The Effects of Latitudinal Shear on Equatorial Waves. Part II: Applications to the Atmosphere

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ABSTRACT

The analytical and numerical methodology of Boyd (1978) is applied to observed atmospheric waves. It is found that the structure and vertical wavelength of the stratospheric Kelvin wave of 15-day period and the tropospheric Kelvin wave of 40–50 day period are both negligibly affected by even the strongest shear. In contrast, the shear of the quasi-biennial oscillation can decrease the wavelength of the stratospheric $n=0$ mixed Rossby-gravity wave of 5-day period by 60% and produce changes of 50–100% in wave fluxes and velocities. The structure of synoptic-scale easterly waves ($n=1$ Rossby waves of 5-day period) is not drastically altered by shear, but the wavelength is tripled. This makes it unlikely that one can construct a quantitative wave-CISK theory of this mode without including latitudinal shear.

1. Introduction

The importance of equatorial waves in the tropical lower stratosphere, tropical troposphere and equatorial oceans has been reviewed by Wallace (1973), Holton (1970a) and Wunsch (1977), respectively. The mean current varies strongly with latitude in all these regions. Nonetheless, although Holton (1970b), Andrews and McIntyre (1976a,b) and Simmons (1978) have made limited investigations along the same line, this present two-part work is the first comprehensive study of the effects of latitudinal shear on equatorial waves. Theory and methods are presented in the first part and representative numerical cases for atmospheric equatorial waves are described here in Part II.

As explained in Part I, the method of multiple scales implies that the local vertical wavelength, the latitudinal structure and the interrelationships of various wave fields at a given height are determined solely by the wind profile at that height. Therefore, the effects of latitudinal shear on equatorial waves are quasi-one-dimensional. To emphasize this, all the results presented here are given in the form of solutions to the one-dimensional latitudinal eigenvalue equation [Eqs. (3.14) or (3.15) of Part I]. The sole connection between the wave at different levels is an overall amplitude factor, which varies slowly with height on the same scale as the mean zonal wind and which may be calculated, once solutions to the eigenvalue equation are known, by using the generalized wave-action equation as explained in Part I (Sections 2 and 8).

As noted in the Abstract, this paper can be read independently of Part I since one can gain a good feeling for the importance or unimportance of latitudinal shear for various classes of equatorial waves merely by inspecting the graphs and numbers. However, to understand the quasi-one-dimensional nature of these effects, the origin of the analytic approximations that will be quoted in later sections, the Hermite spectral method, and the alternatives when the method of multiple scales is not valid (as seems to be the case in the ocean), it is necessary to read Part I.

Holton (1970b) carried out a purely numerical study of the Kelvin wave using wind profiles which varied exponentially in latitude. Since his investigation included vertical shear both with and without critical surfaces but was limited by the restriction that the wind be symmetric about the equator, his work is complementary to that reported here. His results are in good agreement with mine and are described further in Section 3.

Andrews and McIntyre (1976a,b) showed how to include weak horizontal shear at first explicit order in their multiple scales perturbation theory (in contrast to the scheme described in Part I, which effectively includes it at *lowest* order in the vertical shear expansion parameter). Since the primary purpose of their work is to derive and analyze theorems on wave-mean flow interaction and wave fluxes and their technique for extending the method of multiple scales to all orders, their treatment of horizontal shear effects is limited, but some perceptive comments on the relationship between the net acceleration and latitudinal shear are noted in Section 3 also.

Simmons (1978), using a spherical one-dimensional model in which the mean wind is assumed to be a function of latitude only, has made some numerical

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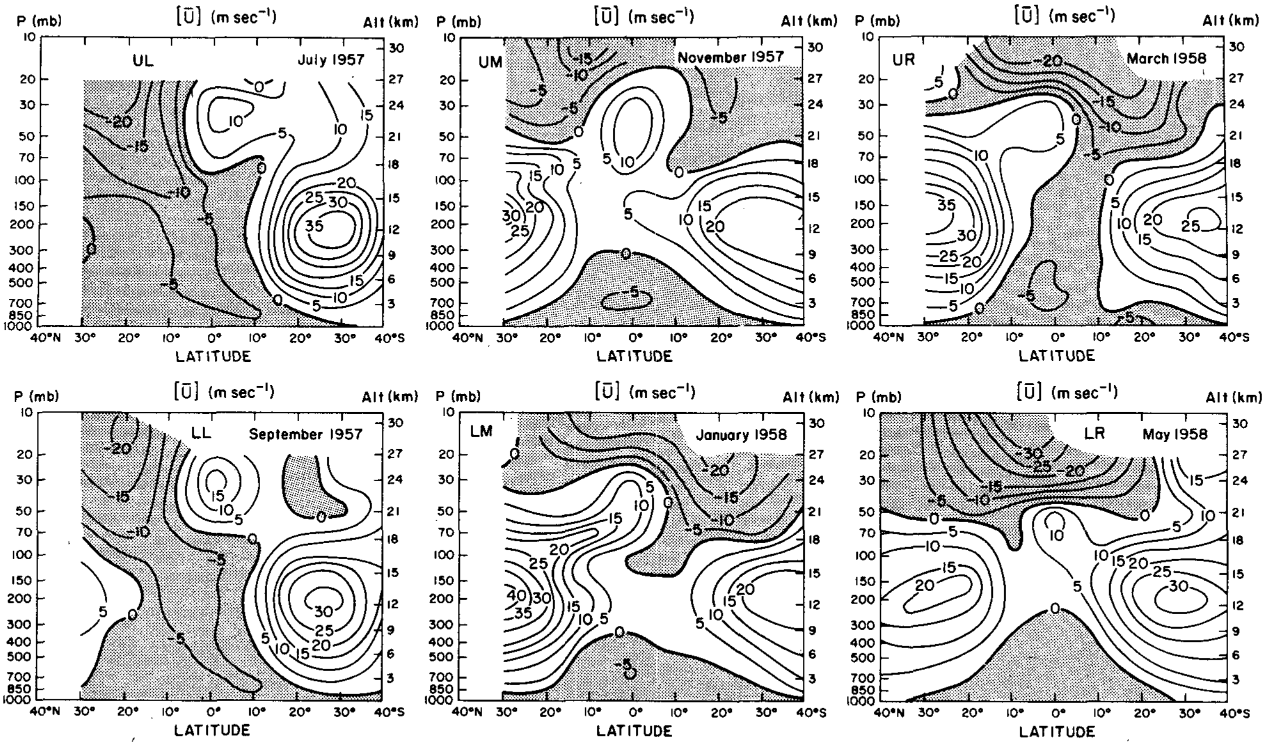

FIG. 1. Mean zonal wind (m s⁻¹) at 2-month intervals. Fig. 10.24a of Newell *et al.* (1974).

calculations for equatorial waves although the bulk of his paper is concerned with the quite different topic of stratospheric long waves. His results, which were independently obtained, are in good qualitative agreement with those presented here. He noted, for example, that empirically, the zonal wind has a node roughly where the gradient of absolute vorticity vanishes; the theoretical explanation for this is discussed in Section 6 of Part I.

Observations of the mean wind and the philosophy of the simple models used here are discussed in the next section. The effects of latitudinal shear on observed Kelvin, mixed Rossby-gravity and Rossby waves are described in Sections 3, 4 and 5. A few concluding remarks are made in the final section.

2. Observations

Fig. 1 shows the observed mean zonal winds in the tropical lower stratosphere at two-month intervals from Fig. (10.24a) of Newell *et al.* (1974). Two conclusions stand out. First, the mean wind is highly variable in height, latitude, and time. Although I shall not present cross sections to prove it, tropospheric tropical winds are also highly variable. Second, defining

$$\gamma \equiv \frac{\partial \bar{U}}{\partial y}, \tag{2.1}$$

$$\delta \equiv \frac{\partial^2 \bar{U}}{\partial y^2}, \tag{2.2}$$

where $\bar{U}$ is the mean zonal wind, one finds that γ is often as large as $O(10^{-5}\ s^{-1})$ and δ as large as $O(8.8 \times 10^{-12}\ s^{-1}\ m^{-1})$, and these values will be used below.

As noted in the Introduction, results will be presented in one-dimensional form. One justification for doing this is provided by the applicability of the method of multiple scales to equatorial waves in the tropical lower stratosphere. The temporal variability and irregularity of the mean wind supply a second reason. If one carried out a "realistic" treatment by solving the wave equation using the instantaneous zonal winds as displayed in Fig. 1, the results would display a myriad of small-scale features that would never be exactly repeated. For similar reasons, the theoretical models of Part I assume that the horizontal dependence of the mean wind at a given height can be adequately approximated by a parabola over the range of interest, even though especially for a wave of broad north–south extent such as the Kelvin, the shape of the winds in Fig. 1 is often considerably more complicated over the latitude band where the wave has significant amplitude. It would be straightforward to extend the Kelvin wave perturbation theory of Part I to a wind profile which was approximated by a polynomial of higher degree, but this would contribute little to our understanding of the phenomenon. Although two-dimensional height-latitude sections of equatorial waves in observed winds are not without interest, especially when critical surfaces are present, if only to gain a feeling of how the small-scale irregularities in the wind alter the wave and wave-mean flow interaction, the philosophy adopted here is that it is better to strive for insight in a simple framework than to chase the will o'the wisp of a protean "realism." Previous studies of the effects of shear by Lindzen (1970) and Holton (1970b) have adopted similarly idealized wind profiles for similar reasons.

3. The Kelvin wave

Two species of Kelvin wave have been identified in the atmosphere. In the lower stratosphere as reviewed by Wallace (1973) and Holton (1975), a Kelvin wave of zonal wavenumber 1 to 2 (a wavelength of 20 000 to 40 000 km) and a ground-based period of about 15 days (8 days relative to the wind) is the dominant eddy disturbance in this region of the atmosphere during the easterly phase of the quasi-biennial oscillation. A Kelvin mode of similar wavelength (~40 000 km) but longer period (40–50 days) has also been observed in the tropical troposphere as thoroughly discussed in Chang (1977). Before presenting numerical results, it is useful to briefly review some theoretical analysis.

In Section 5 of Part I, the following first-order perturbation formulas for the Kelvin wave were derived:

$$v^1 = ik[(\gamma_0+\delta_0 y/2)\beta^{-1}]u^0, \tag{3.1}$$

$$u^1 = -k^2[(\gamma_0 y+\delta_0 y^2/4)\beta^{-1}]u^0, \tag{3.2}$$

$$\phi^1 = -k^2[(\gamma_0 y+\delta_0 y^2/4)\beta^{-1}]\phi^0, \tag{3.3}$$

$$\epsilon^1 = (\delta_0/2\beta)\epsilon_0. \tag{3.4}$$

A full list of symbols is given in Appendix A of Part I; the subscripts on γ and δ refer to the equatorial values of these quantities and the superscripts on u, v, ϕ and ϵ refer to the perturbation order. Eqs. (3.1)–(3.4) are fully discussed in Part I, and here a few remarks will suffice.

It is obvious that for fixed wind profile, that is for fixed γ_0 and δ_0, the size of each first-order field depends strongly on the zonal wavenumber. The observed Kelvin waves in the stratosphere and troposphere, although of quite different periods, are both zonal wavenumber 1, i.e., one wavelength is equal to the earth's circumference and $k=1/a$, where a is the earth's radius. Letting θ be latitude (rad) and Ω be the angular frequency of the earth, we can rewrite (3.1)–(3.3) for zonal wavenumber 1 as

$$v^1 = i\left(\frac{\gamma_0}{2\Omega}+\frac{\delta_0}{2\beta}\theta\right)u^0, \tag{3.5}$$

$$u^1 = -\left(\frac{\gamma_0}{2\Omega}\theta+\frac{\delta_0}{4\beta}\theta^2\right)u^0, \tag{3.6}$$

$$\phi^1 = -\left(\frac{\gamma_0}{2\Omega}\theta+\frac{\delta_0}{4\beta}\theta^2\right)\phi^0. \tag{3.7}$$

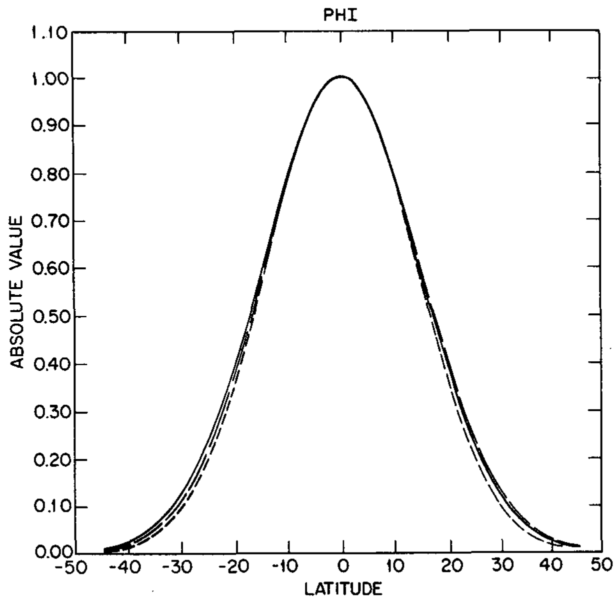

FIG. 2. Geopotential amplitude normalized to a maximum value of 1.0 m² s⁻¹ for the stratospheric Kelvin wave with no shear (solid line), linear shear (long dashes) and parabolic shear (short dashes).

Three nondimensional parameters appear in (3.5) through (3.6) and they are all small:

$$\theta \approx 1/4, \tag{3.8}$$

$$\gamma_0/2\Omega \approx 1/15, \tag{3.9}$$

$$\delta_0/2\beta \approx 1/10. \tag{3.10}$$

Eq. (3.8) is based on the observed halfwidth of the stratospheric Kelvin wave; for its tropospheric counterpart, θ is smaller because of the larger period and narrower latitudinal extent of the latter. Eqs. (3.9) and (3.10) are based on the observational estimates of the previous section.

From substituting (3.8)–(3.10) into (3.5)–(3.7), it is clear that the effects of latitudinal shear on Kelvin waves of low zonal wavenumber are extremely small. To illustrate this, three cases were run numerically for the zonal wavenumber 1 stratospheric Kelvin wave of 15 day period, and Figs. 2–4 show the superimposed wavefields.[1] The solid line is no shear; the short dashes are for $\gamma_0=0.5\times10^{-5}$ s^{-1}, $\delta_0=0$, and the long dashes for $\gamma_0=0$, $\delta_0=0.375\beta$. The maximum value of the north–south velocity is only 3% of the maximum value of the zonal wind even for the most extreme case, and the relative sizes of the other wave fields are smaller. The change in eigenvalue is approximately 15% for the

[1] Note that for all graphs in this paper (i) the absolute value is plotted; (ii) the geopotential is normalized to a maximum value of 1.0, and this determines the normalization of the other fields; and (iii) the curves were generated by a numerical model that solves the meridional structure equation in spherical geometry. This is why v is nonzero in Fig. 2, even when there is no shear.

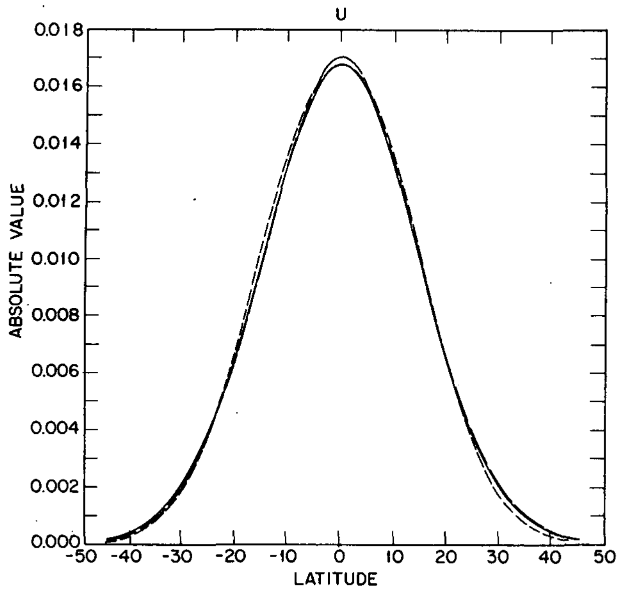

FIG. 3. As in Fig. 2 except for the zonal wind (m s^{-1}).

parabolic shear, but since the vertical wavelength is proportional to $\epsilon^{-\frac{1}{2}}$, this implies less than a 10% change in vertical wavelength. Although I will not discuss the tropospheric Kelvin wave in detail, it is easy to verify using (3.5)–(3.7) that it also is only weakly affected by shear.

The very small values of the first-order fields are surprising, but have been confirmed by numerical simulations. Holton (1970b) studied Kelvin waves in coupled horizontal and vertical shear (although with the restriction that the wind be symmetric about the equator) both with and without critical surfaces. For either case, he found that even very strong shear had little effect on the Kelvin wave. When the critical surface was concave up (the only case he considered), he found the wave was effectively stopped at the height at which the Doppler-shifted frequency vanished at the equator, even though the height of the critical surface rose steeply away from the equator. Andrews and McIntyre (1976a, Appendix B) show that under modest restrictions, the multiple-scales approximation is valid right up to the critical surface and this has been confirmed (in the absence of horizontal shear) by Lindzen's (1970) numerical work. Further, in contrast to the mixed-Rossby wave discussed in the next section, the first-order (in γ_0 and δ_0) solutions do not grow in magnitude relative to the zeroth-order solutions as $\omega_0 \rightarrow 0$, implying that the horizontal shear perturbative approximation is also valid as the Kelvin wave approaches its critical surface and that latitudinal shear effects remain small. Thus, Holton's results agree with theory. For the two most important observed Kelvin waves—the 15 day period wave in the lower stratosphere and the 40–50 day period wave in the troposphere—latitudinal wind shear can be safely neglected as far as wave *structure* is concerned.

Andrews and McIntyre (1976a) note, however, that the net acceleration of the mean zonal wind by a dissipating wave is proportional to ω^{-1} times various wave correlations. Even if latitudinal shear does not significantly change Kelvin wave structure so that the correlations are still roughly symmetric about the equator, the asymmetry in ω^{-1} will cause a corresponding asymmetry in the mean flow acceleration. They add that this "contributes a tendency for the y profile of U to be (algebraically) unstable to incipient irregularities and asymmetries, i.e., a tendency for $|\omega|$ to diminish most rapidly where it is already smallest." This remark applies to all classes of equatorial waves, not just the Kelvin, although for other modes the ω^{-1} factor is overshadowed by changes in the structure of the wave itself. There is no firm evidence, however, that this algebraic instability mechanism is important in the atmosphere—and none that it is not.

4. The mixed Rossby-gravity wave

As reviewed in Wallace (1973) and Holton (1975), a mixed Rossby-gravity wave of zonal wavenumber 4 (i.e., a wavelength of about 10 000 km) and a period relative to the ground of about 5 days, is the dominant wave in the tropical lower stratosphere during the westerly phase of the quasi-biennial in the same way that the Kelvin mode is dominant during the quasi-biennial's easterly phase. Unfortunately, this similarity also extends to the fact that ϵ is sufficiently close to k^2/ω_0^2 over most of the stratosphere for this mode that, as for the Kelvin wave, one cannot neglect the apparently singular terms in (3.14) of Part I, which creates severe analytic difficulties. Near a critical level, however, ϵ becomes large, the vertical wavelength short,

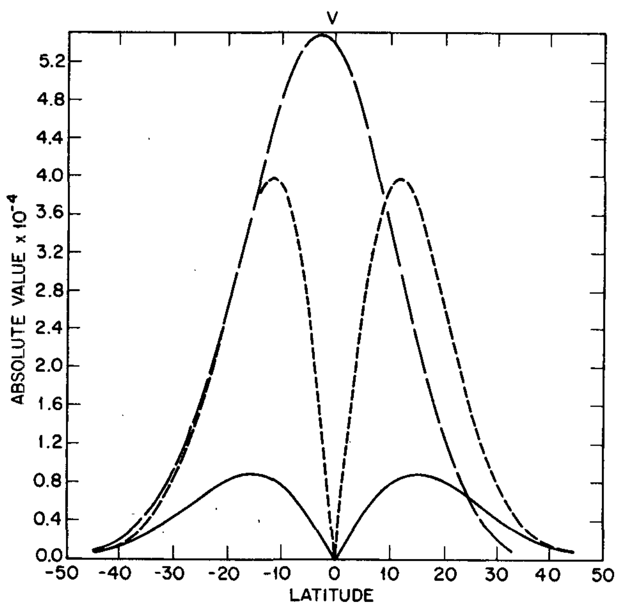

FIG. 4. As in Fig. 2 except for the meridional wind (m s^{-1}).

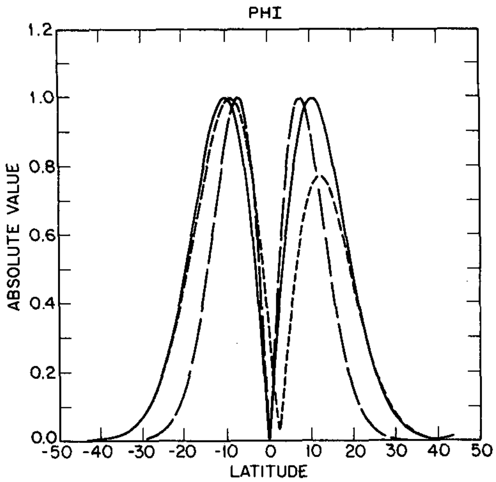

FIG. 5. Geopotential amplitude normalized to a maximum value of 1.0 m^2 s^{-1} for the stratospheric mixed Rossby gravity wave away from the critical surface (Case I) with no shear (solid line), linear shear (short dashes) and parabolic shear (long dashes).

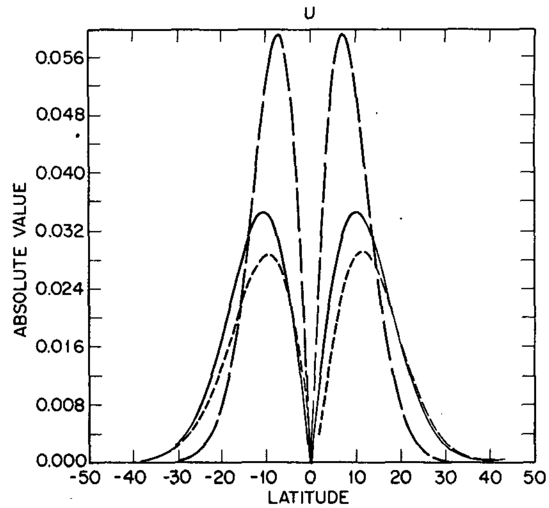

FIG. 6. As in Fig. 5 except for the east–west wind (m s^{-1}).

and the mixed Rossby-gravity wave behaves like an ordinary gravity wave so that the "gamma" plane formulas of Section 6 of Part I are accurate. Away from the neighborhood of the critical surface, however, the treatment will be numerical, so it is convenient to consider these two situations—near and away from a critical level—separately.

a. Away from the critical surface

Figs. 5–8 show the geopotential, zonal and meridional wind, and net acceleration[2] for Case I. For all three runs (and for those of Case II below), the zonal wavenumber is 4, and linear friction and cooling with equal coefficients of 10^{-8} s^{-1} were used to compute the net acceleration. All runs for Case I had the same equatorial frequency: $\omega_0 = 1.82\times10^{-5}$ s^{-1}. The solid line is for no shear; the short dashes for $\gamma_0 = 0.45$ 10^{-5} s^{-1}, $\delta_0 = 0$; and the long dashes are for $\gamma_0 = 0$, $\delta_0 = -0.375\beta$. The magnitudes of the shear are the same as in the previous section, but the sign of δ_0 has been reversed to reflect the fact that the mixed Rossby-gravity and Kelvin waves are observed during different phases of the quasi-biennial oscillation. (The sign of γ_0, of course, is irrelevant because it merely interchanges the roles of the Northern and Southern Hemispheres without altering the underlying physics.)

In startling contrast to the Kelvin wave, the mixed Rossby-gravity wave is strongly affected by both the quasi-parabolic and quasi-linear components of the wind. The shear of the model quasi-biennial oscillation, for example, increases the eddy north–south wind by more than 40%, the zonal wind by nearly 100% and the net acceleration by 270%, while decreasing the latitudinal scale by about a quarter. The effects of quasi-linear shear are less dramatic but still noticeable in the figures.

The nondimensional eigenvalues (ϵ^*) are 1007., 2511., and 927., respectively.[3] Thus the idealized quasi-biennial oscillation has decreased the vertical wavelength by about 60%.

b. Near the critical level

In contrast with the Kelvin wave, the qualitative response of the mixed Rossby-gravity wave to shear changes as $\omega_0 \rightarrow 0$. As discussed in Part I (Section 4), pure Rossby waves normally satisfy the inequality

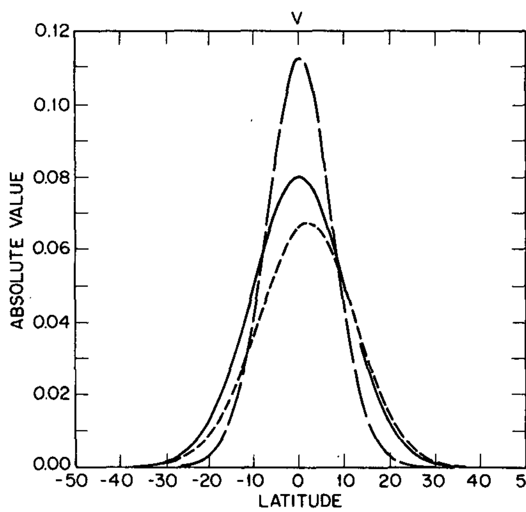

FIG. 7. As in Fig. 5 except for the north–south wind (m s^{-1}).

[2] The "net" acceleration is the total time rate of change of the mean zonal wind due to the wave, not merely the sum of the horizontal and vertical momentum flux convergences. This concept is explained more fully in Part I and in Boyd (1976).

[3] These are the values calculated numerically on the sphere. Without shear, the beta-plane $\epsilon^* = 1024$.

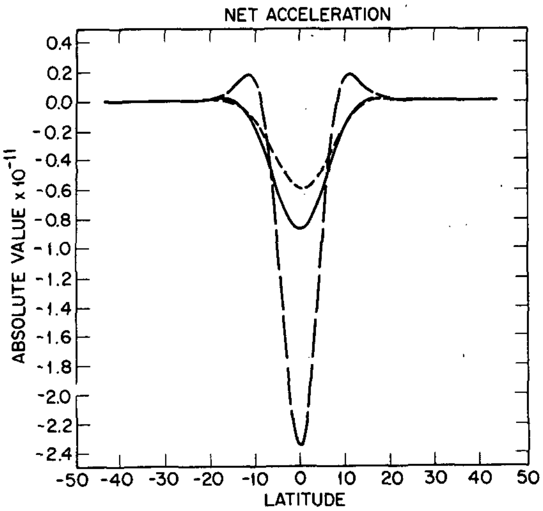

FIG. 8. As in Fig. 5 except for the net acceleration (m s^{-2}).

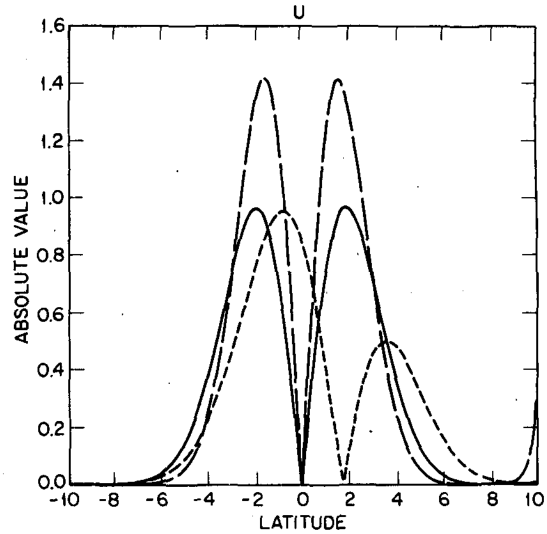

FIG. 10. As in Fig. 9 except for the zonal wind (m s^{-1}).

$\epsilon \ll k^2/\omega_0^2$, whereas pure gravity waves (for small zonal wavenumber) satisfy the opposite $\epsilon \gg k^2/\omega_0$. Thus, away from the critical level, the observed mixed Rossby-gravity wave, with $\epsilon \approx k^2/\omega_0^2$, is "mixed" in the most literal possible sense, but as $\omega_0 \to 0$, ϵ becomes very large and the wave behaves more and more like a pure gravity wave in the sense that $\epsilon \gg k^2/\omega_0^2$ and ϵ is proportional to ω_0^{-4}, not to ω_0^{-2}, as is true of Rossby waves. This justifies the use of the analytic gravity wave theory of Section 6 of Part I. The second qualitative change is that as the critical level is approached, the quasi-linear component of the wind becomes increasingly important in comparison with the quasi-parabolic.

The reason for this is that the component of ω which is proportional to δ_0 is *symmetric* about the equator, and therefore cannot change the *qualitative* shape of

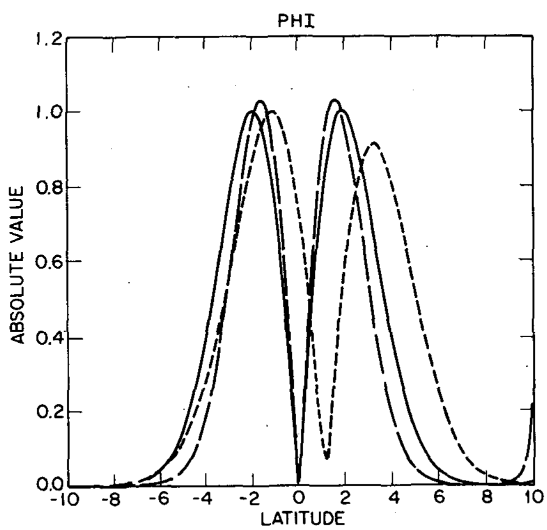

FIG. 9. As in Fig. 5 except for the stratospheric mixed Rossby-gravity wave near its critical level (Case II).

the various wave fields. The principal *quantitative* effect is to replace β^2 by $\beta(\beta-\delta_0)$ in the usual gravity wave eigenvalue formula. Since the latitudinal scale of the wave depends only on the one-fourth power of the eigenvalue, it would take a large value of δ_0 indeed to noticeably alter the north–south scale.

As shown in Section 6 of Part I, however, the linear portion of the wind can change the qualitative shape of the wave fields by shifting the turning points (the latitudes where the wave changes from meridionally oscillatory to meridionally exponential behavior) a couple of degrees into the hemisphere of higher Doppler-shifted frequency, in effect bodily transporting the v and ϕ fields a couple of degrees with little or no change in shape. In addition to this translation, however, the u field is distorted. The zonal wind peak in the hemisphere of lower Doppler-shifted frequency becomes larger while the other local peak becomes smaller as γ_0 increases. For reference, I give the results of the "gamma plane" approximation when $\delta_0=0$. As explained in Part I, this approximation is accurate when the nondimensional parameter χ $(=k\omega_0/\beta)$ is small, even if γ is as large as $O(\omega_0)$:

$$\epsilon^{\frac{1}{2}}=\beta/(\omega_0^2+\gamma_0^2/4), \tag{4.1}$$

$$\xi=(y-\gamma_0/2\beta)\beta^{\frac{1}{2}}\epsilon^{\frac{1}{4}}, \tag{4.2}$$

$$v=e^{-\xi^2/2}, \tag{4.3}$$

$$u=\frac{ie^{-\xi^2/2}}{\omega_0}[\tfrac{1}{2}\gamma_0-(\omega_0^2+\gamma_0^2/4)^{\frac{1}{2}}\xi], \tag{4.4}$$

$$\phi=\frac{-i\beta^{\frac{1}{2}}\xi}{\omega_0\epsilon^{\frac{1}{4}}}e^{-\xi^2/2}. \tag{4.5}$$

From these expressions, one can see explicitly that the lateral translation is $\gamma_0/2\beta$ in meters and that v and ϕ

are unchanged in shape while, in contrast, the antisymmetry of the zonal wind with respect to $\xi=0$ is destroyed.

Figs. 9–12 show the wave fields for Case II, which is identical in every way to Case I except that ω_0, the equatorial Doppler-shifted frequency is 0.46×10^{-5} s^{-1}, one-fourth its value for Case I. The effects of linear shear are clearly much greater than those of parabolic shear, the reverse of the situation for Case I, but both are significant.

The linear shear shifts all the wave fields about a degree and a half into the hemisphere of higher frequency. The southern peak of the zonal wind is twice as large as the northern, but the shapes of v and ϕ are little changed. The north–south wind, however, is reduced by about 35%. The net acceleration is dramatically altered—it is zero at the equator, which is where it is largest in the absence of shear—and its maximum value is halved.

Since $\delta_0=-0.375\beta$, the gamma plane formula

$$\epsilon=\beta(\beta-\delta_0)/\omega_0^4 \tag{4.6}$$

predicts that parabolic shear will increase ϵ by about 40%, so it is hardly surprising that the effects of such shear are still important even close to a critical surface. The net acceleration, for example, is more than doubled. The predicted change in latitudinal scale is only about 10%, however, and the figures confirm that changes in the y profile of wave fields are large for linear shear and very small for parabolic.

By comparing the exact nondimensional eigenvalues, which are 8.03×10^5, 4.46×10^5 and 1.22×10^6, respectively, with the predictions of the gamma-plane theory, one finds errors of 25% or roughly 2χ, where $\chi\equiv(k\omega_0/\beta)=0.125$ for Case II. Because of these errors, Figs. 9–12 were computed numerically, but although it is straightforward to extend the gamma-plane approximation to higher order in χ, the lowest order theory is quite adequate for qualitatively understanding Case II and has the virtue of great simplicity.

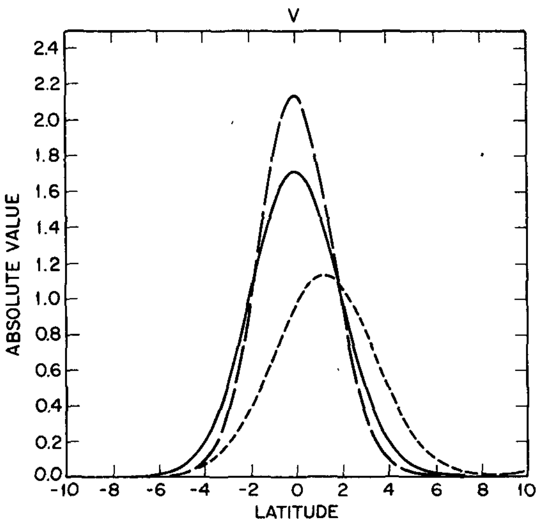

FIG. 11. As in Fig. 9 except for the meridional wind (m s^{-1}).

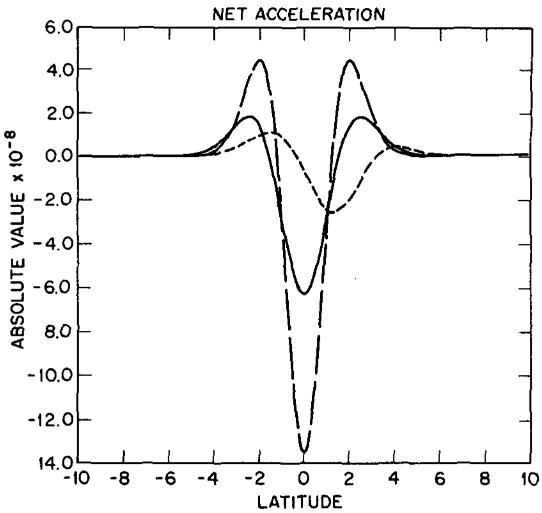

FIG. 12. As in Fig. 9 except for the net acceleration (m s^{-2}).

The winds of the lower stratosphere are highly variable, so there will be times (and levels) for which γ_0 and/or δ_0 will be quite small. Nonetheless, it is clear that horizontal shear such as that associated with the quasi-biennial oscillation of the mean wind can double or halve various wave fields. To a numerical modeler attempting to check the results of his program by using linear wave theory, for example, such dramatic changes would be extremely important.

5. Rossby waves

Synoptic-scale easterly waves are a major phenomenon of the tropical troposphere. Holton (1970a) has identified these waves as[4] $n=1$ Rossby modes with a zonal wavenumber of about 10 (wavelength of 4000 km) which propagate with a phase speed of about -3 m s^{-1} relative to an equatorial mean zonal wind in the eastern Pacific of about -6 m s^{-1}. Reed and Recker (1970) have made a thorough observational study of these waves. The wind in the tropical troposphere varies strongly with height, latitude and longitude, wave amplitudes are large enough so that linearity is a poor approximation, and, as discussed in Stevens *et al.* (1977), forcing and dissipation due to cumulus convection are extremely important for these waves, so a full treatment would be extremely complicated and its complexity would obscure the basic issue under discussion here: how does latitudinal shear affect these waves? Consequently, although the method of multiple scales cannot be justified for the troposphere, it will be instructive to employ the same one-dimensional model

[4] The mode number terminology is explained in Holton (1975) and in Part I.

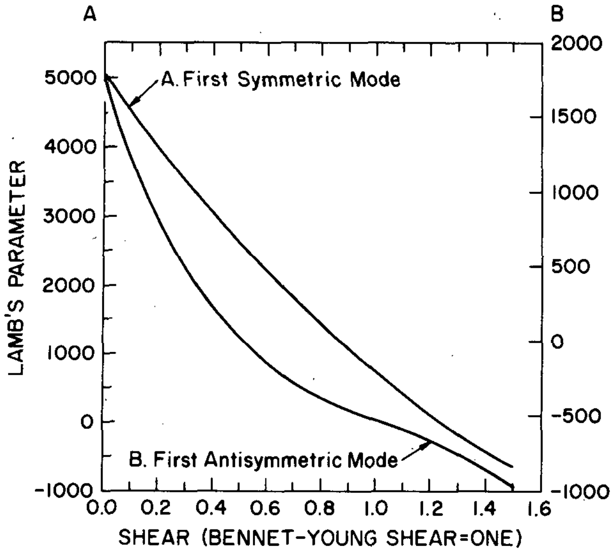

FIG. 13. Nondimensional eigenvalue ϵ^* (Lamb's parameter) versus the shear parameter S for the lowest two tropospheric Rossby waves.

with a mean wind which depends on latitude only which was used in the previous sections.[5]

Following Mak (1969) and Bennet and Young (1971) (who employed a similar model to study the propagation of midlatitude waves into the tropics), tropospheric winds will be approximated by a parabola

$$\bar{U}-c=3+56S\theta^2 \text{ [m s}^{-1}\text{]}, \tag{5.1}$$

where θ is latitude and $S=1$ is the Mak-Bennet-Young profile for the vertically averaged wind. Although the primary interest is focused on the $n=1$ mode for $S=1$, it is helpful, in order to fully understand shear effects, to consider S as a variable parameter and to look at the $n=2$ mode as well.

The nondimensional eigenvalues ϵ^* for these two modes are plotted as a function of S in Fig. 13. For $S=1$, the wavelength of the $n=1$ mode has been tripled and the eigenvalues for the $n=2$ and all higher modes are *negative*, implying that these modes are vertically trapped and decay exponentially with height. When the frequency is constant, all waves with negative ϵ are planetary in extent [for $\epsilon=0$, the horizontal streamfunction is a spherical harmonic as shown in Haurwitz (1940)] and the tripled wavelength of the lowest mode would imply a 70% increase in latitudinal scale. Fig. 14, which compares the $n=1$ geopotential for $S=0$ with the geopotential for $S=1.23$ (the shear strength for which $\epsilon^*=0$), tells a very different tale. Far from being a spherical harmonic, the $n=1$ mode is only slightly wider at $\epsilon^*=0$ than it is when there is no shear, and for the observed shear, $S=1$, the difference is proportionately smaller. How is this possible?

If one writes Eq. (4.13) of Part I as

$$V_{yy}+q(y)v=0, \tag{5.2}$$

then for the wave to be equatorially trapped it is necessary and sufficient that 1) $q(y)<0$ for all y except for a small neighborhood about the equator and 2) $q(y)>0$ for some neighborhood about $y=0$. Where $q(y)$ is negative, the solutions of (5.2) have exponential behavior. The boundary conditions impose the further requirement on the eigenfunctions that they decay exponentially as $|y|\rightarrow\infty$. Where $q(y)$ is positive, the eigenfunctions are sinusoidal, and a small region of this type of behavior is necessary so that the eigenfunctions can make a smooth transition from exponentially growing as y increases for $y\ll 0$ to exponentially decreasing for $y\gg 0$.

When the frequency is constant, one has

$$q(y)=\frac{k\beta}{\omega_0}-k^2-\epsilon\beta^2y^2. \tag{5.3}$$

Conditions 1) and 2) above can only be met if ϵ is large and positive and $(k\beta/\omega_0-k^2)$ is positive. When shear is present and $\epsilon=0$,

$$q(y)=k\beta/\omega(y)-k^2. \tag{5.4}$$

As before, $(k\beta/\omega_0-k^2)$ must be positive, but if $\omega(y)$ increases as $|y|$ increases, then for large enough y, $q(y)$ can be made negative by $(-k^2)$ and the $(-\epsilon\beta^2y^2)$ term is not needed. The eigenfunction will decay as $\exp(-k|y|)$ rather than as a Gaussian, however, and this slower rate of decay with shear than without is

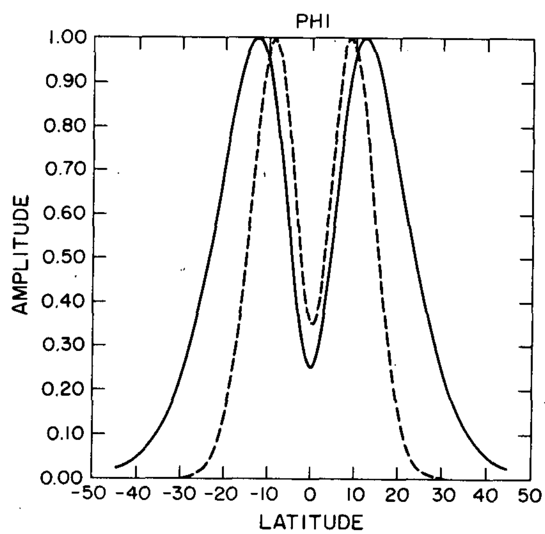

FIG. 14. Geopotential amplitude normalized to 1.0 $m^2 s^{-2}$ for the first symmetric ($n=1$) Rossby wave for $S=0$ (dashed line) and $S=1.23$ (solid line).

[5] In earlier sections, however, the understanding was that vertical shear could be incorporated into the model by the method of multiple scales. Here, the one-dimensional eigenvalue equation is used by itself as a heuristic model.

clearly evident in Fig. 14. Though the location of the peaks is little altered, the wave is considerably broader for $S=1.23$ than for $S=0$.

To examine nonzero values of ϵ, note that with $\gamma_0=0$, Eq. (4.12) of Part I with Taylor expansion of the coefficient to second order in y shows

$$q(y)=(-k^2+k\beta/\omega_0)-[\epsilon\beta^2+\epsilon\delta_0(k^2/2\omega_0^2-3\epsilon)]y^2. \quad (5.5)$$

From the relations of Appendix B, Part I, it is trivial to show that the turning points of (5.5) depend only on the term which is independent of y. Since $(-k^2+k\beta/\omega_0)$ is independent of the shear, however, this implies that to a first approximation, shear will not change the shape of the wave except possibly for large values of $|y|$ where the Taylor series of $\omega^{-1}(y)$, which is implicit in (5.5), is invalid. The coefficient of y^2 in (5.5) must also be independent of δ_0 (because the other term is, and the eigenvalue relation must hold for all δ_0) and this implies that ϵ must decrease rapidly as δ_0 (or equivalently S) increases. Alas, the observed wind shear is so strong that low-order perturbation theory is not quantitively accurate. For $S=1$, the first- and second-order eigenvalues are $\epsilon^{*1}=-672$ and $\epsilon^{*2}=1480$, which compare poorly[6] with the exact eigenvalues $\epsilon^*=736$; this is why a numerical approach has been used to draw Figs. 13 and 14. Nonetheless, Eq. (5.5) is qualitatively correct: the $n=1$ Rossby mode responds to parabolic shear principally through drastic reductions in the eigenvalue with little or no change in its structure except at large $|y|$.

Because of the imperfections of the model, such modest changes in wave structure are probably not too important (except possibly by their absence). The large change in ϵ, however,—a factor of 9— is probably extremely important since the vertical wavelength is a crucial parameter in wave-CISK theories of this mode. It therefore seems likely that latitudinal shear is an essential ingredient for modeling this wave.

6. Concluding remarks

In this two-part work, latitudinal shear effects have been studied in isolation. The logical followup is to use the present results in building and analyzing more complicated models of waves and wave-related phenomena. The drastic change in the vertical wavelength of the tropospheric, $n=1$ Rossby wave is especially significant in this regard because, as a first approximation, the vertical velocity is proportional to the product of the geopotential and the wavelength. Since vertical motion creates the convection that sustains the wave, it is clear that large changes in the vertical wavelength will drastically alter the quantitative dynamics of the wave. Thus, latitudinal shear is probably an essential ingredient in realistic wave-CISK theories of this mode.

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[6] For $S=0$, $\epsilon=5100$, so when compared with the *change* in ϵ^* due to shear rather than with ϵ^* itself, these approximations do have a modest accuracy.

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