy energy and eddy enstrophy if we are not interested in the consequences of artificial, random two-dimensional stirring?

Let us look at this problem from a different perspective. Imagine that we want to study problems in climate dynamics by computing the general circulation of nondivergent, barotropic flow of an inviscid fluid. Would the results make any sense? Clearly not—a barotropic model merely advects vorticity around; it cannot begin to represent the energetics of the atmosphere's general circulation because it does not include dynamical interactions between the temperature field and the flow field. If isotherms and isobars are parallel everywhere, there is no temperature advection and no vorticity amplification in developing cyclones. The flow in a nondivergent, barotropic model cannot be baroclinically unstable and cannot feed on the mean meridional temperature gradient. We conclude that two-dimensional turbulence will have trivial energy cycles unless we provide a mechanism by which the flow field interacts in a sensible way with the temperature field.

If a nondivergent, one-level model of atmospheric motion is to have nontrivial energy cycles, it must somehow imitate baroclinic effects. It must allow vorticity amplification in developing disturbances and it must lead to conversion of eddy potential energy into eddy kinetic energy. In addition, it should be able to maintain mid-latitude westerlies by a reverse energy cascade and it should relate all of the above features to the poleward eddy flux of heat and the mean temperature contrast between the equator and the poles.

In this paper, we will attempt to show that nondivergent, two-dimensional flow of an almost incompressible, nearly inviscid gas with buoyant forcing is capable of exhibiting behavior that resembles that of the general circulation in the earth's atmosphere. Obviously, two-dimensional turbulence in which baroclinic forcing is replaced by buoyant forcing cannot be expected to represent accurately the subtle interactions between vertical and horizontal fields of motion that make atmospheric dynamics so interesting. Nevertheless, it turns out that the similarities are surprisingly close in many respects. We will, therefore, employ the terminology of climate dynamics when we discuss the statistical characteristics of the flow and temperature fields. Also, we will not hesitate to borrow adjectives from the arsenal of dynamic meteorology when the occasion warrants such use. In particular, we will use the adjective "baroclinic" frequently without quotation marks, though "buoyant" would be more appropriate from a formal point of view.

This is an exploratory paper. In order to keep the analysis relatively simple, we shall employ a Cartesian beta plane and assume that the flow is of infinite extent in the zonal direction but confined between rigid walls at two latitudes. The only source of energy is the temperature difference between the two boundaries; it is maintained by diabatic heating and cooling. The mean heating rate and the mean temperature field are taken to be independent of time and of longitude; whenever averages are needed, we shall take them both over time and over latitude "circles." Surface topography, continents and oceans are absent; surface friction will be represented by linear Rayleigh terms.

2. Governing equations

We shall investigate the dynamical and statistical characteristics of the following system of equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv-\frac{K_s}{h^2}u \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu-\frac{K_s}{h^2}v+\frac{g^*}{T}\theta \tag{3}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=R. \tag{4}$$

In the Boussinesq approximation employed here, the first law of thermodynamics is replaced by the temperature equation (4); for the time being we will take the thermal diffusivity to be negligible. The source term R in (4) represents diabatic heating and cooling; it will have to be positive near the lower wall of the zonal channel and negative near the upper wall. The equations of motion include linear surface-friction terms; K_s is the surface exchange coefficient and h represents the scale height of the atmosphere. The acceleration of gravity g^* is directed toward the equator; this makes sure that warm air is forced toward the pole, while cool air "sinks" toward the equator. The magnitude of g^* will be scaled shortly in such a way that the amplification rate of unstable buoyant waves corresponds to that of unstable baroclinic waves in the earth's atmosphere. The fluid is nearly inviscid; we ignore viscous diffusion terms in the equations of motion, but we shall make provisions for the viscous dissipation of enstrophy caused by the presumed existence of an enstrophy cascade.

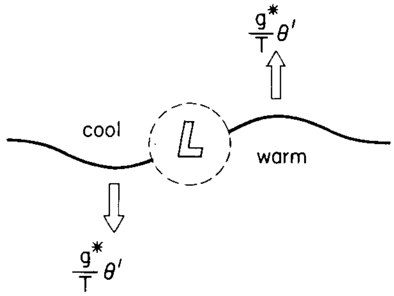

FIG. 1. As warm air begins to move poleward and cool air moves toward the equator, the zonal temperature gradient $\partial\theta/\partial x$ becomes positive. This intensifies the vorticity in the low-pressure center because the buoyancy forces exert a torque.

Terms similar to the buoyancy force in (3) occur in primitive equations written for sloping coordinate systems. The analogy with Fleagle's (1955) equations in stream-surface coordinates is particularly close; the effective acceleration of gravity (g^*) corresponds to the true acceleration (g), multiplied by the slope of stream surfaces. That slope is typically about 1:1000; this suggests that g^* should be about 10^{-2} m s^{-2}. This estimate will be verified in the next section.

The analogy should not be carried too far, however. Energy conversions in the atmosphere depend on the difference in slope between isentropic surfaces and stream surfaces (Holton, 1972); that difference depends on horizontal scale and vertical hydrostatic stability. By comparison, g^* must be taken constant; it does not take into account that isentropic surfaces may have different slopes. The shortcomings of the model should be kept in mind.

We investigate the vorticity dynamics associated with the equations given above. The relative vorticity ζ is defined by

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{5}$$

The evolution of ζ is governed by the following equation (we use $f=f_0+\beta y$):

$$\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}=-\beta v+\frac{g^*}{T}\frac{\partial\theta}{\partial x}-\frac{K_s}{h^2}\zeta. \tag{6}$$

In addition to a surface friction term and the beta effect (advection of planetary vorticity), this equation incorporates a "baroclinic" source term. It is easy to see that this term would be absent if we had decided to make the buoyant forcing terms in (2) and (3) proportional to $\partial\theta/\partial x$ and $\partial\theta/\partial y$, respectively. The somewhat awkward buoyancy term in (3) is necessary if we want to incorporate vorticity amplification in our model.

Fig. 1 shows how the zonal temperature gradient in an unstable baroclinic wave is related to the intensification of the disturbance.

It is convenient to introduce a wind field which corresponds to a balance among the three principal forces involved. We define the "thermal wind" $\mathbf{v}_T=(u_T,v_T)$ by

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv_T, \tag{7}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu_T+\frac{g^*}{T}\theta. \tag{8}$$

Strictly speaking, $\mathbf{v}_T$ is the geostrophic-buoyant balance wind. However, as we shall see shortly, $\mathbf{v}_T$ in a certain sense imitates the behavior of the upper level wind field in the earth's atmosphere. The choice of adjective thus is not altogether unjustified.

From (7) and (8) we obtain by differentiation

$$\frac{g^*}{T}\frac{\partial\theta}{\partial x}=\beta v_T+f\nabla\cdot\mathbf{v}_T, \tag{9}$$

$$\frac{g^*}{T}\frac{\partial\theta}{\partial y}=\beta u_T-f\zeta_T+\frac{1}{\rho}\nabla^2 p. \tag{10}$$

Here ζ_T is the thermal vorticity, defined by

$$\zeta_T=\frac{\partial v_T}{\partial x}-\frac{\partial u_T}{\partial y}. \tag{11}$$

Substituting (9) into (6), we obtain

$$\frac{d\zeta}{dt}=-\beta(v-v_T)+f\nabla\cdot\mathbf{v}_T-\frac{K_s}{h^2}\zeta. \tag{12}$$

If we now define the unbalanced wind $\mathbf{v}_A$ by

$$\mathbf{v}=\mathbf{v}_T+\mathbf{v}_A, \tag{13}$$

then because $\mathbf{v}$ is nondivergent,

$$\nabla\cdot\mathbf{v}_A=-\nabla\cdot\mathbf{v}_T. \tag{14}$$

The vorticity equation thus can be written also in the form

$$\frac{d\zeta}{dt}=-\beta v_A-f\nabla\cdot\mathbf{v}_A-\frac{K_s}{h^2}\zeta. \tag{15}$$

This states that the vorticity of a fluid parcel is changed by three processes. The first of these is the advection of planetary vorticity on the beta plane by the meridional component of the unbalanced wind, the second is the amplification of planetary vorticity caused by the convergence of the unbalanced wind, and the third is vorticity decay by surface friction. By contrast, Eq. (12) states that the vorticity is intensified by the divergence of the thermal wind. The unbalanced wind thus corresponds to low-level winds in a growing baroclinic disturbance, while the thermal wind imitates the divergence of the upper level winds in a developing storm. Since the thermal wind is defined on the basis of a force balance, it is fair to state that the intensification of a storm is generated by the divergence of the thermal wind.

Fig. 2 illustrates the geometry of the force vector diagram for a developing wave; Fig. 3 shows how this corresponds to divergence of the thermal wind and convergence of the unbalanced wind.

3. Stability analysis

It is useful to verify the qualitative interpretation developed above by performing a linear stability analysis. Clearly there is no need to study the isentropic stability characteristics of the model, since they are identical to those of a nondivergent barotropic model. For our "baroclinic" stability analysis, we assume that there is a uniform mean zonal flow $\bar{u}$ and a uniform mean temperature gradient $\partial\bar{\theta}/\partial y$. We use a perturbation streamfunction ψ' and a temperature perturbation θ' defined by

$$\psi' = a_\psi \exp[ik(x-ct)], \tag{16}$$

$$\theta' = a_\theta \exp[ik(x-ct)+i\phi]. \tag{17}$$

Here ϕ is the phase difference between the temperature wave and the streamfunction wave, k the zonal wavenumber and c the complex phase speed.

Substituting these expressions into the linearized vorticity and temperature equations and neglecting surface friction terms, we obtain the following results. The wavenumber of neutral stability is given by

$$-\frac{g^*}{T}\frac{\partial\bar{\theta}}{\partial y} = \left(\frac{\beta}{2k_0}\right)^2, \tag{18}$$

all waves with $k \geqslant k_0$ propagate at a speed

$$c_r = \bar{u} - \frac{\beta}{2k^2}, \tag{19}$$

and the amplification rate of those waves is given by

$$\alpha^2 = -\frac{g^*}{T}\frac{\partial\bar{\theta}}{\partial y} - \left(\frac{\beta}{2k}\right)^2. \tag{20}$$

Here α is the imaginary part of ck.

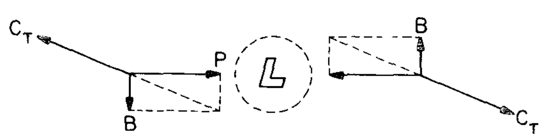

FIG. 2. The balance of forces east and west of a developing cyclone in the Northern Hemisphere. The pressure gradient P is directed toward the low-pressure center, the buoyancy force B is directed toward the pole in the warm sector and toward the equator in the cold sector, and the Coriolis force C_T of the "thermal" wind balances the other two.

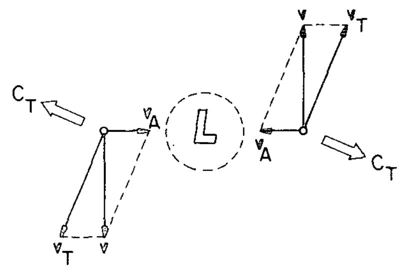

FIG. 3. Divergence of the thermal wind and convergence of the unbalanced wind around a developing storm, seen from a frame of reference moving with the mean flow. For simplicity, it is assumed that the meridional component of $\mathbf{v}_A$ is zero. The direction of $\mathbf{v}_T$ is determined by the force balance sketched in Fig. 2.

The air gets colder toward the pole, and the mean meridional temperature gradient destabilizes the flow. Eq. (18) states that the wavenumber of neutral stability is reached when the imaginary Brunt-Väisälä frequency of the flow equals the "Rossby frequency" $\beta/2k$. Eq. (19) indicates that unstable waves move westward relative to the mean zonal flow, and that their relative propagation speed is one-half of that of Rossby waves on a barotropic beta plane. Eq. (20) shows that the amplification rate of baroclinic waves increases as the wavenumber increases, and that the amplification rate of very short waves is approximately equal to

$$\alpha_\infty = \left(-\frac{g^*}{T}\frac{\partial\bar{\theta}}{\partial y}\right)^{\frac{1}{2}}. \tag{21}$$

Unfortunately, the system has no high-wavenumber cutoff corresponding to the Rossby radius of deformation in atmospheric motion. It is hard to speculate just how serious this defect is, but we may take some comfort from the fact that the general circulation model of GFDL has similar problems (Gall, 1976). Also, the linear stability characteristics of the system do not necessarily correspond on a one-to-one basis with its nonlinear stability characteristics; randomizing tendencies at small wavelengths may very well create an effective nonlinear cutoff at high wavenumbers. This would have to be tested by computer simulations.

In principle it would be possible, at least in a spectral solution method of the model equations, to postulate an appropriate spectral cutoff for g^*. The point is that g^* controls the rate of conversion of potential energy into kinetic energy; if g^* were dependent on wavenumber it would imitate the stabilization of short waves by the vertical stability of the earth's atmosphere. At the time of writing, we have not yet discovered a simple and internally consistent solution to this problem.

Eqs. (18)–(21) permit a rough estimate of the value of g^* needed to get values of α and k_0 that correspond to those in the earth's atmosphere. We put $\partial\bar{\theta}/\partial y = -3$ K (1000 km)$^{-1}$, $T = 300$ K and $\alpha_\infty = 10^{-5}$ s^{-1} (roughly 1

TABLE 1. Orders of magnitude of the major parameters and variables in the system of equations.

Parameters	Eddy fluxes	Velocities	Vorticities	Temperatures
$\beta \sim 10^{-11}$ m^{-1} s^{-1}	$\overline{u'v'} \sim 10$ m^2 s^{-2}	$\bar{u} \sim 10$ m s^{-1}	$\bar{\zeta} \sim 10^{-6}$ s^{-1}	$\sigma_\theta \sim 3$ K
$f \sim 10^{-4}$ s^{-1}	$\overline{v'\zeta'} \sim 10^{-5}$ m s^{-2}	$\sigma_u \sim 10$ m s^{-1}	$\sigma_\zeta \sim 10^{-5}$ s^{-1}	$\partial\bar{\theta}/\partial y \sim 3\times10^{-6}$ K m^{-1}
$g^* \sim 10^{-2}$ m s^{-2}	$\overline{\theta'v'} \sim 10$ mK s^{-1}	$\sigma_v \sim 6$ m s^{-1}	$\partial\bar{\zeta}/\partial y \sim 10^{-11}$ m^{-1} s^{-1}	
$h \sim 10^4$ m	$\overline{\theta'u'} \sim 10$ mK s^{-1}	$\partial\bar{u}/\partial y \sim 10^{-5}$ s^{-1}		
$K_s \sim 10^2$ m^2 s^{-1}				
$\tilde{R} \sim 10^{-5}$ K s^{-1}				
$T \sim 300$ K				

lay^{-1}). We thus need to use $g^*=10^{-2}$ m s^{-2}. The corresponding critical wavelength is about 12 500 km if $\beta \approx 10^{-11}$ (m s)$^{-1}$. Roughly speaking, waves above zonal wavenumber 2 (mid-latitude) are unstable. The effective acceleration of gravity g^* is about one-tenth of one percent of g; this corresponds to horizontal motion across isentropic surfaces whose slope is about 1:1000. These estimates appear to correspond fairly well with the numbers used in dynamic meteorology (see Table 1 for a complete list).

We can verify the validity of the numbers given above by scaling the source term in the vorticity equation. The divergence of the thermal wind should be of order 10^{-6} s^{-1} in order to be realistic. Therefore, $(g^*/T)\partial\theta/\partial x$ should be of order 10^{-10} s^{-2} in unstable baroclinic waves. If we estimate $\partial\theta/\partial x \approx 3\,K$ (1000 km)$^{-1}$, and take $T=300\,K$ as before, we also obtain $g^* \approx 10^{-2}$ m s^{-2}. The corresponding buoyant acceleration in the equations of motion is about 3×10^{-5} m s^{-2} if the temperature differences are small (1 K) and about 3×10^{-4} m s^{-2} if they are very large (10 K). The buoyancy forces thus tend to be fairly small (but not *very* small!) compared to the Coriolis and pressure gradient forces. The flow is quasi-geostrophic all the time, and the buoyancy forces are sufficiently strong to overcome the weak retarding effects of surface friction (see also next section).

4. Mean flow and eddy kinetic energy

We now turn to an analysis of the general circulation of our model. The climate of the model is defined by the meridional profile of the mean zonal wind $\bar{u}(y)$, the mean temperature distribution $\bar{\theta}(y)$, and the various variances and covariances of the fluctuations associated with the eddy motion. All averages used here are temporal averages of zonal averages; therefore the partial derivatives of all averaged variables with respect to x and t are zero.

The flow is confined between rigid boundaries at $y=\pm\frac{1}{2}L$. Since the velocity field is nondivergent, $\bar{v}$ has to be zero everywhere. Maintenance of the mean zonal wind requires that the convergence of the poleward eddy flux of zonal momentum be equal to the surface friction force, i.e.,

$$\frac{\partial\bar{u}}{\partial t}=0=-\frac{\partial}{\partial y}\overline{u'v'}-\frac{K_s}{h^2}\bar{u}. \tag{22}$$

Here $u'=u-\bar{u}$, $v'=v$ because $\bar{v}=0$, and averages are indicated by overbars. The mean momentum balance would be trivial in the absence of surface friction because the flow is nondivergent and does not allow circulations in the meridional-vertical plane. We also note that the eddy momentum flux in (22) has to be up the gradient, because the eddies have to supply the mean zonal flow with momentum in order to make up for the loss of momentum caused by surface friction.

Since the convergence of the eddy momentum flux is equal to the vorticity flux $\overline{v'\zeta'}$, Eq. (22) can also be written as

$$0=\overline{v'\zeta'}-\frac{K_s}{h^2}\bar{u}. \tag{23}$$

This equation suggests that mid-latitude cyclones ($\zeta'>0$ in the Northern Hemisphere) migrate toward the pole—because the mean zonal wind at middle latitudes blows from the west, storm tracks must be oriented toward the northeast.

The value of K_s can be estimated by taking the spindown time constant of the mean zonal flow to be of order 10^6 s (10 days) in the absence of an eddy vorticity flux. This means that $K_s\,h^{-2}$ has to be about 10^{-6} s^{-1}. If we take the scale height h to be 10 km, we conclude that K_s must be about 10^2 m^2 s^{-1}. Alternatively, if the kinematic surface stress is of order 0.1 (m s^{-1})^2, then the frictional deceleration is of order 10^{-5} m s^{-2} for a flow whose depth is 10 km. If $\bar{u}$ is of order 10 m s^{-1}, this again implies that $K_s/h^2 \approx 10^{-6}$ s. These estimates also determine the order of magnitude of $\overline{v'\zeta'}$. We find

$$\overline{v'\zeta'} \approx 10^{-5}\ \mathrm{m\ s^{-2}}. \tag{24}$$

Since $\sigma_v \approx 6$ m s^{-1} and $\sigma_\zeta \approx 10^{-5}$ s^{-1} (Blackmon *et al.*, 1977), the correlation between v' and ζ' is fairly poor (throughout this paper the symbol σ is used to represent the standard deviation of fluctuations). If the meridional scale of variation of $\overline{u'v'}$ is about 1000 km, the order of magnitude of $\overline{u'v'}$ must be

$$\overline{u'v'}\approx 10 \text{ m}^2\text{ s}^{-2}. \tag{25}$$

Since $\sigma_u\approx 10$ m s^{-1} and $\sigma_v\approx 6$ m s^{-1}, the correlation between u' and v' also is rather small (Blackmon *et al.*, 1977).

It is useful to determine the equations governing the mean thermal wind and the mean unbalanced wind. We note that $\bar v_T=0$ because $\partial\bar p/\partial x=0$; since $\bar v=0$, we also have $\bar v_A=0$. Since $\bar u_T$ is a function of y only, the mean thermal wind is nondivergent: $\nabla\cdot\bar{\mathbf{v}}_T=0$. The zonal component of the mean thermal wind is given by

$$0=-\frac{1}{\rho}\frac{\partial\bar p}{\partial y}-f\bar u_T+\frac{g^*}{T}\bar\theta; \tag{26}$$

the value of $\bar u_A$ is determined by

$$f(\bar u-\bar u_T)=f\bar u_A=-\frac{\partial}{\partial y}\sigma_v^2. \tag{27}$$

The pressure has extrema at latitudes close to those at which $\bar u_T$ changes sign; this creates a subtropical high pressure belt south of the midlatitude westerlies in the Northern Hemisphere and a low pressure belt north of the jet stream.

In our model the meridional profile of $\overline{v'\zeta'}$ is exactly the same as that of $\bar u$. This condition imposes severe constraints on the profiles of $\bar u$, $\bar\zeta$, $\overline{u'v'}$ and $\overline{v'\zeta'}$ across the zonal channel. If the mean wind profile is symmetric about the centerline of the channel ($y=0$), the various meridional profiles must look like the ones sketched in Fig. 4. For future reference, we note that there are two regions (shaded) near the boundaries of the channel where the sign of $\overline{u'v'}$ opposes that of $\partial\bar u/\partial y$, while there are also two narrow belts (solid black) where the sign of $\overline{v'\zeta'}$ opposes that of $\partial\bar\zeta/\partial y$.

Since $K_s\,h^{-2}\approx 10^{-6}$ s^{-1}, the surface friction terms in the primitive equations are of order 10^{-5} m s^{-2}. This is two orders of magnitude smaller than the leading terms; it would thus be justified to ignore surface friction for the purpose of short-term forecasts. However, surface friction cannot be neglected in studies of the climate of the model for the simple reason that (22) and (23) would become trivial if friction were absent.

We now are ready for a first look at the energetics of the general circulation. The equation for the maintenance of eddy kinetic energy is obtained by straightforward manipulation. It reads

$$\frac{\partial}{\partial t}(\overline{\tfrac{1}{2}q^2})=0=-\overline{u'v'}\frac{\partial\bar u}{\partial y}+\frac{g^*}{T}\overline{\theta'v'}-\frac{K_s}{h^2}\overline{q^2}-\frac{\partial}{\partial y}(\overline{\tfrac{1}{2}q^2v'+p'v'/\rho}). \tag{28}$$

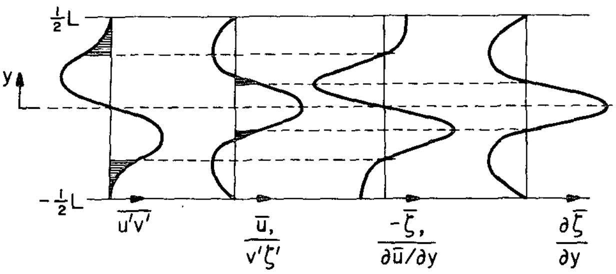

FIG. 4. Schematic distributions of mean zonal wind, mean relative vorticity, mean vorticity gradient, eddy momentum flux and eddy vorticity flux associated with a symmetric zonal-wind profile in the channel between $y=-\frac{1}{2}L$ and $y=\frac{1}{2}L$. Since the boundaries are rigid, all eddy fluxes must vanish there. The relevance of the shaded regions is explained in the text.

Here $\overline{\tfrac{1}{2}q^2}=\tfrac{1}{2}(\sigma_u^2+\sigma_v^2)$ is the mean kinetic energy per unit mass. With the aid of the definitions of the thermal and unbalanced winds, (28) can also be written as

$$\frac{\partial}{\partial t}(\overline{\tfrac{1}{2}q^2})=0=-\overline{u'v'}\frac{\partial\bar u}{\partial y}+f(\overline{u'v_A'}-\overline{v'u_A'})-\frac{K_s}{h^2}\overline{q^2}-\frac{\partial}{\partial y}(\overline{\tfrac{1}{2}q^2v'}). \tag{29}$$

The kinetic energy budget (28) demonstrates the energy conversion processes that are necessary to maintain the general circulation of the model. Fig. 4 shows that the sign of $\overline{u'v'}$ is almost everywhere the same as that of $\partial\bar u/\partial y$, except near the two zonal walls where both the flux and the gradient are small. The first term of (28) thus is a sink for eddy kinetic energy. This conversion of eddy kinetic energy into the kinetic energy of the mean zonal flow represents one aspect of the reverse energy cascade in two-dimensional flow; it is required to maintain the mid-latitude jet against surface friction. The model thus faithfully imitates one of the principal energy conversions of the atmosphere's general circulation. We note that the conversion rate is largest in the belts where the mean zonal wind changes sign. In the subtropical high pressure belt, for example, both the eddy momentum flux and the mean shear are large (Fig. 4). The conversion rate $\overline{u'v'}\;\partial\bar u/\partial y$ thus has a very pronounced peak there. This suggests that the eddy kinetic energy will be fairly small in the subtropics, and that the flux divergence term of (28) will have to make up for some of the loss. The flux of eddy kinetic energy is likely to be down the energy gradient.

The only source term in (28) is the conversion rate $(g^*/T)\overline{\theta'v'}$ of eddy potential energy into eddy kinetic energy. The mean heat flux is from the equator toward the pole, so that $\overline{\theta'v'}>0$ everywhere, with a broad maximum near the center of the channel if the distribution of the diabatic heating is symmetric with respect to $y=0$. Eq (29) sheds more light on the dynamical processes involved. As is evident from Figs. 2

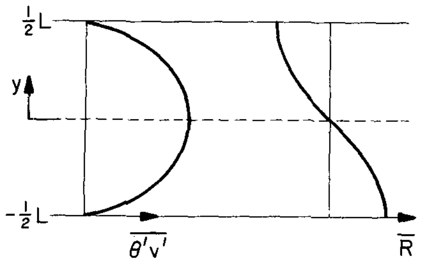

FIG. 5. The distribution of the poleward eddy heat flux can be specified at wish by selecting an appropriate distribution of the mean diabatic heating rate $\bar{R}$. Energy conservation requires that the integral of $\bar{R}$ across the channel be zero.

and 3, positive values of v' in growing baroclinic eddies tend to be associated with negative values of u_A', and vice versa. The term $-f\overline{v'u_A'}$ thus tends to be positive in active storms. In the same way, the eastward acceleration of the flow in the warm sector of a developing cyclone must be associated with positive values of v_A', because $du/dt=fv_A'-(K_s/h^2)u$. The deceleration in the cold sector, on the other hand, is associated with negative values of v_A'. The term $f(\overline{u'v_A'})$ is thus also likely to be positive in the development stage of mid-latitude storms. We note, however, that it is difficult to speculate on the sign of $f(\overline{u'v_A'}-\overline{v'u_A'})$ outside active baroclinic disturbances.

It is not difficult to see that the principal source and sink terms in (29) will roughly balance each other. Since v_A' is typically about 10% of v', we must estimate that $\overline{u'v'}\approx 10\ \overline{u'v_A'}$. The mean relative vorticity $\bar{\zeta}$, on the other hand, is of order 10^{-5} s^{-1}, while $f\approx 10^{-4}$ s. The term $\overline{u'v'}\,\bar{\zeta}$ balances the term $f\,\overline{u'v_A'}$ because the two factors of ten involved cancel each other. It appears possible to maintain an approximate balance.

Eventually all kinetic energy is dissipated by surface friction. If we include the frictional energy losses of the mean zonal flow, an overall kinetic energy balance [which ignores the flux-divergence terms of (28) because the fluxes vanish at the boundaries of the channel] may be approximated by

$$\frac{g^*}{T}\overline{\theta'v'}\approx\frac{K_s}{h^2}(\bar{u}^2+\sigma_u{}^2+\sigma_v{}^2). \qquad (30)$$

If $K_s/h^2=10^{-6}$ s^{-1}, $\bar{u}=10$ m s^{-1}, $\sigma_u=10$ m s^{-1}, $\sigma_v=6$ m s^{-1}, $g^*=10^{-2}$ m s^{-2} and $T=300$ K, we obtain

$$\overline{\theta'v'}\approx 10 \text{ m K s}^{-1}. \qquad (31)$$

This estimate agrees with the data of Blackmon *et al.* (1977). Assuming that the correlation coefficient between θ' and v' is about ½ (a value borrowed from free convection in the atmospheric boundary layer) and that $\sigma_v\approx 6$ m s^{-1}, we find that σ_θ must be about 3 K. This seems altogether reasonable.

5. Heat flux and temperature variance

The poleward eddy flux of heat is determined by the diabatic heating term R in (4). Taking the average of (4), we find

$$\frac{\partial}{\partial y}(\overline{\theta'v'})=\bar{R}. \qquad (32)$$

Since the heat flux must vanish at the boundaries of the zonal channel, the meridional profiles of $\overline{\theta'v'}$ and $\bar{R}$ look like those sketched in Fig. 5. The shape of the heat-flux profile is fairly insensitive to the distribution of $\bar{R}$ because the former is an integral of the latter. The meridional scale of variation of $\overline{\theta'v'}$ near the boundaries presumably is about 1000 km. Since $\overline{\theta'v'}\approx 10$ m K s^{-1} by virtue of (31), the extrema of $\bar{R}$ are of order 10^{-5} K s^{-1}. This corresponds to diabatic heating and cooling rates of about 1 K day^{-1} near the boundaries of the channel.

If we ignore fluctuations in the diabatic heating rate R, the maintenance equation for temperature variance reads

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\sigma_\theta{}^2)=0=-\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{\theta'\theta'v'})-\overline{\theta'v'}\frac{\partial\bar{\theta}}{\partial y}-\epsilon_\theta. \qquad (33)$$

The first term on the right-hand side of (33) is a flux-divergence term; it can only redistribute temperature variance inside the zonal channel because the flux must vanish at the boundaries. The second term of (33) is a source for temperature variance because $\overline{\theta'v'}>0$ and $\partial\bar{\theta}/\partial y<0$ everywhere if the lower wall is heated and the upper wall is cooled. It is clear that we need a sink term, too.

The only sink term in (33) is ϵ_θ, which stands for the destruction of temperature variance caused by the spectral cascade of temperature fluctuations in an almost nonconducting gas. In two-dimensional turbulence, the cascade mechanism for scalar contaminants is similar to the enstrophy cascade (Kraichnan, 1976) because stochastic flow fields in two dimensions tend to intensify scalar fluctuation gradients in much the same way as vorticity gradients. The cascade rate presumably is determined by the large-scale dynamics of the flow field, not by the exact value of the thermal diffusivity if the latter is sufficiently small. This means that the numerical diffusivity needed for the solution of the primitive equations on a grid with finite mesh size will not significantly harm the dynamics if the spatial resolution is adequate.

Eq. (31) states that $\overline{\theta'v'}$ must be of order 10 m K s^{-1}. If the mean temperature gradient $\partial\bar{\theta}/\partial y$ is of order 3 K (1000 km)$^{-1}$, the source term in (33) must be of order 3×10^{-5} K^2 s^{-1}. In view of the analogy with the enstrophy budget (which will be discussed later), it appears reasonable to estimate ϵ_θ as

$$\epsilon_\theta\approx\tfrac{1}{3}\sigma_\theta{}^2\sigma_\zeta. \qquad (34)$$

With $\sigma_\theta \approx 3$ K and $\sigma_\xi \approx 10^{-5}$ s, the dissipation term must be of order 3×10^{-5} K^2 s^{-1}. This agrees with our estimate for the source term. We conclude that the temperature variance cascade is vigorous enough to maintain a balance in (33).

The key feature of the temperature variance equation is that the potential energy available in the mean meridional temperature gradient is converted into eddy potential energy. In this model eddy potential energy is not proportional to the temperature variance, but to the standard deviation of temperature. There appears to be no immediate analogy with the concept of available potential energy in atmospheric dynamics. It is here in particular that we pay a high price for the simplicity of our Boussinesq model.

6. Flux maintenance dynamics

So far our discussion has skirted around the closure problem of turbulence dynamics. It is easy enough to write budget equations for velocity and temperature variance, but those equations contain the covariances $\overline{u'v'}$ and $\overline{\theta'v'}$. If we appeal to some gradient-transfer hypothesis (turbulent "mixing" or "diffusion"), it seems reasonable to expect that $\overline{\theta'v'}$ must be positive when $\partial\bar{\theta}/\partial y$ is negative. However, the eddy momentum flux $\overline{u'v'}$ is not directed *down* the gradient $\partial\bar{u}/\partial y$, but *up* the gradient most everywhere. How is a countergradient momentum flux maintained? It seems inevitable to seek wisdom from the maintenance equations for $\overline{\theta'v'}$ and $\overline{u'v'}$. However, those equations will introduce yet other covariances, and it is not at all clear that sensible closure assumptions can be made at that level of sophistication. Nevertheless, we are convinced that climate models will have to deal with the physics of flux maintenance, and that it will be necessary to face squarely the parameterization problems that arise in this context.

The maintenance equation for $\overline{\theta'v'}$ is

$$\frac{\partial}{\partial t}(\overline{\theta'v'})=0=-\sigma_v^2\frac{\partial\bar{\theta}}{\partial y}+\frac{g^*}{T}\sigma_\theta^2-\frac{K_s}{h^2}\overline{\theta'v'}-\frac{\partial}{\partial y}(\overline{\theta'v'v'})-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial y}}-f\overline{\theta'u'}. \quad (35)$$

Fortunately, the structure of this equation is more transparent if the thermal balance equations (7) and (8), and the definition (13) of the unbalanced wind, are used. We obtain

$$\frac{\partial}{\partial t}(\overline{\theta'v'})=0=-\sigma_v^2\frac{\partial\bar{\theta}}{\partial y}-f\overline{\theta'u_A'}-\frac{K_s}{h^2}\overline{\theta'v'}-\frac{\partial}{\partial y}(\overline{\theta'v'v'}). \quad (36)$$

An order-of-magnitude analysis shows that the surface friction term can be ignored compared to the leading source term. The latter represents turbulent "mixing" of the mean temperature field: the mere presence of σ_v^2 tends to create a poleward heat flux if $\partial\bar{\theta}/\partial y$ is negative.

The flux divergence term of (36) represents redistribution of $\overline{\theta'v'}$ inside the channel (the flux vanishes at the boundaries). Since surface friction is an insufficient sink, the Coriolis term in (36) must be the major sink term. This poses a problem. In growing baroclinic disturbances warm air ($\theta'>0$) tends to be associated with negative values of u_A', while cool air ($\theta'<0$) tends to be correlated with positive values of u_A' (see Figs. 2 and 3). In developing storms the term $-f\overline{\theta'u_A'}$ is thus positive, adding to the effects of the mixing term and creating very large local values of $\overline{\theta'v'}$ because there is no appropriate sink. This implies that a significant fraction of $\overline{\theta'v'}$ must be associated with developing storms.

On the average, however, the Coriolis term in (36) must be negative. This suggests that it probably switches sign as a storm enters its decay stage, and that the flux-divergence term evens out the differences. Numerical experiments with the general circulation of the model may give the data necessary to decide what is happening. They may also give further hints for suitable closure approximations.

If the maintenance of $\overline{\theta'v'}$ leads to difficult questions, what about the maintenance of $\overline{u'v'}$? The budget equation for $\overline{u'v'}$ reads

$$\frac{\partial}{\partial t}\overline{u'v'}=0=-\sigma_v^2\frac{\partial\bar{u}}{\partial y}+\frac{g^*}{T}\overline{\theta'u'}-2\frac{K_s}{h^2}\overline{u'v'}+f(\sigma_v^2-\sigma_u^2)-\frac{1}{\rho}\overline{\left(v'\frac{\partial p'}{\partial x}+u'\frac{\partial p'}{\partial y}\right)}-\frac{\partial}{\partial y}(\overline{u'v'v'}). \quad (37)$$

The first term in (37) tends to create a down-gradient momentum flux, and it seems reasonable to expect that the buoyancy term—which features the zonal eddy heat flux—must be the principal source of countergradient momentum fluxes. However, it will be hard to make further progress before the results of general circulation experiments with this model are available. The ultimate goal, of course, is to find a parameterized version of the entire system of flux-maintenance equations which does justice to the energetics of the general circulation.

In the budget for $\overline{u'v'}$, the major source term involves the zonal eddy heat flux. The maintenance equation for $\overline{\theta'u'}$ is

$$\frac{\partial}{\partial t}\overline{\theta'u'}=0=-\overline{u'v'}\frac{\partial\bar{\theta}}{\partial y}-\overline{\theta'v'}\frac{\partial\bar{u}}{\partial y}-\frac{K_s}{h^2}\overline{\theta'u'}-\frac{\partial}{\partial y}(\overline{\theta'u'v'})-\overline{\frac{\theta'}{\rho}\frac{\partial p'}{\partial x}}+f\overline{\theta'v'}. \quad (38)$$

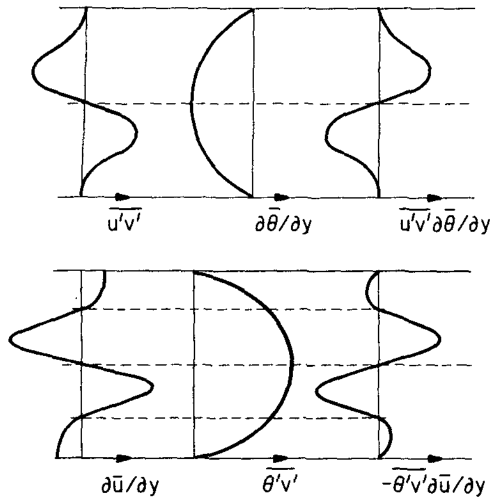

FIG. 6. The meridional distributions of the first two terms in the budget equation for the zonal eddy heat flux $\overline{\theta' u'}$. Apart from the immediate vicinity of the boundaries, the two curves look quite similar; numerical estimates show that the values of $\overline{\theta' v'}\ \partial\bar{u}/\partial y$ and $\overline{u'v'}\ \partial\bar{u}/\partial y$ are comparable.

South of the jet-stream axis, $\overline{u'v'}$ and $\partial\bar{u}/\partial y$ are positive (Fig. 4); they reverse sign on the other side of the jet. The poleward heat flux $\overline{\theta' v'}$ is positive everywhere and the mean temperature gradient $\partial\bar{\theta}/\partial y$ is negative. The term $-\overline{u'v'}\ \partial\bar{\theta}/\partial y$ thus is positive south of the jet axis, while the term $-\overline{\theta' v'}\ \partial\bar{u}/\partial y$ is negative there. The roles of the two terms are reversed on the poleward flank of the jet. The distributions of the first two terms in (38) are sketched in Fig. 6. The two curves are strikingly similar, except for the way in which they approach the boundaries of the zonal channel. It is very tempting to postulate that the principal balance mechanism in (38) is

$$\overline{u'v'}\frac{\partial\bar{\theta}}{\partial y}\approx -\overline{\theta' v'}\frac{\partial\bar{u}}{\partial y}. \qquad (39)$$

Order-of-magnitude estimates confirm that these two terms are comparable. The crude balance expressed by (39) implies that

$$\frac{\overline{u'v'}}{\partial\bar{u}/\partial y}\approx -\frac{\overline{\theta' v'}}{\partial\bar{\theta}/\partial y}. \qquad (40)$$

These ratios define the meridional eddy exchange coefficients for momentum and heat. We find

$$K_M\approx -K_H. \qquad (41)$$

Since $\overline{\theta' v'}\approx 10$ m K s^{-1} and $\partial\bar{\theta}/\partial y\simeq 3\times 10^{-6}$ K m^{-1} (Table 1), the eddy heat diffusivity is of order 3×10^6 m^2 s^{-1}. This value agrees with those used in mean-motion models of the general circulation (Green, 1969).

A negative eddy viscosity, however, does not begin to explain the complicated energetics of the general circulation. We must refrain from further speculations until the results of numerical experiments become available.

7. Enstrophy maintenance and cascade dynamics

We now turn to a further discussion of the vorticity dynamics of the model. The enstrophy of the eddy motion is governed by

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\sigma_\zeta^2)=0=-\left(\beta+\frac{\partial\bar{\zeta}}{\partial y}\right)\overline{v'\zeta'}+\frac{g^*}{T}\overline{\zeta'\frac{\partial\theta'}{\partial x}}$$
$$-\frac{\partial}{\partial y}(\overline{\tfrac{1}{2}\zeta'\zeta' v'})-\frac{K_s}{h^2}\sigma_\zeta^2-\chi. \qquad (42)$$

Here χ represents viscous destruction of enstrophy, based on the presumed existence of an enstrophy cascade. We recall that the fluid is assumed to be almost inviscid and almost nonconducting, so that all viscous terms and heat diffusion terms in the system of equations are negligible, except for those related to the spectral cascades of enstrophy and temperature variance.

The first term in (42) depends on the poleward gradient of the mean absolute vorticity $\bar{\zeta}_a=f+\bar{\zeta}$. The flow is barotropically stable everywhere if the sign of $\partial\bar{\zeta}_a/\partial y=\beta+\partial\bar{\zeta}/\partial y$ is the same all through the channel. If this is the case (there is no reason to expect otherwise), there is a very rapid loss of enstrophy to the mean zonal flow in the center of the channel (Fig. 4) because $\overline{v'\zeta'}$ and $\partial\bar{\zeta}/\partial y$ both have maxima there. There are, however, small gains near the two walls. The overall picture, of course, is consistent with a reverse energy cascade; it is worth noting, however, that the geometry of the situation is such that a reverse enstrophy cascade, away from the wavenumbers of greatest baroclinic forcing, is not excluded.

The second term of (42) represents baroclinic forcing; it is the primary source of eddy enstrophy in the system. Since the eddy enstrophy suffers losses at both extremes of the spectrum, it could not survive without the baroclinic source term.

The orders of magnitude of some of the terms in (42) are (Table 1)

$$\left(\beta+\frac{\partial\bar{\zeta}}{\partial y}\right)\overline{v'\zeta'}\approx 10^{-16}\ \mathrm{s}^{-3} \qquad (43)$$

$$\frac{K_s}{h^2}\sigma_\zeta^2\approx 10^{-16}\ \mathrm{s}^{-3} \qquad (44)$$

$$\chi\approx\tfrac{1}{3}\sigma_\zeta^3\approx 3\times 10^{-16}\ \mathrm{s}^{-3}. \qquad (45)$$

The factor $\frac{1}{3}$ in the estimate for χ is based on the assumption that the enstrophy cascade is fairly inefficient. We have to keep in mind, however, that the estimate for X involves raising σ_ζ to the third power; the number given in (45) can easily be off by a factor of 10.

With the aid of the thermal wind equations, two of the terms in (42) can be put in the alternative form

$$-\beta\overline{v'\zeta'}+\frac{g^*}{T}\overline{\zeta'\frac{\partial\theta'}{\partial x}}=-\beta\overline{v'_A\zeta'}-f\overline{\zeta'(\nabla\cdot\mathbf{v}'_A)}. \qquad (46)$$

The first of the two terms on the right-hand side is of order 10^{-17} s^{-3} if ζ' is no better correlated with v'_A than with v' (recall that $v'\approx 10v'_A$); we will neglect it. The second term is estimated by taking the convergence of the thermal wind to be of order 10^{-6} s^{-1} and assuming that the correlation coefficient between ζ' and $\nabla\cdot\mathbf{v}'_A$ is about one-third. This yields

$$-f\overline{\zeta'(\nabla\cdot\mathbf{v}'_A)}\approx 3\times 10^{-16}\ \text{s}^{-3}. \qquad (47)$$

The baroclinic generation of eddy enstrophy is thus capable, at least to within an order of magnitude, to balance the losses due to the enstrophy cascade. The flux-divergence term in (42) is of order 10^{-16} s^{-3}; it can handle the redistribution needed to cope with the large loss of eddy enstrophy to the mean zonal flow in the center of the channel. It appears that the entire system of equations is dynamically consistent, at least at the level of accuracy obtainable without support of hard data from numerical experiments.

The source and sink terms in (42) do pose other problems, however. An inertial subrange with a conservative enstrophy cascade can exist only if there are no sources or sinks of enstrophy at scales of motion that are small compared to those containing most of the kinetic energy, or if at each wavenumber the various sources and sinks balance each other exactly. The first alternative is ruled out here because the surface friction term $(K_s/h^2)\sigma_\zeta^2$ does not discriminate among scales of motion. If, for example, the kinetic energy spectrum has a k^{-3} range, then surface friction removes enstrophy at the same rate in each octave of that range. In that case a conservative enstrophy cascade can be maintained only if the spectrum of the baroclinic forcing term in (59) exactly balances that of the frictional losses and if the enstrophy removal by the mean vorticity gradient becomes negligible at small scales. The first condition implies that the contributions to $\overline{\zeta'\partial\theta'/\partial x}$ made by each octave of the spectrum must be independent of wavenumber; the second means that the contributions to $\overline{v'\zeta'}$ associated with small scales of motion must negligible.

The linear stability analysis in Section 3 suggests that the growth rate of small waves may be roughly independent of wavenumber. If that conclusion could be carried over to the nonlinear stability characteristics of the system, the spectrum of $\overline{\zeta'\partial\theta'/\partial x}$ would have the shape necessary to balance frictional losses throughout the inertial subrange. A qualitative balance, however, is not good enough; also, there is no justification for assuming that the behavior of turbulence is similar to that of linearized waves.

The spectrum of $\overline{v'\zeta'}$, on the other hand, probably will behave in the desired manner. In a k^{-3} range, the contribution to $\overline{v'\zeta'}$ made by each octave of the spectrum is proportional to the central wavelength of that octave if the coherence between v' and ζ' is independent of wavenumber. If there is any tendency toward small-scale isotropy, however, the coherence will decrease as the wavelength decreases ($\overline{v'\zeta'}=0$ in isotropic turbulence). The spectrum of $\overline{v'\zeta'}$ may thus be expected to decrease quite rapidly with increasing wavenumbers, and it is not unlikely that the reverse spectral enstrophy flux associated with $\overline{v'\zeta'}(\beta+\partial\bar{\zeta}/\partial y)$ will be quite small at small scales of motion.

The issues raised here clearly deserve further analysis. In two-dimensional turbulence with thermal forcing and surface friction, will there be any small-scale isotropy at all? Will the energy spectrum exhibit a k^{-3} subrange? Will the temperature variance exhibit a spectrum similar to that of enstrophy? Are there any ranges in which the spectral fluxes of enstrophy and/or energy are negligible? Are there ranges in which nonzero spectral fluxes violate conservation principles? At this stage we have no answers to these questions: we hope that they will be studied by researchers better versed in the spectral theory of two-dimensional turbulence.

We have assumed that enstrophy and temperature variance are cascaded toward the smallest scales of motion, where they are annihilated by molecular friction and heat conduction. In numerical integrations of the primitive equations, the molecular diffusivities can be replaced by the numerical diffusion coefficients which are needed to avoid discontinuities at scales comparable to the mesh size of the integration grid. This should not affect the behavior of the system significantly if the mesh size is sufficiently small, because the cascade rates of enstrophy and temperature variance are determined by the large-scale dynamics of the flow field. The diffusivities involved determine the smallest scale of motion; they do not significantly affect the large-scale dynamics of the system at sufficiently large Reynolds numbers.

The very concept of spectral cascades of enstrophy and temperature variance, however, implies that the largest vorticity and temperature gradients will be associated with scales of motion that are just large enough to avoid rapid attenuation of gradients by diffusion. In other words, the system will generate "fronts" characterized by appreciable vorticity and temperature changes, and most of the dissipation of enstrophy and temperature variance will occur in frontal zones. This line of reasoning adds another dimension to the study of frontogenesis and frontolysis. Will frontogenesis be rapid enough to ensure that enstrophy and temperature variance are destroyed at the required rates? What is the life cycle of a typical frontal zone in our model? Will the mid-latitude cyclones in the model develop cold and warm fronts similar to those associated with storms in the earth's atmosphere? Will the presence of frontal zones affect the shape of the various spectra at large wavenumbers? We emphasize that these are questions related to the maintenance of the general circulation; a statistically steady regime cannot be maintained without adequate sinks for enstrophy and temperature variance.

8. Conclusions

In this paper we have explored the general circulation of a one-level model of atmospheric motion equipped with a thermodynamic energy cycle which faithfully imitates several features of the thermodynamics of atmospheric motion. The model employs two basic equations, a vorticity equation and a Boussinesq approximation to the first law of thermodynamics. In complexity our model is thus comparable to a two-level baroclinic system of equations. The model obviously is at a disadvantage compared to multilevel baroclinic models, because it cannot represent the interactions between the vertical and horizontal fields of motion that help to shape the general circulation in the earth's atmosphere. On the other hand, it seems logical to expect that barotropic models might perform much better if they were equipped with a simulated thermodynamic cycle of the kind explored in this paper. Since the additional computer time involved would not be large, it appears that some experimentation in this direction would be useful.

The equations studied here illustrate the energetics of the general circulation in a particularly simple way. A reverse energy cascade is needed to maintain the mean zonal flow against surface friction, eddy kinetic energy has to be maintained by baroclinic effects, the simulated thermodynamics is necessary to maintain a poleward heat flux, and so on. The model appears to be energetically consistent in all features that have been investigated here, and it shows clearly that the maintenance of the general circulation depends in part on the spectral cascades of enstrophy and temperature variance. Because the model has fairly realistic energy cycles, it may help to give new momentum to the study of the spectral theory of two-dimensional turbulence. No longer is it necessary to restrict that theory to isotropic fields with artificial forcing at preassigned wavenumbers. Our equations also show which covariances are involved in the maintenance of the flow field; this provides a background for attempts to develop mean-motion models of the general circulation.

At this stage, it is not clear whether the problems we have encountered will seriously affect the applicability of the model. Of particular concern is the absence of a high-wavenumber cutoff in the baroclinic forcing term. Without numerical experiments, we have no way of knowing whether the nonlinear characteristics of the model will concentrate most of the baroclinic effects in a narrow range of wavenumbers. This issue is related to the absence or presence of isotropy at small scales. Here, too, our insight is inadequate at this time.

Acknowledgments. A first draft of this paper was written while the author was a 1976 Summer Guest at the Research Establishment Risø, Denmark; the hospitality of Dr. N. E. Busch and his coworkers is acknowledged with gratitude. The paper grew toward its final form during a six-month stay at the University of Washington; Drs. J. R. Holton and J. M. Wallace provided much-needed impedance. Thanks are due also to Dr. J. A. Dutton at the Pennsylvania State University, whose insistence on energetic consistency helped to move the author out of the doldrums of two-dimensional turbulence kinematics. This research was supported by the Atmospheric Sciences Section of the National Science Foundation under Grants DES76-13357(PSU) and DES-ATM76-00855(UW).

REFERENCES

Batchelor, G. K., 1969: Computation of the energy spectrum in homogeneous two-dimensional turbulence. *Phys. Fluids*, **12**, Suppl. 2, 233–239.

Blackmon, M. L., J. M. Wallace, N. C. Lau and S. L. Mullen, 1977: An observational study of the Northern Hemisphere wintertime circulation. *J. Atmos. Sci.*, **34** (in press).

Fjørtoft, R., 1953: On the changes in the spectral distribution of kinetic energy for two-dimensional, non-divergent flow. *Tellus*, **5**, 225–230.

Fleagle, R. G., 1955: Instability criteria and growth of baroclinic disturbances. *Tellus*, **7**, 168–176.

Gall, R. L., 1976: A comparison of linear baroclinic instability theory with the eddy statistics of a general circulation model. *J. Atmos. Sci.*, **33**, 349–373.

Green, J. S. A., 1969: Transfer properties of the large-scale eddies and the general circulation of the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **96**, 157–185.

Holton, J. R., 1972: *An Introduction to Dynamic Meteorology.* Academic Press, 319 pp.

Kraichnan, R. H., 1967: Inertial ranges in two-dimensional turbulence. *Phys. Fluids*, **10**, 1417–1423.

Kraichnan, R. H., 1976: Eddy viscosity in two and three dimensions. *J. Atmos. Sci.*, **33**, 1565–1536.