On the Radiative Damping of Atmospheric Waves

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ABSTRACT

General expressions for computing time constants for radiative decay of harmonic temperature perturbations in planetary atmospheres are developed. Previous studies have not taken into account the inhomogeneity of planetary atmospheres or, more important, the role of the planetary surface. These omissions lead to significant overestimates of the radiative time constant in a number of cases. Four spatial scales are shown to be generally important: the scale height of the atmospheric absorber, the absorption mean-free-path for thermal radiation, the altitude above the planetary surface, and the wavelength of the temperature perturbation. Atmospheric inhomogeneity is particularly important when the radiation mean free path and the vertical wavelength divided by 2\pi both exceed the absorber scale height. The surface is very important for decay of waves at altitudes less than a few radiation mean free paths. The effect of the surface depends on a comparison of the surface response time and the lifetime of the atmospheric perturbation. The surface response time depends on the conducting and emitting properties of the surface material and on the strength of turbulence in the planetary boundary layer. Highly conducting surfaces have very long response times and lead to much shorter radiative time constants than insulated surfaces which have very short response times. The additional influence of chemical reactions and phase changes on radiative damping is discussed and several assumptions inherent in the development of the general expressions for time constants are evaluated. Terrestrial examples are used for purposes of illustration but the development is kept sufficiently general so that the results remain applicable to most situations on other planets.

1. Introduction

The time constant for decay of temperature perturbations by purely radiative processes in the atmosphere is an important practical and conceptual quantity in dynamical meteorology. It plays significant roles in classical studies of the onset of cellular convection (Goody, 1956; Spiegel, 1960), the onset of turbulence (Townsend, 1958), the radiative-diffusive boundary layer (Goody, 1964, p. 350), and the propagation of linear waves (Lindzen and McKenzie, 1967; Dickinson, 1969). It is also an important quantity for scale analysis of the thermodynamic equation (Gierasch et al., 1970) and provides a rational basis for the widely utilized "Newtonian cooling" law in atmospheric models (e.g., Stone, 1972; Cunnold et al., 1975).

The definition of a radiative time constant first involves a linearization of the relationship between the perturbation temperature decay rate and the perturbation temperature itself. One method for obtaining this time constant is empirical; it entails computer studies of the damping of very small perturbations from some "standard" temperature profile in a full radiative transfer model of the atmosphere (Dickinson, 1973). This method should provide very reliable estimates but suffers from the fact that the resultant time constants are valid only when the atmospheric temperatures closely resemble the "standard" profile. The method also becomes very cumbersome if we also want to know the dependence of the radiative time constant on the spatial scales of the temperature perturbations.

An analytical approach first developed by Spiegel (1957) provides a useful complement to the empirical method and also provides physical insight by directly showing the importance of spatial scales in determining radiative time constants. In this approach any arbitrary temperature perturbation is first expressed as a Fourier sum of simple-harmonic perturbations. The problem then reduces to one of determining the time constant for decay of a single simple-harmonic temperature perturbation in the atmosphere. Spiegel (1957) specifically obtained an analytical solution for an infinite homogeneous grey-absorbing medium. This solution was developed primarily to approximate conditions in stellar atmospheres but it has also been subsequently utilized in either a grey or non-grey absorbing context to provide a first approximation for radiative time constants in planetary atmospheres (Goody, 1964; Goody and Belton, 1967; Gierasch and Goody, 1969; Gierasch et al., 1970; Stone, 1972). This extension to planetary atmospheres has proved extremely beneficial but the accuracy of the procedure has not been investigated in any great detail.

In this paper less restrictive assumptions than those chosen by Spiegel (1957) will be made. The aim will
be to obtain expressions for radiative time constants which take into account the special characteristics of planetary atmospheres. These will include an exponential decrease of density with altitude and, more important, the presence of an absorbing, radiating and conducting surface layer. We will also discuss the influence of chemical reactions and phase changes on these time constants and critically discuss terms omitted from the perturbation analysis.

The homogeneous situation studied by Spiegel (1957) indicated that the radiative time was essentially constant when the radiation mean free path was much greater than the wavelength of the atmospheric temperature perturbation (the "transparent" case), while it increased as the square of the perturbation wavelength when the radiation mean free path was much less than the perturbation wavelength (the "opaque" case). We will demonstrate that this conclusion is correct for planetary atmospheres only in a restricted case. This restricted case requires that the radiation mean free path be much less than both the scale height for atmospheric absorbers and the vertical displacement from the planetary surface. In other cases the Spiegel (1957) formula may lead to errors in the radiation time constant which exceed an order of magnitude.

Our approach to this problem will initially involve a first-order perturbation analysis appropriate for the planetary problem. We will then present numerical solutions which will demonstrate how the time constant varies with a number of important parameters. The problem of the surface response to atmospheric waves will then be analyzed in some detail. Finally, we will discuss a number of limitations and complications to the theory, and present a few simple terrestrial applications. We have kept the analysis sufficiently general so that the results remain applicable to a wide variety of situations on both the earth and other planets.

2. Linear first-order perturbation theory

The volume radiative heating rate $q_0(0)$, at a point defined by a position vector $r = 0$ in a plane-parallel atmosphere in local thermodynamic equilibrium, is simply obtained by integrating the general radiative transfer equation derived in Chandrasekhar (1960) over the angles $\theta$ and $\phi$ as defined in Fig. 1 and over all frequencies $\nu$, i.e.,

$$q_0(0) = \rho(0) C_\nu \frac{\partial T_0(0)}{\partial t},$$

$$= \int_0^\infty \int_0^{2\pi} \int_0^\pi \alpha(0)[I^0_\nu(0, \theta, \phi) - B^0_\nu(0)] \times \sin \theta d\theta d\phi d\nu + \int_0^\infty \alpha(0) J_\nu(0, \theta_0, \phi_0) d\nu \right\}.$$ (1)

Here $r_\nu$ is the distance to the ground in the direction defined by $\theta$ and $\phi$, $B^0_\nu(T_{\nu 0})$ is the unperturbed Planck function for the ground at temperature $T_{\nu 0}$, and

$$T_\nu(\nu) = \exp \left[ - \int_0^\infty \alpha_\nu(\nu) d\nu \right],$$ (3)

$$= \exp \left[ - r_\nu(\nu) \right].$$

where $r_\nu(\nu)$ is the unperturbed optical depth measured from $r = 0$. In (2) we assume no incident diffuse radiation at the top of the atmosphere. Finally, the incident unperturbed direct solar radiation intensity $J_\nu(0, \theta_0, \phi_0)$ required in (1) is simply obtained by Beers law, viz.,

$$J_\nu(0, \theta_0, \phi_0) = J_\nu(\infty, \theta_0, \phi_0) T_\nu(\infty, \theta_0, \phi_0).$$ (4)

We now consider a harmonic temperature perturbation at time $t = 0$ in the atmosphere with amplitude $T' \ll T_0(\theta)$, wavenumber vector $K$, and origin at $r_0$, i.e.,

$$T(\theta) = T_0(\theta) + T' \exp[iK \cdot (r - r_0)].$$ (5)
The resultant change in the Planck function is linearly approximated by retaining only the first two terms in a Taylor expansion:

\[ B_0(x) \approx B_0^{0}(x) + \left( \frac{\partial B_0(x)}{\partial T} \right) \tau_0 \exp[iK \cdot (r - r_0)]. \tag{6} \]

For the moment we assume that alterations to \( \sigma_r(x), \rho_o(x), \rho(x) \) and \( C_p \) due to the imposed temperature perturbation are small enough to be neglected. We caution that this assumption may under certain circumstances prove inaccurate. Absorption cross sections are generally temperature-dependent and if phase changes or chemical reactions govern the absorbing gas concentration then this latter concentration may be strongly temperature-dependent. Changes in the heat capacity may result from phase changes or from activation of more molecular degrees of freedom as the temperature increases. We will also have to make certain assumptions about the ground which will be affected by the atmospheric temperature changes. To provide a suitable case for study we will assume here that the ground temperature changes are in phase with the air just above the ground; i.e.,

\[ T_o(x_0) = T_o(x) + T_o \exp[iK \cdot (x - x_0)]. \tag{7} \]

Under certain circumstances, for example when the surface is an excellent heat conductor, then \( T_o \approx 0 \). In other circumstances, for example when the surface is a very poor conductor but an excellent emitter, we can assume radiative equilibrium to obtain \( T_o \). We will expand on these various assumptions and their limitations in some detail in a later section.

With the above cautionary statements in mind we can now express the radiative heating rate in the perturbed atmosphere at \( r = 0 \) as

\[
q(0) = \rho(0)C_p\frac{\partial}{\partial t} \{ T(0) + T \exp[-iK \cdot r_0] \}
\]

\[
= q_0(0) + \int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \{ \alpha_r(0) \left[ \int_0^\infty T^0 \exp[iK \cdot (r - r_0) - \tau_0] \alpha_r \frac{\partial B_0}{\partial T} T \exp[-iK \cdot r_0] \right] \}
\]

\[
\times \sin \theta d\theta d\phi d\nu + \int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \{ \alpha_r(0) \left[ \int_0^\infty T^0 \exp[iK \cdot (r - r_0) - \tau_0] \alpha_r \frac{\partial B_0}{\partial T} \right] \}
\]

\[
\times \sin \theta d\theta d\phi d\nu + \int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \{ \alpha_r(0)T_o \exp[iK \cdot (r - r_0) - \tau_0] \}
\]

\[
\times \frac{\partial B_0}{\partial T} \left( \frac{T_o(0)}{T_o(0)} \right) \sin \theta d\theta d\phi d\nu \tag{8}
\]

Utilizing (1), and dividing through by \( \rho(0)C_pT^0 \exp[-iK \cdot r_0] \), we then obtain the rate of radiative decay of the amplitude of the temperature perturbation and thus the radiative time constant \( t_{rad} \):

\[
\frac{1}{t_{rad}} = \int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \{ \alpha_r(0) \left[ \int_0^\infty T^0 \exp[iK \cdot (r - r_0) - \tau_0] \alpha_r \frac{\partial B_0}{\partial T} \right] \}
\]

\[
\times \sin \theta d\theta d\phi d\nu + \int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \{ \alpha_r(0)T_o \exp[iK \cdot (r - r_0) - \tau_0] \}
\]

\[
\times \frac{\partial B_0}{\partial T} \left( \frac{T_o(0)}{T_o(0)} \right) \sin \theta d\theta d\phi d\nu \tag{9}
\]

To obtain \( T'_o/T' \) when the surface is a blackbody in radiative equilibrium we simply equate the net downward flux of perturbation radiation into the surface

\[
\int_0^\infty \int_0^{\tau/2} \int_0^{\tau/2} \cos \theta \left[ \int_0^\infty T' \exp[iK \cdot (r - r_0) - \tau_0] \alpha_r \frac{\partial B_0}{\partial T} \right] \sin \theta d\theta d\phi d\nu, \tag{10}
\]
with the net upward flux of perturbation radiation emitted by the surface

$$\int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \left\{ T'_s \exp \left[ -i \mathbf{K} \cdot \mathbf{r}_s \right] \frac{\partial B_0}{\partial T}(T_s) \right\} \sin \theta d\theta d\phi d\nu. \tag{11}$$

This leads to

$$\frac{T'_s}{T'} = \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \left\{ \int_0^\infty \exp \left[ i \mathbf{K} \cdot \mathbf{r} - \tau_s \right] \frac{\partial B_0}{\partial T}(T_s) \right\} \sin \theta d\theta d\phi d\nu / \int_0^\infty \frac{\partial B_0}{\partial T}(T_s) d\nu. \tag{12}$$

The general expression (9) can be simplified without too much loss of generality. We will show later that $\sigma$, and $\partial B_0/\partial T$ are roughly constant throughout the relevant portions of the atmosphere and can therefore be replaced in (9) by appropriate spatially-averaged values $\overline{\sigma}$ and $\overline{\partial B_0/\partial T}$. Also, we can often assume that the density of the absorbing gas is horizontally invariant over the relevant horizontal scales and varies in the vertical with a constant density scale height $H_\alpha$. That is

$$\alpha \approx \rho_\alpha(0) \overline{\sigma} \exp(-z/H_\alpha),$$

where $z$ is altitude measured from the level $r=0$. Thus (9) becomes

$$1 = \frac{4\pi \rho_\alpha(0)}{\rho(0)C_p} \int_0^\infty \frac{\partial B_0}{\partial T}(T'_s) \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right] \right] \right]$$

$$= \frac{4\pi \rho_\alpha(0)}{\rho(0)C_p} \int_0^\infty \frac{\partial B_0}{\partial T}(T'_s) \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

$$= \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

$$= \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

$$= \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

$$= \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

$$= \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

In an infinite homogeneous atmosphere (i.e., $r_s \rightarrow \infty$, $H_\alpha \rightarrow \infty$) this expression after some manipulation simplifies to the familiar result obtained by Spiegel (1957) for a grey atmosphere and generalized to the non-grey case we are considering here by Goody (1964, p. 347), i.e.,

$$1 = \frac{4\pi \rho_\alpha(0)}{\rho(0)C_p} \int_0^\infty \frac{\partial B_0}{\partial T}(T'_s) \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

We will be particularly concerned in this paper with a comparison of the $t_{rad}$ values implied by (13) and (14). From a physical viewpoint $t_{rad}$ is given by the difference between a perturbation damping term

$$\frac{4\pi \rho_\alpha(0)}{\rho(0)C_p} \int_0^\infty \frac{\partial B_0}{\partial T}(T'_s) \left[ \left( 1 - \frac{\rho_\alpha(0)\overline{\sigma}}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right) \right]$$

which describes the emission of perturbation radiation by the air parcel to its surroundings, and a perturbation amplifying term

$$\frac{4\pi \rho_\alpha(0)}{\rho(0)C_p} \int_0^\infty \frac{\partial B_0}{\partial T}(T'_s) \left[ S(n) \right]$$

which describes the absorption by the air parcel of perturbation radiation from its surroundings. For the finite inhomogeneous atmosphere the dimensionless amplification parameter $S(n)$ is the real part of

$$\frac{\rho_\alpha(0)\overline{\sigma}}{K} \int_0^{2\pi} \int_0^{\pi/2} \int_0^\nu \exp \left[ i \mathbf{K} \cdot \mathbf{r} - (z/H_\alpha) - \tau_s \right] \sin \theta d\theta d\phi d\nu \right]$$

while for the infinite homogeneous atmosphere it is

$$S(n) = \frac{\rho_\alpha(0)\overline{\sigma}}{K} \tan^{-1} \left( \frac{K}{\rho_\alpha(0)\overline{\sigma}} \right). \tag{18}$$
Note that $0 \leq \nu \leq 1$ and if $\nu \rightarrow 1$ for all $\nu$ at which the atmosphere has appreciable emission then $\tau_{\text{rad}} \rightarrow \infty$. If we utilize (18) as an approximation in a finite inhomogeneous atmosphere we will therefore expect the maximum error as $\nu \rightarrow 1$.

Before moving to actual numerical evaluation of $S(\nu)$ we should mention that the treatment here has neglected multiple scattering of radiation. Providing $\alpha$, is not affected by the perturbation we do know that multiple scattering of the incoming solar radiation will have no effect on $\tau_{\text{rad}}$. This is because solar radiation, whether direct or diffuse, will then have no influence on the perturbation field since atmospheric emission at visible wavelengths is negligible. In addition, providing the scattering cross section is much less than the absorption cross section for the relevant infrared wavelengths, a case can be made for neglecting multiple scattering of the planetary infrared radiation also.

3. Numerical solution

In order to study situations relevant to planetary atmospheres we will need to specify typical values for the absorber scale height $H_a$, the wavelength of the temperature perturbation $2\pi/K$, and the radiation mean-free path $[\rho_\nu(0)\sigma_\nu]^{-1}$. These parameters, in addition to the altitude above the surface, $\rho(0)$ and $C_p$, serve to determine $\tau_{\text{rad}}$ as defined by (13) or (14).

With a few exceptions (e.g., ozone in the terrestrial troposphere) atmospheric absorbers generally have a density scale height less than or equal to the total gas density scale height $H$. On the other hand the radiation mean free path will vary from very small values in very opaque atmospheres to essentially infinite values in transparent atmospheres. The horizontal and vertical wavelengths for the temperature perturbations caused by diabatic heating can obviously be equated with the horizontal and vertical scales for the heating. For example, for radiative heating by the sun, the horizontal scale is roughly equal to the planetary radius $a$, provided the absorber density is horizontally invariant. The vertical scale depends on both $H_a$ and on the rate of decrease of intensity for the appropriate solar photons with altitude; the resultant value is typically $\lesssim H$. For latent heating, the horizontal scale is usually $\lesssim a$, particularly on the earth where considerable organization of cumulus convection is evident. The vertical scale for latent heating should be similar to the density scale height of the condensing gas. Temperature perturbations associated with baroclinic waves will most often have horizontal scales $\gtrsim$ the Rossby radius of deformation and vertical scales $\gtrsim H$. On the other hand, perturbations associated with internal gravity waves will often have vertical scales $\ll H$ and a wide spectrum of horizontal wavelengths ranging from mesoscales to planetary scales depending on their forcing mechanisms.

The wide range of variability in these various parameters obviously prevents us from presenting numerical computations which will be relevant to all cases. We have therefore chosen for detailed study the specific case of a horizontally invariant temperature perturbation. In practical terms the results will be applicable to any disturbance where the horizontal scales of the perturbation are much greater than the vertical scale and much greater than the radiation mean free path. In addition, we will restrict ourselves to consideration of two situations: $r_v=0$ and $r_v=\infty$.

In practical terms the situation $r_v=0$ provides an estimate for $\tau_{\text{rad}}$ at distances above the ground much less than the radiation mean free path, while the situation $r_v=\infty$ provides an estimate for distances much greater than the mean free path.

For the horizontally invariant case

$$K_{\nu} = n \tau_{\nu},$$

(19)

where $\mu = \cos \theta$ and $n$ is the vertical wavenumber. Thus when $r_v=\infty$ the amplification parameter is from (17)

$$S(\nu) = \frac{1}{2} \rho_\nu(0) \sigma_\nu \int_0^\infty \left[ \int_0^1 \cos(\mu r) \exp[-f(\mu, r)] d\mu \right] dr,$$

(20)

Alternatively, if $r_v=0$ the amplification parameter is

$$S(\nu) = \frac{1}{2} \rho_\nu(0) \sigma_\nu \int_0^\infty \left[ \int_0^1 \cos(\mu r) \exp[-f(\mu, r)] d\mu \right] dr,$$

(21)

when the ground is an excellent conductor, and

$$S(\nu) = \frac{1}{2} \rho_\nu(0) \sigma_\nu \int_0^\infty \left[ \int_0^1 \mu \cos(\mu r) \exp[-f(\mu, r)] d\mu \right] dr + \frac{\rho_\nu(0) \sigma_\nu}{\partial T} \left[ \int_0^\infty \left[ \int_0^1 \mu \cos(\mu r) \exp[-f(\mu, r)] d\mu \right] dr \right] \partial T / \partial T,$$

(22)
when the ground is an insulated blackbody in radiative equilibrium. In (20)–(22) we use the notation
\[
f(\mu, r) = \frac{\mu r}{H_\alpha} + r_s(\mu, r),
\]
\[
= \frac{\mu r}{H_\alpha} + \frac{\rho_0(0)\bar{\sigma}_s}{H_\alpha} [1 - \exp(-r_\mu/H_\alpha)].
\]
(23)

To carry out the integration over distance \( r \) we transform to a convenient dimensionless variable \( R \) where
\[
R = \left[ \frac{2 \rho_0(0)\bar{\sigma}_s}{n} \right] r.
\]
(24)

In a homogeneous medium \( R \) is simply one-half of the optical depth of the path \( r = 0 \) to \( r = r \). As we will discuss shortly in more detail, \( \exp[-f(\mu, r)] \) becomes vanishingly small when \( R > 1 \). Thus we expect that contributions to the integrals in (20)–(22) from regions where \( R > 1 \) will be very small. The other term in the integrands in (20)–(22), namely \( \text{cos}(n r_\mu) \), will under certain circumstances be rapidly oscillating with increasing \( r \) and therefore present problems in numerically evaluating the integrals. The dimensionless wavelength \( R_0 \) of this oscillation is obtained from
\[
\frac{2n|\rho_0(0)\bar{\sigma}_s|}{\rho_0(0)\bar{\sigma}_s} = 2\pi,
\]
which gives
\[
R_0 = \frac{\rho_0(0)\bar{\sigma}_s}{|\rho_0(0)\bar{\sigma}_s| n} \geq \frac{\rho_0(0)\bar{\sigma}_s}{n}.
\]
(26)

Clearly if \( R_0 > 1 \), i.e., \( \rho_0(0)\bar{\sigma}_s/n > \pi^{-1} \), then \( \text{cos}(n r_\mu) \) oscillates sufficiently slowly with \( r \) for us to expect few problems in the numerical evaluations in this case. However, when \( R_0 < 1 \), i.e., \( \rho_0(0)\bar{\sigma}_s/n < \pi^{-1} \), we expect many oscillations in the important domain \( 0 < R < 1 \). At first sight we would therefore anticipate requiring a very high resolution grid for the integration over \( R \) in this case. However, as we will see shortly, when \( \rho_0(0)\bar{\sigma}_s/n < \pi^{-1} \) then \( S(\nu) < 1 \); that is, the actual value assumed for the amplification parameter has negligible influence on \( t_{\text{rad}} \) in this case.

For the evaluation of the integrals in (20)–(22) we have experimented with several numerical quadratures based on orthonormal polynomials of varying types and degrees. For reasons which will become apparent in the following section, (20) will be closely approximated by (18) provided \( n \gg H_\alpha^{-1} \). We can therefore use this expected equality to test the accuracy of our numerical procedures. For this purpose we computed the quantity \( \delta \), which is the ratio of \( 1 - S(\nu) \) defined by (20) to \( 1 - S(\nu) \) defined by (18), as a function of \( \rho_0(0)\bar{\sigma}_s/n \) for \( n = 1000/H_\alpha \gg H_\alpha^{-1} \). Theoretically this quantity should be close to unity but as expected from the above discussion, small deviations from \( \delta = 1 \) appeared in all quadratures tested when \( \rho_0(0)\bar{\sigma}_s/n \ll \pi^{-1} \). In addition, for \( \rho_0(0)\bar{\sigma}_s/n > 100 \) the accuracy becomes increasingly poor for all quadratures due to the fact that \( S(\nu) \) is approaching unity and we are subtracting two numbers of almost equal magnitude when computing \( \delta \). Based on our tests we have utilized a 16-point Gaussian quadrature with \(-1 < \mu \leq 1\) for the integration over angle and a 96-point Gaussian quadrature with \( 0 \leq R \leq 10 \) for the integration over radial optical distance for all the computations we will present here.

We first examine the situation where \( r_s = \infty \). Referring to (20) we have plotted in Fig. 2 the dimensionless expression \( [1 - S(\nu)]^{-1} \) as a function of \( \rho_0(0)\bar{\sigma}_s/n \) for a number of values of the dimensionless quantity \( nH_\alpha \). In this same illustration we have also plotted \( [1 - S(\nu)]^{-1} \) obtained utilizing (20) when \( H_\alpha = \infty \). In this case (20) reduces to the Spiegel expression (18) which is of course invariant to \( nH_\alpha \). The results in Fig. 2 show us that the Spiegel expression provides a very good approximation to the exact expression when \( nH_\alpha > 1 \), i.e., when the vertical wavelength of the temperature perturbation is much less than the scale height of the absorber. This result is of course expected on physical grounds and should in one form or another be applicable under more general circumstances. However, when \( nH_\alpha < 1 \), use of the Spiegel expression will often lead to very significant errors.

This latter point is better illustrated if we make the traditional grey approximation to provide physical significance to the expression plotted in Fig. 2. In this case (13) simplifies to
\[
t_{\text{rad}} = \frac{\rho(0)C_\nu}{16(\sigma)\rho_0(0)\sum T_0^\delta (1 - \langle S \rangle)},
\]
(27)
where \( T_0^\delta \) is the spatially averaged value of \( T_0^\delta \) and \( T_\nu^\delta \), \( \langle \sigma \rangle \) is the grey absorption cross section defined as an appropriately weighted average of \( \bar{\sigma}_s \) over frequency, \( \Sigma \) is Stefan's constant, and \( \langle S \rangle \) is simply \( S(\nu) \) with \( \bar{\sigma}_s \) replaced by \( \langle \bar{\sigma} \rangle \). Thus \( t_{\text{rad}} \) is directly proportional to \( [1 - \langle S \rangle]^{-1} \).

We can conveniently look at the two limiting possibilities considered by Spiegel (1957). The first, which we will refer to as the transparent perturbation case, occurs when the radiation mean free path is much greater than the wavelength of the perturbation. In this case
\[
\rho_0(0)\bar{\sigma}_s/n < 1,
\]
(28)
and we see from Fig. 2 that the Spiegel formula is adequate for all values of \( nH_a \) since \( S(\sigma) \approx 0 \) whether we use (18) or (20). Also \( t_{rad} \) equals a constant minimum value which is independent of \( n, \rho_a(0)/\sigma \) and \( H_a \).

The second possibility, which we will refer to as the opaque perturbation case, occurs when the radiation mean free path is much less than the wavelength of the perturbation. In this case

\[
\rho_a(0)(\sigma)/n > 1. \quad (29)
\]

Here, Fig. 2 implies that the Spiegel formula provides a reasonable approximation only if \( \rho_a(0)(\sigma)/n \) exceeds a critical value which increases very rapidly as \( nH_a \to 0 \). For example, when \( nH_a = 0.1 \) this critical value is \( \sim 10^2 \) while when \( nH_a = 0.01 \) it is \( \sim 10^3 - 10^4 \). In physical terms, once the perturbation is sufficiently opaque the inhomogeneity of the atmosphere no longer influences the radiative damping rate. Thus as \( \rho_a(0)(\sigma)/n \to \infty \) the curves for all \( nH_a > 0 \) approach the Spiegel curve asymptotically. The errors for moderate \( \rho_a(0)(\sigma)/n \) values can however be extremely large; for example, for \( nH_a = 0.1 \) and \( \rho_a(0)(\sigma)/n = 10 \) the Spiegel formula overestimates \( t_{rad} \) by a factor of 23. We conclude that considerable caution should be practiced in using the Spiegel formula even in very opaque atmospheres whenever the vertical wavelength of the temperature perturbation exceeds \( 2\pi \) times the scale height of the absorber. Fortunately, this latter situation is a relatively rare occurrence. Thus for radiative decay of atmospheric waves on planets without surfaces the Spiegel formula often provides a reasonable first approximation.

The reason for these results is readily apparent if in (20) we carry out the integration over angle once by parts and denote the results by \( Q(\sigma, r) \); that is,

\[
Q(\sigma, r) = \frac{\sin n\pi r}{n\pi r} \left[ e^{-f(1,r)} + e^{-f(-1,r)} \right] + \int_{-1}^{1} \left( \frac{\sin n\pi \mu}{n\pi \mu} \right) \times f'(\mu, r) e^{-f(\mu, r)} d\mu. \quad (30)
\]
If \( f(\mu, r) \) were independent of \( \mu \) then integration of \( Q(\nu, r) \) over \( r \) would lead to the Spiegel expression (18) for \( S(\nu) \). Since \( f(\mu, r) \to \rho_a(0) \sigma, r \) as \( H_a \to \infty \), we always expect Spiegel's expression to be valid for sufficiently large \( H_a \). In addition, as the vertical optical depth above the level \( \rho_a(0) \sigma, H_a \to \infty \), the important contributions to \( S(\nu) \) will ultimately come from regions where \( r \ll H_a/\mu \) and in this region \( f(\mu, r) \approx \rho_a(0) \sigma, r \). Thus we might also expect Spiegel's expression to be valid for sufficiently large values of \( \rho_a(0) \sigma, H_a \). In addition, the smaller the value of \( H_a \), the greater the value of \( \rho_a(0) \sigma, \) necessary before this required large value of \( \rho_a(0) \sigma, H_a \) is reached. Both of these conclusions are mirrored in the numerical results which show that the Spiegel expression is valid when \( nH_a > 1 \), or when \( \rho_a(0) \sigma, n \) exceeds a certain critical value which increases as \( nH_a \) decreases.

If we apply the grey approximation it is therefore apparent that two limiting cases in addition to (28) and (29) must now be discerned; one when

\[
\rho_a(0) \langle \sigma \rangle H_a \gg 1, \tag{31}
\]

which will be referred to as the opaque atmosphere case and one when

\[
\rho_a(0) \langle \sigma \rangle H_a \ll 1, \tag{32}
\]

which will be referred to as the transparent atmosphere case. These two cases, which depend on a comparison of the radiation mean free path and the absorber density scale height, are quite distinct from Spiegel's opaque and transparent perturbation cases discussed above. In the opaque atmosphere case the Spiegel expression (14) is valid whether the perturbation itself is opaque \([n \ll \rho_a(0) \langle \sigma \rangle]\) or transparent \([n \gg \rho_a(0) \langle \sigma \rangle]\). However in the transparent case if the perturbation is also transparent, then \( \langle S \rangle \approx 0 \) so the difference between (13) and Spiegel's expression (14) would not lead to a significant difference in \( t_{\text{rad}} \). If, however, the perturbation were even moderately opaque \([\rho_a(0) \langle \sigma \rangle \gtrsim n]\), then \( \langle S \rangle \approx 1 \) and the error in utilizing (14) instead of (13) becomes substantial.

In conclusion, the criterion for use of the Spiegel expression (14) for computing \( t_{\text{rad}} \) for a disturbance of arbitrary wavelength is simply that the atmosphere be opaque as defined above. If the atmosphere is transparent and the perturbation transparent then \( \langle S \rangle \approx 0 \) and \( t_{\text{rad}} \) equals its minimum value as given
in (27). In all other cases (13) should be integrated numerically in order to evaluate $t_{rad}$.

We now examine the situation where $r_g = 0$ and the ground is an excellent conductor. In Fig. 3 we have plotted the function $\frac{1 - S(v)}{1 - S(v)}$ in a manner similar to Fig. 2 but utilizing (21) in place of (20). In this same illustration we have also plotted the analogous function obtained utilizing (21) when $H_a = \infty$, namely

$$[1 - S(v)]^{-1} = \left[1 - \frac{\rho_n(0)\sigma_v}{2n} \tan^{-1}\left[\frac{n}{\rho_n(0)\sigma_v}\right]\right]^{-1},$$

(33)

which is again invariant to $nH_a$. We should also emphasize that (33) implies that $0 \leq S(v) \leq \frac{1}{2}$, i.e., $1 \leq [1 - S(v)]^{-1} \leq 2$. In physical terms, $S(v) \leq \frac{1}{2}$ implies that $t_{rad}$ very close to a conducting ground must always remain finite. This result follows from the fact that we have assumed that the conducting ground is unaffected by the atmospheric temperature perturbation and therefore emits no perturbation radiation.

This situation should be contrasted to that in the infinite atmosphere discussed by Spiegel (1957) and Goody (1964) where $S(v) \leq 1$ and thus infinite $t_{rad}$ values become possible.

The results in Fig. 3 demonstrate that (33) provides a good approximation to the exact expression if $nH_a \gg 1$. In this respect the similarity with the results in Fig. 2 is clear. This similarity is further emphasized if we again utilize the grey approximation together with our definitions of opaque and transparent atmospheres and perturbations given above. For transparent perturbations $\langle S \rangle \approx 0$ for all values of $nH_a$, and $t_{rad}$ equals the identical minimum value found for it in the analogous $r_g = \infty$ situation. For opaque perturbations $\langle S \rangle = \frac{1}{2}$ once $\rho_n(0)\sigma_v/n$ exceeds a critical value which increases rapidly as $nH_a \to 0$; that is, $\langle S \rangle = \frac{1}{2}$ once the atmosphere becomes opaque. At the same time $t_{rad}$ approaches a maximum value equal to twice the minimum value found for transparent perturbations.

![Fig. 4. Computations relevant to levels just above an insulated surface in an inhomogeneous atmosphere. The solid lines have the same significance as their counterpart in Fig. 2. The dashed line is obtained using Spiegel's (1957) formula for an infinite homogeneous atmosphere. Computed values for $nH_a \geq 10$ (not shown) are essentially identical to those obtained from Spiegel's formula. The computations presented here all assume the grey approximation and should therefore be interpreted in the manner indicated in the caption of Fig. 2.](image-url)
We finally examine the situation where \( r_0 = 0 \) but the ground is an insulated blackbody in radiative equilibrium. Eq. (22) is particularly cumbersome but considerable simplification without too much loss of generality is obtained if we assume the grey approximation. With this approximation (22) becomes

\[
\langle S \rangle = \frac{1}{2} \rho_0(0) \langle \sigma \rangle \int_0^\infty \left[ \int_0^1 (1+2\mu) \cos(n r_0 \mu) \right] d\mu dr, \tag{34}
\]

where \( \langle f(\mu, r) \rangle \) is \( f(\mu, r) \) with \( \sigma \) replaced by \( \langle \sigma \rangle \). In Fig. 4 we plot \( [1-\langle S \rangle]^{-1} \) obtained using either (34) or the grey version of the Spiegel expression (18). As one would expect intuitively the results are qualitatively similar to those obtained in the \( r_0 = \infty \) case presented in Fig. 2. Thus the Spiegel formula provides a useful first approximation \( \rho_0(0) \langle \sigma \rangle / n \ll 1 \) or provided \( \rho_0(0) \langle \sigma \rangle / n \) exceeds a critical value which depends on \( nH_a \) (i.e., provided the atmosphere is opaque). Indeed, the only important difference between the cases presented in Figs. 2 and 4 is that \( t_{\text{rad}} \) appears to be uniformly somewhat greater in the \( r_0 = \infty \) case in comparison to the \( r_0 = 0 \), insulated ground case.

The significance of the assumed response of the ground to atmospheric temperature perturbations is very evident in comparing the results in Figs. 3 and 4. In particular, the radiative time constant for opaque perturbations is predicted to be several orders of magnitude greater than a conducting surface than over a conducting surface. This distinction between surfaces will of course decrease once the altitude exceeds a few radiation mean free paths but it is clear that the surface often plays a very crucial role in the radiative damping of atmospheric waves. In Section 4 we will therefore further examine the physics of the surface response in some detail.

This is a useful juncture at which to briefly illustrate some of the points discussed thus far. We will refer to the terrestrial troposphere in which the absorber is mainly water vapor. To a sufficient approximation \( H_a = 2 \) km and at the ground \( \rho_0(0) \langle \sigma \rangle \approx 2 \text{ km}^{-1} \) (Goody, 1964, p. 332) and \( \rho(0) C_\rho = 1.2 \times 10^4 \text{ ergs K}^{-1} \text{ cm}^{-2} \). Placing \( T_0 = 296 \text{ K} \) in (27) we therefore have in the grey approximation\(^1\) \( t_{\text{rad}} \approx (2.55 \times 10^8) / (1-\langle S \rangle) \) seconds. Utilizing (18) and Figs. 2–4, we therefore have for \( n^{-1} = 0.5 \) km:

\[
\begin{align*}
t_{\text{rad}} \text{ (Spiegel)} & \approx 1.2 \times 10^8 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = \infty & ) \approx 1.2 \times 10^8 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = 0; \text{ conducting ground}) & \approx 4.0 \times 10^4 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = 0; \text{ insulated ground}) & \approx 7.8 \times 10^4 \text{ s}
\end{align*}
\]

while for \( n^{-1} = 5 \) km we have

\[
\begin{align*}
t_{\text{rad}} \text{ (Spiegel)} & = 7.7 \times 10^4 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = \infty & ) = 2.1 \times 10^4 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = 0; \text{ conducting ground}) & = 4.7 \times 10^4 \text{ s} \\
t_{\text{rad}} \text{ (r}_0 = 0; \text{ insulated ground}) & = 1.1 \times 10^4 \text{ s}.
\end{align*}
\]

For the shorter vertical wavelength \( n^{-1} = 0.5 \) km the Spiegel (1957) formula is adequate provided the altitude exceeds the radiation mean free path (0.5 km). At lower altitudes it overestimates \( t_{\text{rad}} \) by a factor of 2 or 3. Discrepancies are more serious for the longer vertical wavelength \( n^{-1} = 5 \) km.\(^3\) Here the Spiegel formula would overestimate \( t_{\text{rad}} \) by a factor of 3.7 even at altitudes considerably exceeding 0.5 km while near a conducting ground it overestimates the time constant by two orders of magnitude.\(^8\) We shall illustrate that well-stirred oceanic waters can be considered as a conducting surface in this context, while snow, sand, ice, soil, etc., can generally be considered as insulating surfaces. It is certainly apparent that radiative time constants computed assuming an infinite homogeneous grey medium often lead to serious overestimates of radiative damping rates in the terrestrial atmosphere.

4. Surface response to atmospheric waves

In our solutions presented in the previous section, the assumed response of the surface to atmospheric temperature perturbations was shown to be very influential in determining \( t_{\text{rad}} \) in certain situations. Two essential surface properties need to be assessed—the amplitude and the phase lag for the perturbation temperature of the surface radiating layer relative to the air just above the surface. In our earlier calculations we assumed that either the amplitude was given by a radiative equilibrium assumption with no phase lag or that the amplitude was negligibly small. In this section we will show under what conditions these assumptions are valid and the circumstances in which more complex boundary conditions will be required.

Lettau (1951) has made the classical study of the response of the earth’s surface to periodic heating by solar radiation. Our study will differ from this in several respects. Our surface heating will be due to

\[^1\] Non-grey calculations have been carried out for the Spiegel formula (18) by Goody (1964, p. 336). For \( n^{-1} = 10 \text{ km} \) our grey calculation yields \( t_{\text{rad}} \approx 3.1 \times 10^8 \text{ s} \) compared to \( 1.4 \times 10^9 \text{ s} \) obtained in the non-grey calculations.

\[^3\] This corresponds to an actual vertical wavelength \( 2\pi / n = 31 \text{ km} \). This wavelength is about twice the tropopause altitude and is probably the largest justifiable vertical wavelength in the terrestrial lower atmosphere. As such, this case illustrates the largest expected error for the Spiegel formula resulting from neglect of inhomogeneity in the terrestrial atmosphere, namely, a factor of \( \sim 3.7 \).

\[^8\] At higher levels where \( \langle S \rangle - 1 \) Eq. (27) implies that very small fractional errors in \( \langle S \rangle \) will still lead to substantial errors in \( t_{\text{rad}} \). The effect of the conducting surface is therefore still felt at altitudes a few times the radiation mean free path; the \( r_0 = \infty \) results will of course always become applicable at sufficiently great altitudes in an opaque atmosphere.
perturbation thermal radiation from the atmosphere and we will also explicitly include the perturbation thermal emission by the surface and the transport of perturbation energy by latent heat fluxes. These latter two processes were omitted by Lettau (1951). We will also use a different parameterization for obtaining perturbation sensible heat fluxes.

Utilizing (10) and (11) we can express the net downward flux of perturbation radiation into the surface as

$$4\Sigma T_\circ^3 T'_s \exp(-iK \cdot r_0).$$  \hspace{1cm} (35)

Recall that $T'_s \exp(-iK \cdot r_0)$ is the true surface perturbation temperature only when the ground is a non-conducting blackbody in radiative equilibrium with the atmosphere. Let us suppose that the true perturbation temperature at and below the surface is $T'_s(Z) \exp(-iK \cdot r_0)$ where $Z$ is depth below the surface, and that the ground is a heat conductor with thermal conductivity $K_\circ$. In addition, although we have been concerned here only with radiative heat fluxes in the atmosphere, we will also allow for the fact that perturbation heat can be transferred between the bulk atmosphere and the ground by turbulent fluxes of latent and sensible heat.

Using Kraichnan’s (1962) theory we can approximately express the flux of sensible perturbation heat $\phi_\circ$ between atmosphere and ground as (cf. Gierasch and Goody, 1968)

$$\phi_\circ = -\rho C_v K_\circ \left[ \frac{T'_s - T'_s(0)}{D} \right] \exp(-iK \cdot r_0)$$

$$D = \left[ \frac{K_\circ K_\circ T_0}{16 \rho C_v (T_\circ - T_0)} \right] \left. \right|_{T = 0},$$  \hspace{1cm} (36)

where $A$ is a constant (0.089 for high Prandtl numbers), $g$ the gravitational acceleration, $K_\circ$ the kinematic viscosity, and $K_\circ$ the thermal diffusion coefficient. Formally, the atmospheric perturbation temperature in (36) should refer to the top of the turbulent boundary layer. We have assumed that $2\pi n/\epsilon$ is much greater than the depth of this boundary layer and therefore that the perturbation temperature just above the ground is essentially the required value. Finally, to obtain the latent heat perturbation flux $\phi_L$, we will use Priestley’s (1959, p. 91) approximation

$$\phi_L = \frac{L \left( \frac{\rho L}{C_v \rho} \right)^i}{T'_s(0)} \left. \right|_{T = 0}$$

\hspace{1cm} (37)

\hspace{1cm} 4 It is of course implicit in our discussion that the perturbation heat above the boundary layer is carried by radiative fluxes alone. If $K_H$ is the vertical turbulent diffusion coefficient above the boundary layer we therefore require that $\rho C_v K_\circ \left| \frac{\partial}{\partial \theta} \left( \frac{\rho L}{C_v \rho} \right) \right|_{T = T_\circ} \ll 4 \Sigma T_\circ^3 T'_s(0)$.

Since $T' \sim T'_s$ we therefore require the turbulent diffusion velocity for perturbation heat $n K_\circ \ll 0.5$ cm s$^{-1}$ (for dry air at 1 bar pressure and $T_\circ = 296$ K).

Here $L$ is the latent heat for the phase change at the surface and $\rho L$ the density of the gas involved in the phase change. Continuity of the perturbation heat flux at the surface therefore requires that

$$\left[ 4\Sigma T_\circ^3 + \left( 1 + \frac{\phi_L}{\phi_\circ} \right) \rho C_v K_\circ \phi_L \frac{\partial T'_s}{\partial t} \right] T'_s = \left[ 4\Sigma T_\circ^3 \right] \left. \right|_{T = 0}$$

$$+ \left( 1 + \frac{\phi_L}{\phi_\circ} \right) \rho C_v K_\circ \left. \frac{\partial T'_s}{\partial Z} \right|_{Z = 0}.$$  \hspace{1cm} (38)

For purposes of illustration we suppose that the atmospheric temperature perturbation $T''$ (and hence $T'_s$) is oscillating in time with frequency $\omega$; i.e.,

$$T'_s(t) = T'_s(0)e^{-i\omega t}.$$  \hspace{1cm} (39)

In order to obtain the resultant expression for the true ground temperature $T'_s(Z,t)$ we must solve the thermal diffusion equation in the ground

$$\frac{\partial T'_s}{\partial t} - \frac{K_\circ}{\rho C_v} \frac{\partial^2 T'_s}{\partial Z^2} = 0,$$  \hspace{1cm} (40)

where $\rho_0$ and $C_v$ are the density and heat capacity respectively for the ground material. The boundary conditions are that $T'_s$ be finite for large $Z$ and that (38) be obeyed at $Z = 0$. The appropriate solution to (40) is readily obtained using LaPlace transforms and is

$$T'_s(Z,t) = \frac{-i\omega t - \left( \frac{i\omega \rho_0 C_v}{2K_\circ} \right) \left( 1 - iZ \right)}{U + V - iV}.$$  \hspace{1cm} (41)

where

$$U = \frac{4\Sigma T_\circ^3 + \left( 1 + \frac{\phi_L}{\phi_\circ} \right) \rho C_v K_\circ}{\rho L}$$

$$V = \frac{4\Sigma T_\circ^3 \left( 1 + \frac{\phi_L}{\phi_\circ} \right) \rho C_v K_\circ \phi_L}{\rho L} \frac{\partial T'_s}{\partial \theta} \left. \right|_{T = T_\circ}$$

$$\left. \right|_{T = 0}$$

\hspace{1cm} (42)

are dimensionless numbers which govern the magnitude and phase of the ground perturbation temperature. In most planetary applications $T'' \approx T'_s$ and $\epsilon = 1$ so that $U$ is almost always of order unity. On the other hand, $V$ may vary widely in value.

Taking the real part of (41) at $Z = 0$ we have

$$T'_s(0,t) = T'_s(0) \left( \frac{(U + V) \cos \omega t + V \sin \omega t}{(U + V)^2 + V^2} \right),$$  \hspace{1cm} (43)

which can be compared with the real part of (39), namely,

$$T'_s(t) = T'_s(0) \cos \omega t.$$  \hspace{1cm} (44)
When $V \ll U$ we have
\[ T'_e(0,t) \approx \frac{T'_e(0)}{U} \cos \omega t, \] (45)
a case which requires that the surface material be a very poor conductor, i.e.,
\[ K < 2 \left[ 4 \pi \sum T^3_e + \left( \frac{T_e \sigma K_a}{D} \right)^2 \right] / \omega \rho e C_v. \] (46)

Note that the atmospheric and surface perturbations are in phase for this case. Eq. (22) which we developed earlier therefore corresponds to this case provided $U \approx 1$. This case is therefore favored by good surface emission properties, by strong turbulence in the boundary layer and by poorly conducting surface materials.

The contrasting case occurs when $V \gg U$ whence
\[ T'_e(0,t) \approx \frac{T'_e(0)}{2V} (\cos \omega t + \sin \omega t). \] (47)

This case requires the surface material to be a very good conductor, i.e.,
\[ K > 2 \left[ 4 \pi \sum T^3_e + \left( \frac{T_e \sigma K_a}{D} \right)^2 \right] / \omega \rho e C_v. \] (48)

Note that the surface temperature oscillations lag the atmospheric oscillation by one-eighth of a cycle. However, since $U \approx 1$ then $V \gg 1$ and (47) implies that the amplitude of the surface temperature oscillations is extremely small. Eq. (21) which we used earlier therefore corresponds to this case and is favored by poor surface emission properties, weak boundary layer turbulence and by surface materials which are excellent conductors.

The important parameter therefore appears to be the ratio $V/U$. A useful way of looking at $(V/U)^2$ is to consider it as the ratio of a surface response time $t_e$ defined by
\[ t_e = \pi \rho e C_v K_a \left[ 4 \pi \sum T^3_e + \left( \frac{T_e \sigma K_a}{D} \right)^2 \right]^{-1}, \] (49)
and the period $t_e = 2\pi/\omega$ of the atmospheric temperature oscillation. In the terrestrial situation the contributions to the total boundary layer energy flow by fluxes of radiation, latent heat and sensible heat are generally observed to be of comparable magnitude (e.g., Lenschow, 1973). Our various parameterizations for these fluxes lead to a similar conclusion for the flow of perturbation energy. For example, for $T_{e0} = 296$ K, $T_e = 286$ K and $\epsilon = 1$ we have using (36) for dry air at 1 bar pressure
\[ \frac{\rho e C_v K_a}{4 \pi \sum T^3_e D} = 0.99, \] (50)
while using (37) and (54) for saturated air at 1 bar we have $\phi_l/\phi_a = 0.25$ at 260 K and $\phi_l/\phi_a = 2.5$ at 296 K. Under very stable conditions (i.e., $T_{e0} - T_e = 0$) the turbulent fluxes naturally become negligibly small.

If we temporarily ignore turbulent fluxes in (49) and use data on $\rho e C_v K_a$ values for various surface materials from Priestly (1959, p. 100), we obtain the following $t_e$ values when $\epsilon = 1$:
- $t_e$ (new snow, 260 K) $\approx 29$ min
- $t_e$ (ice, 260 K) $\approx 9.7$ days
- $t_e$ (still water, 296 K) $\approx 2.8$ days
- $t_e$ (stirred water, 296 K) $\approx 184$ days–1510 years
- $t_e$ (dry sand, 296 K) $\approx 5.2$ h
- $t_e$ (organic soil, 296 K) $\approx 3.0$ days.

We can use these to provide rough order-of-magnitude estimates of $t_e$ for the various surfaces. For most atmospheric wave periods of interest over stirred oceanic waters it is apparent that $t_e < t_v$ or $V \gg U$ so radiative time constants computed using (21) are appropriate. The oceanic surface temperature perturbations are much smaller than those in the atmosphere and lag the atmosphere by times $\ll t_v/8$. In contrast, over dry sand and new snow generally $t_e < t_v$ or $V \ll U$ so radiative time constants obtained using (22) are appropriate. The sand and snow surface temperature perturbations are comparable to and in phase with those in the atmosphere. Over surfaces such as ice, still water and organic soil we have the possibility that $t_e \approx t_v$ or $U \approx V$. In such cases the surface temperature perturbations will be somewhat smaller than those in the atmosphere and have a small lag in phase. Radiative time constants obtained using (22) are nevertheless still more appropriate for this intermediate case than those obtained using (21).

If we include turbulent fluxes of equal magnitude to the radiative flux in these considerations we will obviously decrease the $t_v$ values by a factor of 9. This would move the ice, still water and organic soil cases closer to the radiative equilibrium case and make (22) even more appropriate. Inclusion of the turbulent fluxes, however, should not alter our conclusion concerning the use of (21) for stirred oceanic waters.

Before leaving this topic we should add that our discussion has implicitly assumed that the surface is extremely opaque to thermal radiation. In this case the emitted radiation originates from an extremely thin surface layer (e.g., of thickness equal to the wavelength of the thermal radiation). If the effective radiating layer is much thicker, in particular if its thickness exceeds $(2K_a/\omega \rho e C_v)^{1/2}$, then (41) implies we must also take into account the amplitude decay and increasing phase lag for the temperature wave as we go deeper into the surface. The greatest effects will obviously occur for very small $t_e$ and $K_a$ but even
if we assume that the radiating layer for new snow at 260 K is 1 cm thick we will require $\omega > 1.2 \times 10^{-2}$ s$^{-1}$ or $t_s < 9$ min before such effects become important.

5. Other limitations and complications

The specific solutions for $t_{rad}$ developed in earlier sections contained a number of assumptions in addition to those involving the surface response. There are identifiable circumstances when these other assumptions prove poor; the influence of ozone chemistry on radiative damping in the upper atmosphere is a well-known example of such a circumstance and there are others which are outlined briefly below.

a. Perturbations to absorption cross sections

In the original paper by Spiegel (1957), linear perturbations in both $B_\nu$ and $\alpha_\nu$ were included. However, it was found that to the first order in $T'$ the perturbation in $\alpha_\nu$ had no influence on the resulting expression for $t_{rad}$. This fortunate conclusion depended on the fact that in a homogeneous infinite atmosphere the change in the perturbation amplifying term (16) caused by the $\alpha_\nu$ perturbation is exactly balanced by the accompanying change in the perturbation damping term (15). We would expect a similar balance in an inhomogeneous atmosphere whenever the Spiegel expression (18) provides a good approximation to the exact solution. We might also anticipate at least a partial balance of this type in cases where (18) is not valid but we have no guarantee that this balance will be close enough to avoid significant errors.

Another problem arises in considering planetary atmospheres which did not arise in the stellar atmosphere considered by Spiegel, namely, the influence of external solar radiation on radiative damping. Only by ignoring temperature changes in $\sigma_\nu$ and $\rho_\nu$ were we able to remove the dependence of $t_{rad}$ on the solar radiation intensity in planetary atmospheres.

We can gain some feeling for possible errors by comparing the normalized coefficients for $T'$ in the Taylor expansions of $B_\nu$ and $\sigma_\nu$. For the blackbody function this coefficient is

$$\frac{1}{B_\nu} \frac{\partial B_\nu}{\partial T} = \frac{h \nu}{k T^2} \left[ 1 - \exp \left( -\frac{h \nu}{k T} \right) \right]^{-1},$$

$$= 0.011 \text{ K}^{-1}.\quad (51)$$

Here $h$ is Planck’s constant, $k$ Boltzmann’s constant and the numerical value refers to a wavelength of 15 $\mu$m and a temperature of 296 K. For an isolated pressure-broadened Lorentzian line centered at a frequency $\nu_0$ we can readily show that

$$\frac{1}{\sigma_\nu} \frac{\partial \sigma_\nu}{\partial T} = \frac{S \partial T}{\gamma \partial T} + \frac{1}{T} \left[ \frac{\gamma}{\nu_0} \right]^2 \frac{\partial T}{\partial T} - \frac{1}{T^2} \frac{\partial T}{\partial T} \left[ 1 + \left( \frac{\gamma}{\nu_0} \right)^2 \right],$$

$$= 3 \times 153 \frac{1}{T^2} + \frac{1}{T^2} T \frac{1}{1 + \left( \frac{\gamma}{\nu_0} \right)^2} \frac{\partial T}{\partial T} \frac{\partial T}{\partial T},$$

$$= 0.0068 \text{ K}^{-1}$$

where $S$ is the integrated line strength and $\gamma$ the line half-width at half-intensity. The numerical values refer to the contribution to the center of the $Q$ branch of the 15 $\mu$m CO$_2$ band by transitions from initial states with rotational quantum number $J = 16$ at $T = 296$ K. Data were obtained from McClatchey et al. (1973) and we assume the impact theory (Goody, 1964, p. 106) for the pressure broadening. By comparison with (51) it is apparent that neglect of $\sigma_\nu$ perturbations cannot be justified on the basis that (52) is much smaller than (51). A comprehensive analysis including $\sigma_\nu$ perturbations will therefore be necessary before the actual errors involved in neglecting such perturbations can be established.

b. Perturbations to densities

Since $\rho$ only occurs as the ratio $\rho/\rho_0$ in the general $t_{rad}$ expression (13), $\rho$ perturbations can usually be neglected. If the absorbing gas is chemically inert we have for $\rho_0$

$$\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial T} = -\frac{1}{T},$$

$$= -0.0034 \text{ K}^{-1} \text{ (at 296 K)}$$

and thus perturbations in absorber density apparently cannot be neglected. Fortunately some cancellation of the type found for the $\alpha_\nu$ perturbations by Spiegel (1957) will also occur for these $\rho_0$ perturbations. When the absorbing gas is inert we therefore expect that the errors resulting from ignoring $\rho_0$ perturbations will be similar to, but less serious than, those for $\sigma_\nu$ which we have just discussed.

The errors involved in neglecting changes in $\rho_0$ due to phase changes are potentially much more serious. In a saturated air parcel in thermodynamic equilibrium we can utilize the Clausius-Clapeyron equation to obtain

$$\frac{1}{\rho_0} \frac{\partial \rho_0}{\partial T} = \frac{m_a L}{R T^2} \frac{1}{T},$$

$$= 0.057 \text{ K}^{-1}$$

where $L$ is the latent heat of evaporation or sublimation, $m_a$ the absorbing gas molecular weight and $R$ the gas constant. The value quoted in (54) refers to
the liquid to gas phase change for H₂O at 296 K. Although we may again expect some cancellation of the type found by Spiegel (1957), the magnitude of (54) in comparison to (51) and (52) implies that neglect of the ρₚ perturbation will in this case be more serious than neglect of ρₚ perturbations. In addition, unless the phase change occurs at the surface (e.g., ocean or icecap surface) these changes in gas phase concentrations will be accompanied by changes in particulate densities. We will therefore have to take into account the changes in the contributions to ρₚ by both gaseous and particulate absorption.

Finally, we also expect problems whenever the absorbing gas or cloud density is governed by temperature-dependent chemical reactions. The classical example is provided by ozone in the upper atmosphere (Leovy, 1964; Lindzen and Goody, 1965). In this case ozone principally absorbs solar radiation, and since the O₃ density ρₒ₃ is temperature dependent, perturbations to ρₒ₃=ρₒ₃σ₉ must be taken into account. The essential point is illustrated by consideration of the historical Chapman photochemical steady-state model for the ozone layer in which

\[
\frac{1}{\rho_{O_3}} \frac{\partial \rho_{O_3}}{\partial T} = \frac{1}{\rho_{O_3}} \left( 510 \frac{2300}{T^2} \frac{3}{T^2} \right) 
\]

\[
= -0.021 \text{ K}^{-1} \text{ (296 K)}
\]  

(55)

Modern theories take into account catalysis of the recombination of O and O₂ by odd nitrogen, odd hydrogen and chlorine species. Such catalysis in essence reduces the activation energy involved in odd oxygen destruction which is represented by the term 2300/T² in (55). Note, however, that even if this activation energy is zero, (55) is reduced to -0.008 K⁻¹ and not to zero. This point has been recently emphasized by Blake and Lindzen (1973) who studied the influence of odd-nitrogen and odd-hydrogen catalysis on cooling rates in the stratosphere in some detail. The magnitude of (55) relative to (51) makes it clear that the temperature dependence of the ozone density cannot be neglected; indeed Blake and Lindzen (1973) found tₚ was reduced by a factor of 2 or 3 by inclusion of this dependence.

c. Perturbations to heat capacities

The heat capacity of a gas generally depends on the number of degrees of freedom for the gas molecule and on the number of these degrees of freedom which are activated by collisions. Since this latter number increases with increasing temperature then Cₚ for most gases increases with temperature. However, the effect is small. For example, for most organic gases Cₛ typically doubles between 300 K and 800 K or ΔCₛ/(CₛΔT)≈0.001 K⁻¹, while for N₂, Cₛ increases by only 8% over the same temperature range ΔCₛ/(CₛΔT)≈0.0002 K⁻¹.

Phase changes and chemical reactions can often have a much more important effect on the heat capacity. For a system in which a phase change is occurring we can readily show that neglect of the phase change in (1) or (8) requires

\[
\frac{\rho_c}{\rho} \frac{C_p}{C_{pc}} \ll \approx 0.24,
\]

(56)

\[
\frac{\rho_a}{\rho} \frac{C_p}{C_{pc}} \ll \approx 0.007,
\]

(57)

where ρ_c and C_{pc} are respectively the density and heat capacity of the absorber in the condensed phase. The values quoted are for the water/water vapor system in air at 296 K and we have utilized (54) in evaluating (57). In the tropical atmosphere the inequality (56) is always true but the inequality (57) is often not. The effect of including the latent heat term in (1) or (8) will clearly be to increase the effective heat capacity of the system and thus to increase tₚ.

Similar arguments apply to the effect of chemical reactions. If ρ_p and C_{pp} are respectively the density and heat capacity of the products of the chemical reaction and h is the heat of the reaction, then the conditions required for ignoring the chemical reactions are

\[
\frac{\rho_p}{\rho} \frac{C_p}{C_{pp}} \ll \approx 1.23,
\]

(58)

\[
\frac{\rho_a}{\rho} \frac{C_p}{h} \ll \approx 0.016.
\]

(59)

The numerical values refer to the Chapman ozone scheme in which the heat of reaction for formation of O₃ from O₂ is -717 cal g⁻¹ and we have utilized (55) from our previous discussion. In this case ρₚ=ρₒ₃ and since ρₒ₃/ρ ≤ 10⁻⁴ in the upper atmosphere it is clear that perturbation of the heat capacity caused by interchange between chemical potential energy and kinetic energy is negligible. Blake and Lindzen (1973) reach a similar conclusion when odd nitrogen and odd hydrogen chemistry is included.

d. Second-order perturbations

To neglect the second-order term in the Taylor expansion of Bₛ we require at 296 K

\[
T' \ll \frac{\partial B_s}{\partial T} \approx 0.001 \text{ K}^{-1}.
\]

(60)

Provided this inequality holds we can generally be assured that the linear “Newtonian cooling” law is
applicable. In the terrestrial upper atmosphere and in the Martian atmosphere $T'$ may reach values of several tens of degrees and some nonlinearity in the cooling law may therefore result.

\subsection*{Spatial averages}

The fractional error involved in assuming spatially averaged values for $\sigma_r$ and $\partial B_r/\partial T$ can be estimated by a consideration of the ratios of the zeroth and first order terms in appropriate Taylor expansions of these two quantities. The averaging procedure is therefore good provided

\begin{equation}
|T(z)-\bar{T}| \ll \sigma_r \frac{\partial \sigma_r}{\partial T} \approx 147 \text{ K},
\end{equation}

\begin{equation}
|T(z)-\bar{T}| \ll \frac{\partial B_r}{\partial T} \frac{\partial^2 B_r}{\partial T^2} \approx 196 \text{ K},
\end{equation}

where we have used (52) and (60) in obtaining the numerical values. These conditions should be easy to meet at low altitudes. In addition an exponential decrease of absorber density with altitude will generally ensure that any excessively large values for $T(z)-\bar{T}$ at high altitudes will introduce little error into $t_{\text{rad}}$ computations in the lower atmosphere.

\section{Concluding remarks}

We have demonstrated that the conducting and radiating properties of the planetary surface, the altitude above the surface, the strength of turbulence in the boundary layer, the radiation mean-free path, and the scale height of the absorber density must all be considered when evaluating radiative time constants for harmonic atmospheric waves of a particular wavelength. When the exact radiative time constants are approximated by those computed for an infinite homogeneous atmosphere, we have found that neglect of the presence of a conducting surface will lead to the most serious errors. Neglect of an insulated surface and neglect of atmospheric inhomogeneity are of lesser concern particularly when order-of-magnitude estimates are required. Neglect of atmospheric inhomogeneity can formally lead to errors of several orders of magnitude in $t_{\text{rad}}$. However, such errors occur only when both the vertical wavelength of the atmospheric wave exceeds $2\pi$ times the scale height of the atmospheric absorber, and the atmosphere above the level in question is semi-transparent. These conditions are not often met in planetary atmospheres.

We have also demonstrated the circumstances under which additional effects must be included in the radiative damping computations. For example, phase changes and chemical reactions can often have considerable influence on radiative damping rates. Their influence may result from changes in the atmospheric opacity (e.g., by cloud or ozone formation) or in the heat capacity (e.g., by latent heat effects). Except for ozone in the terrestrial atmosphere these effects on $t_{\text{rad}}$ have not yet been treated explicitly. Further work is also required in assessing the relative accuracy of grey and non-grey calculations in inhomogeneous atmospheres above insulated and conducting surfaces. We are presently carrying out such calculations for the terrestrial troposphere and the results will appear elsewhere.

Finally, we emphasize the fact that in an inhomogeneous atmosphere the radiative damping time does vary significantly with altitude, particularly above a conducting surface. This will create problems for waves with long vertical wavelengths since the damping rate of the wave at low and high altitudes may be much faster than that at intermediate altitudes. The wave will thus be deformed with time in such a way as to alter the effective vertical wavelength. The theory developed in this paper will have to be modified to take into account this phenomenon which we expect to have some influence in determining the damping rate of deep atmospheric waves above terrestrial oceans.

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\section*{References}


