Parameterization of Raindrop Size Distributions

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ABSTRACT

The processes of condensation, coalescence and drop breakup tend to produce exponential raindrop size spectra. The intercept n_0 and slope λ of the exponential distribution are given by differential equations expressing the conservation of raindrop concentration and rainwater content M. The differential equations are solved numerically using published experimental data on coalescence efficiency and spontaneous breakup. The number of fragments resulting from a collisional breakup S_0 is taken as a variable parameter. Calculations show that 1) the effects of collisional breakup usually predominate over those of spontaneous breakup, 2) for $dM/dt=0$, a stationary λ results which is a function of S_0, and 3) for $dM/dt>0$, λ tends to a quasi-static value which depends upon S_0 and $(1/M^2)(dM/dt)$ but is close to the stationary value for the same S_0. In each case n_0 is determined by the values of λ and M. Binary interactions, i.e., drop coalescence and collisional breakup, tend to produce raindrop size spectra which have approximately constant λ and an n_0 approximately proportional to M. A method of parameterization for cumulus dynamics models is suggested in which both n_0 and λ are calculated.

1. Introduction

The size distribution of raindrops is of considerable interest to meteorologists for at least three reasons. First, the size distribution is of intrinsic interest because it throws light on the cloud physical processes responsible for the growth of raindrops. Second, radar meteorologists have been interested in the size distribution of raindrops in order to remotely measure rainfall by means of radar. Finally, the size distribution of raindrops is of interest in the numerical simulation of the dynamics of "warm" convective clouds. The dynamics of these clouds is significantly affected by precipitation loading requiring consideration of the size distribution of raindrops. Considerable savings in computer time is possible in the numerical simulation of convective clouds, especially for two- and three-dimensional models, if the size distribution can be parameterized thereby bypassing detailed microphysical calculations of raindrop growth.

A common parameterization of warm rain microphysics used in cumulus dynamics models assumes that the liquid water can be divided into cloud water and rainwater and that the raindrops are distributed in size according to the empirical Marshall–Palmer (1948) distribution

$$N(D)=N_0\exp(-\Lambda D),\quad N_0=0.08\ \text{cm}^{-4},\qquad (1)$$

where $N(D)\Delta D$ is the concentration of drops of diameter D to $D+\Delta D$. The distribution (1) is completely determined by the parameter Λ, which then determines uniquely other parameters, such as the rainfall rate R or the rainwater content M. The parameterization of the microphysics then proceeds by computing the changes in M, assuming that the size distribution always conforms to (1).

The assumption of an exponential raindrop size distribution is supported by both empirical and theoretical evidence. Measurements show that, *on the average*, raindrops are distributed exponentially with size, although the value of N_0 may differ considerably from the Marshall–Palmer value. Close agreement with the Marshall–Palmer value of N_0 is usually found in stratiform or melting band rain where warm rain processes are probably not the major factors in the development of the drop size distribution. In convective situations, on the other hand, large deviations from the Marshall–Palmer value of N_0 have been observed. For example, Waldvogel (1974) made extensive observations of raindrop size spectra, and expressed them in terms of N_0 and Λ of an "equivalent" exponential distribution which gave the water content and radar reflectivity factor computed from the observed distributions; since the observed distributions were approximately exponential, the N_0 and Λ of the equivalent distribution should be approximately equal to the N_0 and Λ which would be derived by fitting an exponential distribution to the observed raindrop size spectra. Waldvogel found the values of N_0 to range over two orders of magnitude, and suggested an association between the value of N_0 and the meteorological situation.

The theoretical basis for assuming an exponential raindrop size distribution lies in the processes of coalescence and breakup of raindrops. Hitschfeld (1955)

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appears to be the first to have suggested that the exponential form of raindrop size distributions may be due to random coalescence. Srivastava (1967) computed the evolution of raindrop size spectra by coalescence between raindrops and found that exponential distributions change only slowly, while narrow distributions develop rapidly into exponential distributions. Srivastava (1971) also calculated the evolution of raindrop size spectra by coalescence and spontaneous breakup, using the data of Komabayasi *et al.* (1964) on the probability of spontaneous breakup of water drops and the size distribution of the fragments resulting from the breakup. The operation of coalescence and spontaneous breakup resulted in equilibrium drop size distributions which were similar to but flatter than observed raindrop size distributions. It was suggested that inclusion of the effects of condensation of water vapor and disintegration of drops on collision may help to produce better agreement between theory and observation. Laboratory observations of the disintegration of water drops on collision, and the size distribution of the resulting fragments, were reported by Brazier-Smith *et al.* (1972) and McTaggart-Cowan and List (1975). Computations of the evolution of drop size spectra by collisional breakup and other processes were reported by Brazier-Smith *et al.* (1973a,b), Young (1975) and Gillespie and List (1976). Young considered both cloud and raindrop sizes, and the processes of activation of cloud condensation nuclei, drop growth by condensation, drop coalescence and breakup. Young (1975) found that collisional breakup dominated over spontaneous disintegration and that the processes considered tended to produce equilibrium distributions in good agreement with the empirical Marshall–Palmer distribution. Gillespie and List (1976) considered the effects of spontaneous disintegration to be negligible. They modeled a one-dimensional steady-state rainshaft, with no vertical air motions, and computed the evolution of raindrop size spectra by coalescence and collisional breakup during fall. Gillespie and List found that exponential distributions remained exponential, although the intercept N_0 and slope Λ changed during fall and tended to equilibrium values. The intercept N_0 (and the radar reflectivity factor) was found to be proportional to the rainfall rate.

From the above, it is seen that both observation and theory tend to support an exponential form of the dropsize distribution [as given in Eq. (1)] but without the restriction to a constant N_0. In this paper we propose a parameterization of raindrop size distribution, primarily for use with numerical cumulus dynamics models, in which the raindrop size distribution is assumed to be exponential, but in contrast to earlier parameterizations both the intercept N_0 and slope Λ are treated as variable parameters. These parameters are determined by calculating the changes in rainwater content and total raindrop concentration by microphysical processes.

2. Governing equations

In this section, we shall state the equation for the development of drop size distribution and use it to derive equations for the total raindrop concentration and rainwater content. Let $f(m,t)$ be the concentration density function for the raindrops, that is, $f(m,t)\Delta m$ is the concentration of raindrops of mass m to $m+\Delta m$ at time t. The time evolution of f will be considered to be governed by the following processes: 1) capture of cloud drops, 2) condensation (evaporation) of water vapor, 3) coalescence between raindrops, 4) collisional breakup of raindrops, 5) spontaneous breakup of raindrops and 6) production of raindrops by coalescence between cloud drops (called autoconversion). The time rate of change of f is given by the kinetic equation

$$\partial f(m,t)/\partial t+\partial[f(m,t)\dot{m}]/\partial m=C+B_c+B_s+A(m,t), \quad (2)$$

where C, B_c, B_s and A represent, respectively, the rate of change of f by processes 3–6 above. The second term on the left-hand side includes the effects of processes 1 and 2, $\dot{m}$ being the rate of change of raindrop mass by these processes with a continuum formulation being assumed for process 1. Explicit expressions for the first three terms on the right-hand side of Eq. (2) are

$$C=(\tfrac{1}{2})\int_0^m f(m',t)f(m-m',t)K(m',m-m')\times q(m',m-m')dm'-f(m,t)\int_0^\infty f(m',t)\times K(m,m')q(m,m')dm', \quad (3)^1$$

$$B_c=(\tfrac{1}{2})\int_0^\infty\int_0^\infty s(m;m',m'')f(m',t)f(m'',t)\times K(m',m'')[1-q(m',m'')]dm'dm''-f(m,t)\int_m^\infty f(m',t)K(m,m')\times[1-q(m,m')]dm', \quad (4)$$

$$B_s=-f(m,t)P(m)+\int_0^\infty Q(m',m)f(m',t)\times P(m')dm', \quad (5)$$

$$K(m',m'')=\pi(r'+r'')^2|V_T'-V_T''|, \quad (6)$$

where r and V_T are the radius and terminal fallspeed of drop of mass m. The term $q(m',m'')$ represents the

[1] In most of the integrals the lower limit of integration is taken as 0; strictly speaking it should be taken as the mass of the smallest drop considered to be a "raindrop" in the parameterization.

probability of coalescence following a collision between m' and m''. We shall take q as given by Brazier-Smith *et al.* (1972):

$$q(m',m'')=\begin{cases} q_0(m',m''), & \text{if} \quad q_0\leqslant 1 \\ 1, & \text{if} \quad q_0>1 \end{cases} \tag{7}$$

where

$$q_0(m',m'')=\frac{2.4\sigma(r'+r'')^{11/3}[(r'^2+r''^2)-(r'^3+r''^3)^{\frac{2}{3}}]}{\rho_w(r'+r'')^2(V'_T-V''_T)^2(r'r'')^6}, \tag{8}$$

and σ and ρ_w are the surface tension and density of water, respectively. The above formulation for q differs from that of Whelpdale and List (1971).

It is seen that the terms on the right-hand side of Eq. (3) represent the usual formulation (e.g., Berry, 1967) for the rate of gain and loss of $f(m,t)$ by coalescence, except for the occurrence of the q term which takes account of the fact that only the fraction $q(m',m'')$ of binary collisions between drops of masses m' and m'' results in coalescence. The fraction $[1-q(m',m'')]$ of collisions between drops of masses m' and m'' results in breakup. This results in a loss to $f(m,t)$ at the rate given by the second term on the right-hand side of (4). It is seen that the two loss terms in (3) and (4) are similar; if these two terms are combined $q(m',m'')$ cancels out which is an expression of the fact that in a binary collision of a drop of mass m with a drop of any mass m', there is a loss of the drop of mass m whether or not the collision is followed by coalescence.

The production of drops of mass m by collisional disintegration is represented by the first term on the right-hand side of (4). The function s is such that $s(m;m',m'')\Delta m$ is the average concentration of drops of m to $m+\Delta m$ resulting from the collisional breakup of a drop of mass m' and a drop of mass m''. Physical considerations dictate that

$$s(m;m',m'')=0, \quad m\geqslant m'+m'', \tag{9a}$$

$$\int_0^\infty ms(m;m',m'')=m'+m''. \tag{9b}$$

The interpretation of the first term on the right-hand side of (4) should now be obvious. This term considers the production of drops of mass m from collisional breakup of all possible drop pairs (m',m''); the factor $\frac{1}{2}$ is required in front of the integral because each drop pair is counted twice in the double integration.

The present formulation of the term B_c differs from that of Brazier-Smith *et al.* (1973a,b) and Gillespie and List (1976). It is helpful to note that the sum of B_c and B_s reduces to the case of pure or perfect (no collisional distintegration) coalescence by putting $q=1$. The same result is also obtained if instead of putting $q=1$, we put

$$s(m;m',m'')=\delta(m-[m'+m'']), \tag{10}$$

where δ is the Dirac delta function.

The term B_s represents the effects of spontaneous breakup. In Eq. (5), $P(m)\Delta t$ is the probability that a drop of mass m will disintegrate spontaneously in time Δt and $Q(m',m)\Delta m$ is the average number of drops of mass m to $m+\Delta m$ produced by the spontaneous distintegration of a drop of mass m'. Expressions for P and Q will be taken as deduced from the data of Komabayasi *et al.* (1964) by Srivastava (1971):

$$P(r)=2.94\times10^{-7}\exp(34r) \quad [\text{s}^{-1}], \tag{11}$$

$$Q(r',r)=(ab/r')\exp(-br/r') \quad [\text{cm}^{-1}], \tag{12}$$

where $a=62.3$, $b=7$ and r is in centimeters in Eq. (11). Physical considerations show that we must have

$$\left.\begin{aligned} &Q(m',m'')=0, \quad m''\geqslant m', \\ &\int m''Q(m',m'')dm''=m'. \end{aligned}\right\} \tag{13}$$

The term $A(m,t)$ in Eq. (2) represents the effect of autoconversion. We shall not need to consider this term explicitly.

An equation for the rainwater content is obtained by multiplying Eq. (2) by m and integrating over m:

$$dM/dt=\dot{M}=\int_0^\infty f(m,t)\dot{m}dm+\int_0^\infty mA(m,t)dm, \tag{14}$$

where the two terms on the right-hand side represent the effects of cloud capture, condensation (evaporation) of water vapor and cloud autoconversion. It has been assumed that the disintegration of raindrops gives rise to raindrops. An equation for the total raindrop concentration N_T is obtained by integrating Eq. (2) over m:

$$dN_T/dt=N_A+N_E+N_1+N_2, \tag{15}$$

where N_A, N_E, N_1 and N_2 represent the rates of increase of raindrop concentration by autoconversion, evaporation, spontaneous disintegration and binary interactions (i.e., collisions and coalescence and collisional breakup), respectively. Using the equations of the previous section it may be shown that

$$N_1=\int f(m,t)P(m)[Q_0(m)-1]dm, \tag{16}$$

$$\begin{aligned} N_2=&-\frac{1}{2}\int\int f(m',t)f(m'',t)K(m',m'')q(m',m'')dm'dm'' \\ &+\frac{1}{2}\int\int f(m',t)f(m'',t)K(m',m'')[1-q(m',m'')] \\ &\times[s_0(m',m'')-2]dm'dm'', \end{aligned} \tag{17}$$

where

$$Q_0(m)=\int_0^m Q(m,m')dm', \tag{18}$$

$$s_0(m',m'')=\int s(m;m',m'')dm, \tag{19}$$

represent, respectively, the average total number of fragments resulting from the spontaneous breakup of a drop of mass m and the collisional breakup of a pair of drops of masses m' and m''. In the latter case $[s_0(m',m'')-2]$ is often referred to as the average number of satellites resulting from the binary interaction.

It is interesting and illuminating to interpret some of the above equations intuitively. Thus, it is clear that the rainwater content can change only by the capture of cloud drops, by condensation (evaporation) and by autoconversion, and that it cannot change as a result of interactions between raindrops; hence, Eq. (14) involves only terms representing the former processes. Eq. (15) for the rate of change of raindrop concentration can also be understood intuitively. The terms N_A and N_E take account of the change of concentration by autoconversion and raindrop evaporation, respectively. The term N_1 [Eq. (16)] represents the rate of increase of concentration by spontaneous distintegration since, on disintegration, one drop of mass m is replaced by $Q_0(m)$ drops. The term N_2 represents the rate of change of concentration by binary interactions. The first term on the right-hand side of Eq. (17) simply represents the fact that in a coalescence two drops are replaced by one drop. The second term on the right-hand side of Eq. (17) represents the fact that in a collisional disintegration two drops are replaced by s_0 drops. The $\frac{1}{2}$ in front of the double integrals in Eq. (17) takes account of the fact that each pair of binary interactions is counted twice inside the double integrals.

The well-known equation for the rate of increase of concentration by perfect coalescence

$$dN_T/dt=-\frac{1}{2}\int\int f(m',t)f(m'',t) \times K(m',m'')dm'dm'' \quad (20)$$

may be seen as a special case of the above equations by putting $q=1$ in Eq. (17). The same result is obtained by putting (10) in (19) which gives $s_0=1$ and then substituting $s_0=1$ in Eq. (17); it is clear that $s_0=1$ implies that in a collisional disintegration two drops are replaced by one drop which is tantamount to a coalescence.

3. Proposed parameterization

We proceed by assuming that the raindrop size distribution is exponential at all times, but that both N_0 and Λ may vary. We shall also express Eq. (1) in terms of the radius rather than the diameter; thus

$$n(r)=n_0\exp(-\lambda r). \quad (21)$$

Then we must have

$$n_0=2N_0,\quad \lambda=2\Lambda. \quad (22)$$

The total concentration and water content of the distribution (21) are given by

$$N_T=n_0/\lambda, \quad (23)$$

$$M=8\pi\rho_w n_0/\lambda^4. \quad (24)$$

From Eq. (24) and substituting (23) into (15) we have

$$d(n_0/\lambda^4)/dt=\dot{M}/8\pi\rho_w, \quad (25)$$

$$d(n_0/\lambda)/dt=N_A+N_E+n_0I_1(\lambda)+n_0^2I_2(\lambda), \quad (26)$$

where

$$I_1(\lambda)=2.94\times10^{-7}a(1-e^{-b})\{\exp[(34-\lambda)r_*]-1\}/(34-\lambda), \quad (27)$$

$$I_2(\lambda)=-I_{21}(\lambda)+I_3(\lambda), \quad (28)$$

$$I_{21}(\lambda)=+(\pi/2)\int\int(r'+r'')^2|V_T'-V_T''|q(r',r'') \times\exp[-\lambda(r'+r'')]dr'dr'', \quad (29)$$

$$I_3(\lambda)=(\pi/2)\int\int(r'+r'')^2|V_T'-V_T''|[1-q(r',r'')] \times\exp[-\lambda(r'+r'')][s_0(r',r'')-2]dr'dr''. \quad (30)$$

Eq. (27) follows from (11), (12), (16) and (18), and Eqs. (29) and (30) follow from (3), (6), (17) and (19). In Eq. (27) r_* is the maximum raindrop radius to be considered. The limits on the integrals in (29) and (30) are to be taken as the minimum raindrop radius to be considered and r_*. In arriving at Eqs. (23) and (24), the upper and lower limits of integration were taken as 0 and ∞: the basic idea and method of the present parameterization are not changed if these limits of integration are taken to be the minimum raindrop radius and r_*, but the resulting equations are then tedious to handle analytically.

It should be noted that the four terms on the right-hand side of Eqs. (15) and (26) correspond to each other in the order written. Further, I_1 and I_2 in Eq. (26) are independent of n_0. The reasons for the proportionality of the third and fourth terms on the right-hand side of Eq. (26) to n_0 and n_0^2, respectively, should be obvious. The third term represents the rate of change of concentration by spontaneous disintegration, and hence is proportional to n_0, while the fourth term represents the rate of change due to binary interactions, and hence is proportional to n_0^2. The further breakdown of I_2 in Eqs. (28) is such that $-n_0^2I_{21}(\lambda)$ represents the rate of decrease of N_T by collisions followed by coalescence, while $n_0^2I_3(\lambda)$ represents the rate of increase of N_T by collisional disintegration.

For later use it is convenient to restate Eq. (30) by defining a new function $S_*(\lambda)$:

$$I_3(\lambda)=S_*(\lambda)I_{22}(\lambda), \quad (31)$$

where

$$I_{22}(\lambda)=(\pi/2)\int\int(r'+r'')^2|V'_T-V''_T|[1-q(r',r'')]$$
$$\times\exp[-\lambda(r'+r'')]dr'dr''. \quad (32)$$

By comparing Eqs. (30), (31) and (32), it is seen that $S_*(\lambda)$ is the average value of $[s_0(r',r'')-2]$ weighted by the product of the drop concentration and the collection kernel, that $n_0^2S_*(\lambda)I_{22}(\lambda)$ is the rate of increase of N_T by collisional disintegration, and since $n_0^2I_{22}(\lambda)$ is the number of collisions per unit volume per unit time that is followed by disintegration, $S_*(\lambda)$ may be interpreted as the average number of satellites.

Eqs. (25) and (26) constitute the basic equations of the proposed parameterization. They can be solved for n_0 and λ provided $\dot{M}$, N_A, N_E, q and s_0 are known.

4. Examples

We give below examples of the use of the proposed parameterization. In all cases, we assume $N_A=N_E=0$, that is, no increase or decrease of drop concentration by autoconversion or evaporation, respectively. The term $\dot{M}$, representing the rate of increase of rainwater content by autoconversion, cloud capture and condensation, will be taken as a prescribed constant in our illustrative examples. The effects of these and other simplifying assumptions will be discussed later. In the first set of examples, we take $\dot{M}=0$ and compute n_0 and λ of the resulting equilibrium distributions. In the second set of examples, we take $\dot{M}>0$ leading to evolving distributions which tend to an asymptotic form as time progresses.

In the computations involved in the following examples, we shall need I_2 [see Eq. (28)]. In evaluating I_3, we make the further simplifying assumption that

$$s_0(r',r'')=S_0+2=\text{constant}. \quad (33)$$

We shall regard S_0 as a parameter which will be varied from case to case. The assumption of a constant S_0 follows the treatment of Young which was based on observational findings of Brazier-Smith *et al.* (1972). With this assumption Eq. (28) becomes

$$I_2=I_2(S_0,\lambda)=-I_{21}(\lambda)+S_0I_{22}(\lambda), \quad (34)$$

where I_{22} is given by (32). With these additional assumptions, Eq. (26) becomes

$$d(n_0/\lambda)/dt=n_0I_1(\lambda)+n_0^2I_2(S_0,\lambda). \quad (35)$$

It may be repeated that the first term on the right-hand side of (35) represents the rate of increase of total concentration by spontaneous disintegration, while the second term represents the rate of increase by binary interactions and that the latter can be decomposed into contributions due to coalescence $[-I_{21}(\lambda)]$ and collisional breakup $[S_0I_{22}(\lambda)]$ as shown in Eq. (34).

The term I_1 is given explicitly by Eq. (27). The double integrals I_{21} and I_{22} were evaluated numerically, the lower and upper limits of integration being taken as 0.01 and 0.30 cm radius, respectively. The terminal fall velocities were taken from the data of Gunn and Kinzer (1949). The numerical integrations were carried out using 8-point Gaussian quadratures with successive interval subdivision until the results of the quadratures agreed to within 1%. The slope parameter λ was varied from 20 to 150. It was found that the double integrals I_{21} and I_{22} could be fairly well approximated by inverse power series in λ with four terms (i.e., $I_{21},I_{22}=c_0\lambda^{-4}+c_1\lambda^{-3}+c_2\lambda^{-2}+c_3\lambda^{-1}$, c_i constants) by dividing the interval $20\leqslant\lambda\leqslant150$ into two parts. Although the power series for I_{21} and I_{22} were determined only for $20\leqslant\lambda\leqslant150$, we have extrapolated these series to $\lambda=160$ in the examples to be presented below.

a. *Stationary distributions*

Three cases involving the following combinations of processes will be considered: 1) perfect coalescence (no collisional breakup, $q=1$) and spontaneous breakup, 2) coalescence and spontaneous and collisional breakup, and 3) coalescence and collisional breakup (binary interactions only).

Since we assumed $\dot{M}=0$, equilibrium is achieved when the left-hand side of (35) is also equal to zero; that is, when

$$I_1(\lambda)+n_0I_2(S_0,\lambda)=0. \quad (36)$$

To appreciate the behavior of solutions of (36), we consider the behavior of I_1 and I_2. Fig. 1 shows I_1 as a function of λ; I_1n_0 gives the rate of production of drops

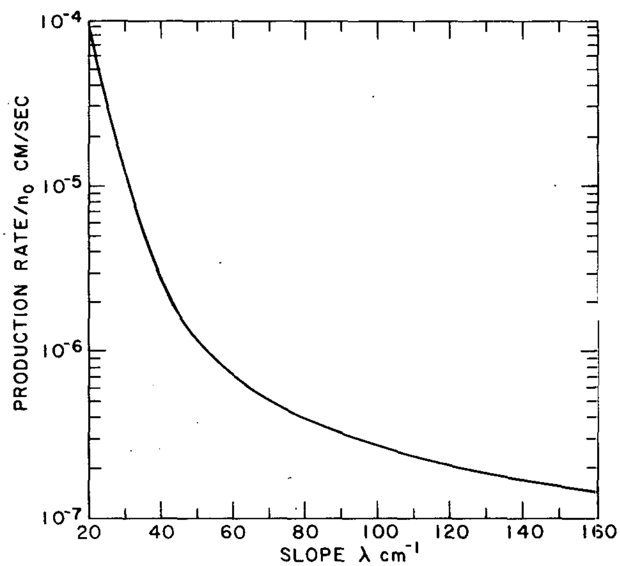

FIG. 1. Normalized rate of production of fragments by spontaneous breakup as a function of λ. Ordinate multiplied by n_0 gives production rate (cm^{-3} s^{-1}).

by spontaneous breakup. Fig. 2 shows contours of I_2 on a (S_0,λ) plot; $n_0^2 I_2$ gives the rate of increase of raindrop concentration by collisional disintegration and coalescence. Hence in contrast to I_1, which is always positive, I_2 assumes both positive and negative values. Along the thick solid line in Fig. 2, there is no change of drop concentration by binary interactions.

For case 1, the equilibrium n_0 is given by

$$n_0 = -I_1(\lambda)/I_2(-1,\lambda). \qquad (37)$$

The solution of (37) is shown in Fig. 3. Since n_0 is determined as a function of λ, M determines both n_0 and λ. It is seen that n_0 is smaller than the Marshall–Palmer ($0.16\ cm^{-4}$) value for $\lambda \lesssim 125\ cm^{-1}$, with a corresponding M of approximately $1.6\times10^{-2}\ g\ m^{-3}$. Calculations show that the maximum M ($0.8\ g\ m^{-3}$), in the range of λ considered, occurs at $\lambda = 20\ cm^{-1}$ with $n_0 = 5.5\times10^{-3}\ cm^{-4}$. A Marshall–Palmer distribution has a constant $n_0 = 0.16\ cm^{-4}$ showing that, if exponential distributions resulted from an equilibrium between perfect coalescence and spontaneous disintegration, their slopes would be much smaller than those of the Marshall–Palmer distribution having the same M, at least for $M \lesssim 1\ g\ m^{-3}$. From his numerical simulations, Young (1975) also found the equilibrium distributions to be much flatter than the Marshall–Palmer distribution.

For case 2, equilibrium n_0 is given by

$$n_0 = -I_1(\lambda)/I_2(S_0,\lambda). \qquad (38)$$

This shows that an equilibrium distribution is possible only to the right of the thick solid line in Fig. 2 if a balance between spontaneous disintegration and binary interactions prevails. To the left of the line, the distribution must evolve because both spontaneous distintegration and binary interactions tend to increase the total drop concentration. The intercept n_0 (Fig. 4) attains a minimum along the dashed line and increases indefinitely as the thick solid line ($I_2 = 0$) is approached. Comparison of Figs. 3 and 4 shows that the effects of spontaneous disintegration predominate over those of collisional disintegration to the right of the dashed line; the reverse is the case to the left of the dashed line. Again calculation shows that M is less than or equal to a few tenths of a gram per cubic meter in the portion of the diagram to the right of the dashed line showing that the effects of binary interactions generally predominate over those of spontaneous disintegration in determining raindrop size spectra, a conclusion arrived at by Young (1975) by numerical calculations.

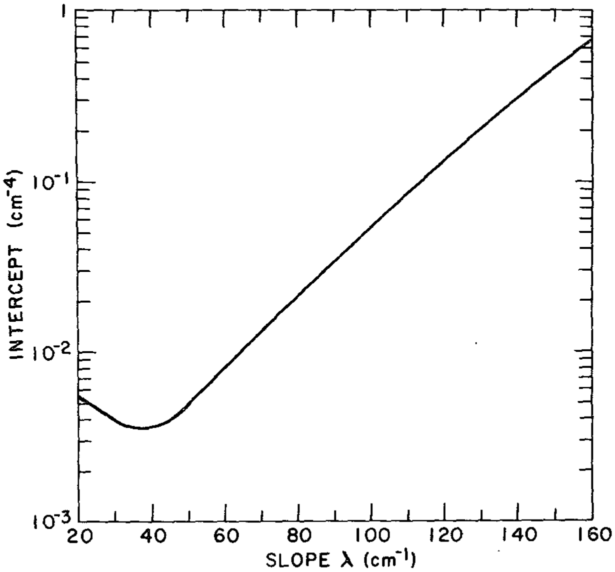

FIG. 3. Equilibrium intercept as a function of slope for perfect coalescence and spontaneous breakup.

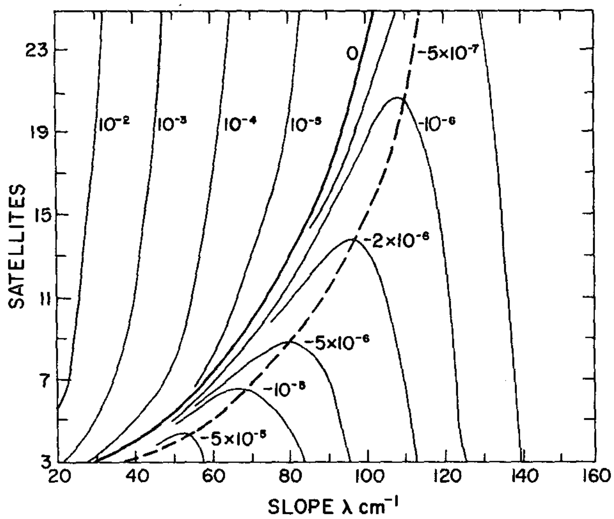

FIG. 2. Normalized rate of production of fragments by collisional disintegration as a function of λ and S_0. Numbers multiplied by n_0^2 give production rate ($cm^{-3}\ s^{-1}$).

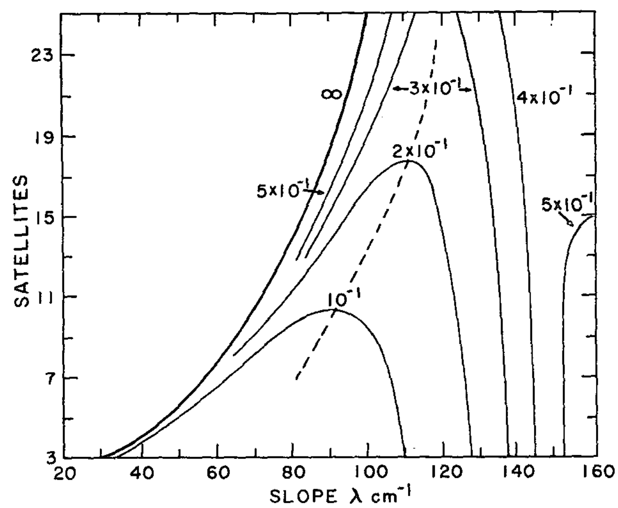

FIG. 4. Contours of equilibrium intercept n_0 (cm^{-4}) as a function of S_0 and λ for coalescence and spontaneous and collisional breakup.

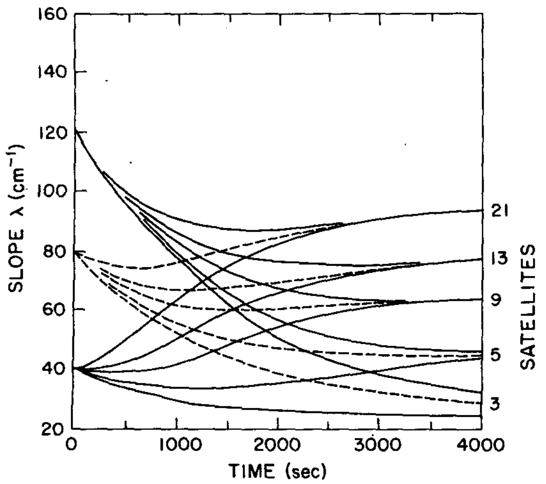

FIG. 5a. Slope as a function of time for $\lambda_0=40$ (solid), 80 (dashed) and 120 cm^{-1} (solid), and $S_0=3$, 5, 9, 13 and 21 as indicated on the right. The water content increased from 0.2 g m^{-3} at $t=0$ to 1.2 g m^{-3} at $t=4000$ s.

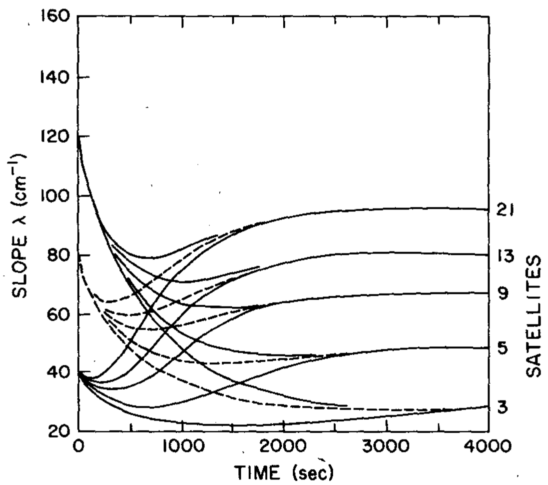

FIG. 5b. As in Fig. 5a except that the water content increased from 0.2 g m^{-3} at $t=0$ to 4.2 g m^{-3} at $t=4000$ s.

Case 3 involves only binary interactions; in this case the equilibrium distribution must satisfy

$$I_2(S_0,\lambda)=0 \tag{39}$$

from which n_0 has completely disappeared. The equilibrium λ is given by the thick solid line in Figs. 2 or 4. Thus, λ is determined for any given S_0 and n_0 must take on a value consistent with the water content (or rainfall rate). This result is not surprising. The rate of change of total drop concentration by all processes considered in this case is proportional to n_0^2; hence, if we equate the rate of change of total drop concentration to zero, only λ is determined by the equation.

Our results for case 3 are similar to those of Gillespie and List (1976) who calculated the evolution of raindrop size spectra by binary interactions only. They found that the rainfall rate and radar reflectivity factor of the equilibrium exponential distributions were proportional to n_0, which implies that the equilibrium distributions had a constant λ.

b. *Evolving distributions*

In this case, we assume $N_A=N_B=0$ and let $\dot{M}$ be a prescribed constant. We shall also put $I_1=0$, in keeping with our previous discussion which has suggested that the effect of collisional breakup generally dominates over that of spontaneous breakup. This was confirmed by performing some of the following calculations *without* assuming $I_1=0$; the results were virtually unchanged from those *with* $I_1=0$.

Under the above assumptions, Eqs. (25) and (26) may be written as

$$M=M_0+\dot{M}t, \tag{40}$$

$$\frac{1}{\lambda}\frac{d\lambda}{dt}=(M/3)[\lambda^5 I_2(S_0,\lambda)/(8\pi\rho_w)-\dot{M}/M^2], \tag{41}$$

where M_0 is the initial value of M at $t=0$. Given S_0, M_0 and λ_0 (initial value of λ), Eqs. (40) and (41) give M and λ as functions of t; n_0 may then be calculated from M and λ.

The qualitative features of the solution of (40) and (41) may be seen from the behavior of the two terms on the right-hand side of (41). Suppose that for a given S_0, we start with a sufficiently large λ_0 (to the right of the thick solid line in Fig. 2); then the right-hand side of (41) will be negative (assuming $\dot{M}\geqslant 0$) and λ will decrease with time. The right-hand side of (41) may become positive only after λ crosses to the left of the thick solid line in Fig. 2. Calculations show that eventually a quasi-static value of λ emerges such that the term on the left-hand side of (41) is much smaller than any of the other terms [i.e., the terms involving I_{21}, I_{22} and $\dot{M}$; see Eq. (34)] on the right-hand side. The calculations also show that the quasi-static value of λ is near the zero (thick solid) line in Fig. 2.

Figs. 5 and 6 show numerical solutions for λ and n_0 as functions of t for various cases. Fig. 5 gives λ for $S_0=3$, 5, 9, 13 and 21 and $\lambda_0=40$ cm^{-1} (solid line), 80 cm^{-1} (dashed line), 120 cm^{-1} (solid line). In Fig. 5a, M changes linearly from 0.2 g m^{-3} at $t=0$ to 1.2 g m^{-3} at $t=4000$ s, while in Fig. 5b it changes from 0.2 g m^{-3} at $t=0$ 4.2 g m^{-3} at $t=4000$ s. Figs. 6a and 6b give the intercept n_0 corresponding to the cases in Figs. 5a and 5b; Fig. 6c is similar to Figs. 6a and 6b except that M changes linearly from 0.2 g m^{-3} at $t=0$ to 16.2 g m^{-3} at $t=4000$ s. It is seen that irrespective of its initial value, λ tends to a constant value which is essentially a function of S_0, being relatively independent of M and $\dot{M}$.

Since λ tends to a constant and M is proportional to n_0/λ^4, $n_0(t)$ shows an "equilibrium" phase in which it is approximately proportional to M. The intercept n_0 is seen to increase with increasing S_0 and M. According to Brazier-Smith *et al.* (1972) $S_0\approx 3$. McTaggart-Cowan and List (1975) found three types of breakup—"necks," "sheets" and "disks"—with the average number of fragments in the three kinds of breakup being such that $S_0=5.1$, 7.0 and 13.3, respectively. The average S_0 considering the frequency of occurrence of the three kinds of breakup was 7.6. The above calculations show that for $S_0=3$, the "equilibrium" n_0 is smaller than the Marshall–Palmer value (0.16 cm^{-4}) except for water contents $\gtrsim 5$ g m^{-3}. For $S_0 \approx 7.6$, n_0 generally exceeds the Marshall–Palmer value in the equilibrium phase of $n_0(t)$, for $M \gtrsim 1$ g m^{-3}, while for smaller M it is comparable to or smaller than the Marshall–Palmer value.

We may compare the above results with the calculations of Young (1975) as given in his Fig. 4; there $S_0=3$, $M=5.65$ g m^{-3}, $\dot{M} \approx 5.65$ g m^{-3} (30 min)$^{-1}$. Calculations show that with these values we should have $\lambda \approx 30$ corresponding to the equilibrium value for $S_0=3$ (see Fig. 2). With $M=5.65$ g m^{-3}, $\lambda=30$ implies $n_0=0.18$ or $N_0=0.09$ cm^{-4} which fits the distribution curve in Young's Fig. 4 fairly closely. (It may be mentioned here that Young's paper gives $\Lambda=9.4$ cm^{-1}, $N_0=0.076$ cm^{-4}, i.e., $\lambda=18.8$ cm^{-1}, $n_0=15.2$ cm^{-4}, which imply $M \approx 30$ g m^{-3}; therefore, we have not used the value of λ given by Young for this comparison.)

5. Discussion

The physical basis for the above parameterization is straightforward. Numerical calculations reviewed in the Introduction showed that the operating physical processes tend to produce exponential drop size spectra similar to the Marshall–Palmer distribution but without the restriction to a constant n_0. Therefore, we started by assuming an exponential distribution with two parameters (n_0 and λ) and set up equations to determine these parameters by calculating the rates of change of total drop concentration and water content. In this way be bypassed the time-consuming detailed calculations of size distributions and the need to know the functions expressing the size distribution of fragments resulting from drop breakup.

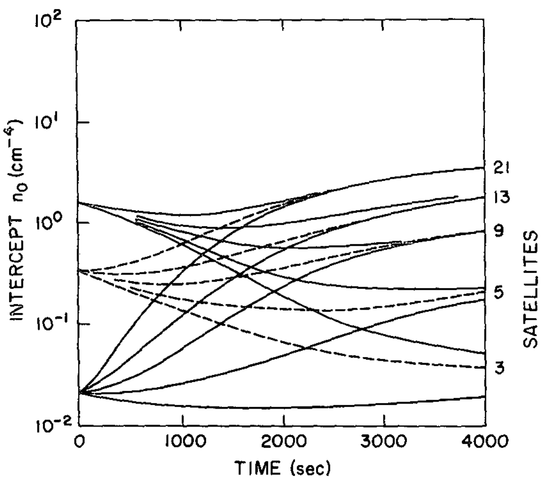

FIG. 6a. Intercept as a function of time for $\lambda_0=40$ (solid), 80 (dashed) and 120 cm^{-1} (solid), and $S_0=3$, 5, 9, 13 and 21 as indicated on the right. The water content increased from 0.2 g m^{-3} at $t=0$ to 1.2 g m^{-3} at $t=4000$ s.

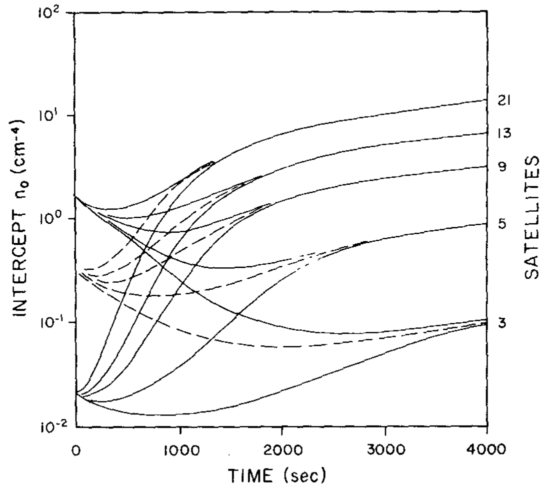

FIG. 6b. As in Fig. 6a except that the water content increased from 0.2 g m^{-3} at $t=0$ to 4.2 g m^{-3} at $t=4000$ s.

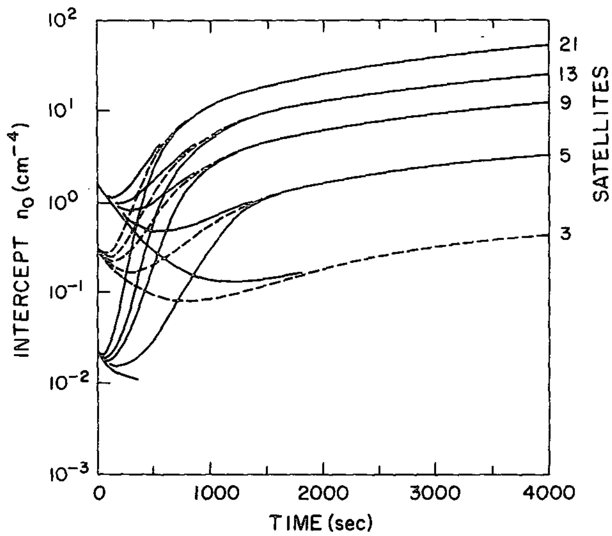

FIG. 6c. As in Fig. 6a except that the water content increased from 0.2 g m^{-3} at $t=0$ to 16.2 g m^{-3} at $t=4000$ s.

a. *Assumptions*

In the light of the above, we can qualitatively discuss the effects of some of the simplifying assumptions made in this study. We assumed $N_A=N_E=0$, that is, the total drop concentration was assumed not to change as a result of autoconversion or evaporation. Actually, these factors can be taken into account if we set up models for N_A and N_E. However, this is outside the scope of the present paper, and it will suffice to note that when precipitation is well developed in a cloud we have $N_E=0$ and N_A is probably small compared to the changes of total concentration by drop-drop interactions.

Another assumption made in the above was $\dot{M}=0$. Again, in a detailed model, we can set up $\dot{M}$ as a function of n_0, λ and cloud water content for cloud capture (e.g., Kessler, 1969) and other processes. However, physical considerations, and the results of above calculations, suggest that such formulations for $\dot{M}$ are not likely to change the qualitative behavior of the above solutions.

A final assumption concerns the constancy of S_0. Gillespie and List showed that $s_0=s_0(r',r'')$. The effects of a variable S_0 may be seen qualitatively. Eq. (35) now becomes

$$d(n_0/\lambda)/dt=n_0I_1(\lambda)+n_0^2I_2(S_*,\lambda), \qquad (42)$$

where S_* is defined by Eq. (31) and

$$I_2(S_*,\lambda)=-I_{21}(\lambda)+S_*(\lambda)I_{22}(\lambda) \qquad (43)$$

replaces Eq. (34). Let us consider case 3 of Section 4a ($\dot{M}=0$, only binary interactions). In this case, n_0 again drops out of the equation and λ is determined by

$$I_2(S_*,\lambda)=0, \qquad (44)$$

that is, we again have a constant λ with n_0 being determined by the prevailing value of M (or rainfall rate). This qualitative conclusion is confirmed by the detailed calculations of Gillespie and List which gave a constant λ.

b. *Implications*

The above calculations showed that for $M\gtrsim 1$ g m^{-3} the effects of spontaneous disintegration are small. Considering only binary interactions, the slope parameter λ tends to a constant stationary value ($\dot{M}=0$) or a quasi-static value ($\dot{M}>0$) which is quite close to the stationary value, and the intercept n_0 adjusts to the prevailing value of M. Thus raindrop size distributions should tend to have approximately a constant slope and an intercept proportional to M. This is in sharp contrast to the properties of the empirical Marshall–Palmer distribution which has a constant intercept and a slope which is consistent with the value of M. In connection with this discrepancy we should note the following: 1) Marshall and Palmer's observations were limited to $R\lesssim 25$ mm h^{-1}, corresponding $M\lesssim 1.2$ g m^{-3}, and as discussed earlier binary interactions alone are not dominant for $M\lesssim 1.0$ g m^{-3}; and 2) the size distributions observed by Marshall and Palmer were probably for melting band rain in which warm microphysics may not be dominant in determining the size distribution. At high water contents binary interactions should dominate, and the question arises if there is any evidence of a constant λ for large rainfall rates. Blanchard and Spencer (1970, Fig. 9) have summarized drop size distributions from various sites for R ranging from 25 to 1550 mm h^{-1}. The outstanding feature of these distribution curves is that they are approximately exponential and parallel to each other giving a constant λ and n_0 proportional to M or R. A fit by eye to the Blanchard and Spencer curves give $\lambda\approx 38$ cm^{-1}. Comparison with Fig. 2 shows that this requires $S_0\approx 4$, not an unlikely value. Thus, there does appear to be some observational evidence which suggests that for $M\gtrsim 1$ g m^{-3}, binary interaction processes tend to control drop size distributions and produce exponential distributions with a constant slope λ.

c. *Parameterization for cumulus dynamics models*

The parameterization of raindrop size distributions suggested here may be used in cumulus dynamics models by extending it to three space dimensions (x,y,z). Eq. (2) becomes

$$\partial f/\partial t+\mathrm{Div}[\mathbf{V}_a f]+\partial(f\dot{m})/\partial t = C+B_s+A+\partial(fV_t)/\partial z+B_c, \qquad (45)$$

where $\mathbf{V}_a$ is the vector air velocity. By integrating this equation we obtain in place of Eqs. (26) and (25)

$$\partial(n_0/\lambda)/\partial t+\mathrm{Div}(\mathbf{V}_a n_0/\lambda)=N_A+N_E+n_0I_1(\lambda) + n_0^2[S_*I_{22}(\lambda)-I_{21}(\lambda)]+\partial(n_0\bar{V}_1/\lambda)/\partial z, \qquad (46)$$

$$\partial(n_0/\lambda^4)/\partial t+\mathrm{Div}(\mathbf{V}_a n_0/\lambda^4)=\dot{M}/8\pi\rho_w + \partial(n_0\bar{V}_3/\lambda^4)/\partial z, \qquad (47)$$

where

$$\bar{V}_1=\bar{V}_1(\lambda)=\int f(m,t)V_t(m)dm \Big/ \int f(m,t)dm = \int V_t(r)e^{-\lambda r}dr \Big/ \int e^{-\lambda r}dr, \qquad (48)$$

$$\bar{V}_3=\bar{V}_3(\lambda)=\int mf(m,t)V_t(m)dm \Big/ \int mf(m,t)dm = \int r^3V_t(r)e^{-\lambda r}dr \Big/ \int r^3e^{-\lambda r}dr. \qquad (49)$$

Eqs. (46)–(49) together with the equations for N_A, N_E, I_1, I_{21}, I_{22} and S_* constitute a complete set of equations for the parameterization of rain development by the warm rain process. In this formulation the median volume diameter of the raindrop size distribution and its effective fall velocity $\bar{V}_1$ and $\bar{V}_3$ are not single-valued functions of M. Hence, this method of parameterization may enable a more realistic simulation of warm rain microphysics for use with cumulus dynamics models.

6. Summary

A review of previous work on the evolution of raindrop size distributions by the processes of condensation, coalescence and breakup of raindrops showed that these processes tend to produce exponential size distributions. Assuming then that raindrop size spectra are exponential, differential equations were derived for the total concentration and water content of the spectra. These equations were solved for the intercept $n_0(t)$ and slope $\lambda(t)$ of the exponential distribution using the experimental data of Komabayasi *et al.* (1964) and Brazier-Smith *et al.* (1972) on the coalescence efficiency, the probability of spontaneous breakup, and the average number of fragments resulting from spontaneous and collisional breakup. The following simplifying assumptions were made: 1) no autoconversion, 2) the rate of increase of rainwater content is a prescribed constant, $\dot{M} \geqslant 0$ and 3) the number of fragments S_0 resulting from a collisional breakup is a constant. Calculations showed that the effects of spontaneous breakup may generally be ignored compared to those of collisional breakup provided $M \gtrsim 1$ g m^{-3}. It was found, ignoring spontaneous disintegration, that (i) for $\dot{M}=0$, the slope λ tends to a stationary value which depends upon S_0, and (ii) for $\dot{M}>0$, the slope λ tends to a quasi-static value which depends primarily on S_0 and to some extent on $\dot{M}/M^2$, and is close to the stationary value for the same S_0. In both cases n_0 takes on a value consistent with the value of λ and M; this implies that n_0 is proportional to M or R. The conclusion is supported to some extent by the data presented by Blanchard and Spencer (1970).

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