NOTES AND CORRESPONDENCE

The Effect of Topography on a Rossby Wave

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ABSTRACT

A system of three Fourier components in a barotropic channel flow is considered to investigate the influence of topography on a neutral Rossby wave. Analytic solutions show how topography induces a nonlinear oscillation of the mean flow-wave system.

1. Introduction

The influence of topography on large-scale atmospheric flows has formed the theme for a large number of meteorological investigations, beginning with the remarkably successful simulation of barotropic flows over mountains by Charney and Eliassen (1949). Since then there have been several more complete numerical investigations (Mintz, 1965; Kasahara and Washington, 1971; Manabe and Terpstra, 1974; and others).

Studies with more complete models are useful for development of operational prediction models as well as general circulation simulations. However, because of their complexity, cause-effect relationships are not always apparent. Perhaps because of this, the current literature includes a number of highly idealized theoretical models.

Paegle and Paegle (1976) show that frequency shifts observed across mountain ranges are also found in simple analytical solutions to barotropic flows across mountains. Grose and Hoskins (1979) use shallow water analogues to demonstrate that the steady solution of the relevant equations linearized about the observed 300 mb zonal flow and perturbed only by topography compare very well with the observed average flow.

Charney and DeVore (1979) use a highly truncated spectral expansion in a barotropic model of flow over terrain. In the presence of topography, one solution is near linear resonance and displays high-amplitude waves with weak zonal flow, and the other, being farther from resonance, exhibits the opposite wave-zonal flow energy distribution. Hart (1979) also investigates such essentially nonlinear behavior emphasizing continuous bifurcations of the solutions. These studies display solution behaviors reminiscent of actual atmospheric transitions between strongly blocked and strongly zonal flows.

The present study investigates the role that topography plays in inducing oscillations in the mean zonal flow as a result of exchange of kinetic energy between the zonal flow and a Rossby wave. This is done by considering a truncated version of the barotropic vorticity equation on a beta plane which isolates this particular interaction. The nonlinear solutions are obtained analytically and discussed in Section 2.

For the case when perturbations from the steady state are small a linearized analysis is pertinent. This is presented in Section 3 to bring out some physical characteristics of the zonal flow-wave interaction which is more readily done from the linearized system. In particular, it is of interest to discuss the conditions for which unstable steady states arise for the topographic profile we have chosen, for which the zonally averaged relief is zero.

It is clear that a latitudinal distribution of topography of height $h(y)$ can destabilize a potential vorticity conserving flow. In the case of a beta plane with the Coriolis parameter given by $f = f_0 + \beta y$, the scale height by $H$, and the zonal flow by $u$, the potential vorticity gradient

$$\frac{\partial^2 u}{\partial y^2} + \beta + \frac{f_0}{H} \frac{\partial h}{\partial y}$$

(1)
replaces the ambient absolute vorticity gradient that distinguishes shear flow instabilities in nondivergent barotropic flows (Kuo, 1949).

Following this classical analysis a necessary condition for flow instability can be shown to be that the meridional potential vorticity gradient (1) becomes zero somewhere in the domain. Therefore, if the absolute vorticity gradient everywhere possesses the same sign it is still possible to have unstable flows when mountains are present as long as \((f_0/H)(\partial h/\partial y)\) locally exceeds \((\partial^2 u/\partial y^2) - \beta\). However, with the possible exception of the Antarctic plateau, the earth's topography exhibits large longitudinal variations, and this criterion for instability cannot be satisfied around a complete latitude circle.

When dissipation is included the time behavior of the system is substantially different (Charney and DeVore, 1979). While unstable states may still be present, the finite-amplitude time evolution shows the flow moving toward one of the possible stable steady states. The stability characteristics of the solution seem to play an important role in determining the nature of this evolution. In particular, integrations of the equations started from conditions close to an unstable solution have shown that the zonal flow undergoes large, finite-amplitude oscillations on time scales of a few days to perhaps a month before it settles into the stable steady state. When the chosen parameter conditions are such that the solution is stable, the flow quickly converges into it with small zonal flow oscillations. We believe that such disparate behavior can be anticipated from the present idealized analysis.

2. Truncated model

An incompressible, homogeneous fluid with a terrain height \(h\) and a top rigid boundary is considered. In this model, potential vorticity \((\xi + f)/(H + h)\) is conserved following the motion. Here \(\xi\) is the relative vorticity and \(H\) the depth of the fluid (Batchelor, 1967).

For large scales, the quasi-geostrophic approximation holds, and assuming a time scale of \(f_0^{-1}\), horizontal and vertical length scales of \(L\) and \(H\), respectively, and a velocity scale of \(L f_0\), and \(H/\bar{h}\), the conservation of potential vorticity can be expressed in nondimensional form as

\[
\frac{\partial}{\partial t} \nabla^2 \psi + \int \left( \frac{\nabla^2 \psi}{H} + \frac{\bar{h}}{f_0} \right) = 0,
\]

where \(\psi\) is the streamfunction.

From (2), conservation of total energy \(E_T\) and total potential enstrophy \(V_T\) in a channel follows, where

\[
E_T = \frac{1}{2} \int_0^{L_x} \int_0^{L_y} \nabla \psi^2 \, dx \, dy,
\]

\[
V_T = \frac{1}{2} \int_0^{L_x} \int_0^{L_y} \left( \nabla^2 \psi + \frac{f}{f_0} + \frac{\bar{h}}{H} \right)^2 \, dx \, dy.
\]

Here \(L_x\) and \(L_y\) represent the zonal and meridional extent of the channel.

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| \(V_2\) | 10.39 | 16.47 | 10.99 |
We follow Lorenz (1963) in choosing a spectral representation for $\psi$ in Fourier space of the form

$$\psi = \sqrt{2} \psi_\alpha \cos(y) + 2 \psi_k \sin(y) \cos(nx)$$

$$+ 2 \psi_L \sin(y) \sin(nx) \quad (5)$$

to represent a wave in the presence of the latitudinally variable zonal flow. This representation is a strong simplification of the flow, but it allows a direct interpretation of the results. Charney and DeVore (1979) have discussed the case with a larger number of degrees of freedom. The selection of only the first latitudinal mode in a channel of $\pi L$ extent in the meridional direction excludes simple barotropic instabilities since $\beta - \partial^2 \tilde{u}/\partial y^2$ does not vanish within the channel.

The following form is chosen for the terrain:

$$\frac{\tilde{h}}{H} = h \sin(y) \cos(nx). \quad (6)$$

The total energy $E_T$ and potential enstrophy $V_T$ for the system given by (5) and (6) are

$$E_T = \pi^2 [\psi_\alpha^2 + (n^2 + 1)(\psi_k^2 + \psi_L^2)], \quad (7)$$

$$V_T = V_1 + V_2, \quad (8)$$

$$V_1 = \pi^2 [\psi_\alpha^2 + (n^2 + 1)^2(\psi_k^2 + \psi_L^2)]$$

$$- (n^2 + 1) \psi_k \psi_{\alpha_{\psi}} + 4 \sqrt{2} \pi^{-1} \beta_0 \psi_k, \quad (9)$$

$$V_2 = \frac{1}{2} \int_0^\pi \int_0^{2\pi} \left( \frac{f}{f_0} + \frac{\tilde{h}}{H} \right)^2 dxdy. \quad (10)$$

The individual terms appearing in the right-hand side of (9) are time-dependent, while those in (10) are constant in time.

Eqs. (7) and (9) may be combined to obtain the time-invariant quantity

$$V = n^2 \psi_\alpha^2 + (n^2 + 1) h \psi_k - 4 \sqrt{2} \pi^{-1} \beta_0 \psi_k. \quad (11)$$

Only the projection of the beta plane on the first meridional Fourier cosine mode is included in $V$. The projection of $\psi_{\alpha_{\psi}}$ on this mode is chosen as $-(4/n) \beta_0 \cos(y)$ to obtain a spectral truncation that precisely conserves potential enstrophy as well as energy.

Equations for the time dependent Fourier coefficients are

$$\dot{\Lambda} = \alpha h \psi_L, \quad \dot{\psi}_k = -\beta \Lambda \psi_L, \quad \dot{\psi}_L = \beta \Lambda \psi_k - \frac{h_1 (\Lambda + \beta/n)}{\alpha (n^2 + 1)}, \quad (12)$$

where

$$\alpha = \frac{8 \sqrt{2} n}{3 \pi}, \quad \beta = \frac{n^2}{n^2 + 1} \quad \text{(Lorenz, 1963)}$$

$$h_1 = \frac{\alpha h}{2}, \quad \Lambda = \alpha \psi_\alpha - \frac{\beta}{n}, \quad \beta = \frac{32}{3 \pi^2} \beta_0 = 1.081 \beta_0 \quad \text{(13)}$$

It can be verified that (11)–(13) conserve $E_T$ and $V_T$ given by (6)–(9).

The time invariance of $V$ is used to obtain analytic solutions to the nonlinear system (11)–(13). This is done by reducing the system to one nonlinear ordinary differential equation

$$\left( \frac{d \Lambda}{dt} \right)^2 = -\frac{\beta^2}{4} \Lambda^4$$

$$+ \left[ \frac{\alpha^2 \beta}{2(n^2 + 1)} \right] V + \frac{\beta^2}{2n^2} \beta^2 - \frac{h_1^2}{(1 + n^2)} \Lambda^2$$

$$- 2 \frac{h_1^2}{(1 + n^2)} \frac{\beta}{n} \Lambda + C = G(\Lambda), \quad (14)$$

where $C$ is a constant determined from the initial conditions.

The roots of the fourth-order polynomial $G(\Lambda)$ are found using Ferrari’s method for a variety of parameter values. The solution of (14) in terms of elliptic functions is obtained once these roots are known. The periods and amplitudes of the solutions for selected parameter values are shown in Table 1. The amplitude of the oscillation increases with increasing initial energy and potential enstrophy as expected. The period of the oscillation decreases with increasing mountain height.

For the case of no mountains ($h_1 = 0$), Eq. (11) indicates that $\Lambda$ (and thus the zonal flow) is constant with time. The amplitude of the wave $A = (\psi_\alpha^2 + \psi_L^2)^{1/2}$ is also constant with time and the streamfunction is

$$\psi = \sqrt{2} \psi_{\alpha_{\psi}} \cos(y) + 2 A_0 \sin(nx - \beta \Lambda_0 t), \quad (15)$$

where the subscript 0 refers to initial values. Eq. (15) describes a Rossby wave which propagates with a frequency equal to $\beta \Lambda_0$. The sign of $\Lambda_0$ gives the direction of propagation of the wave with positive $\Lambda_0$ indicating eastward propagation.

It is clear, therefore, that the oscillation of the mean flow described by (14) is caused by the existence of mountains in the model.
3. Linearized analysis

The equilibrium solution of Eqs. (11)–(13) is given by

\[ \psi_L = 0, \]  
\[ \hat{\psi}_K = \frac{h_1}{\alpha(n^2 + 1)\hat{\beta}} \left( \frac{\hat{\Lambda} + \beta n}{\hat{\Lambda}} \right). \]  

This steady state indicates the intensity that the mode parallel to the mountain contours should have to exactly account for the stretching and contraction of the column of air caused by the mountains in the presence of the zonal flow. For any arbitrary zonal flow it is then always possible to find a wave amplitude (17) which will give steady conditions, provided that the amplitude of the mode that is out of phase with the mountains vanishes. The mountain ridge will coincide with a cyclone or anticyclone depending on the sign of \( \hat{\Lambda} \) as indicated by (17).

A linearized analysis about this steady state shows that for solutions of the form \( e^{\sigma t} \), the equation for \( \sigma \) is

\[ \sigma^3 = \left[ \frac{h_1^2}{(1 + n^2) n\hat{\Lambda}} - (\hat{\beta}\hat{\Lambda})^2 \right] \sigma. \]  

This relationship indicates that an exponential growth of the solution is possible if

\[ \frac{h_1^2}{(1 + n^2) n\hat{\Lambda}} > (\hat{\beta}\hat{\Lambda})^2. \]  

In order that exponential growth takes place \( \hat{\Lambda} \) must be positive [see Eq. (18)], indicating that the zonal flow should be strong enough to produce an eastward propagating Rossby wave in the absence of the mountains. The kinetic energy of deviations from the steady state given by (16) and (17) changes in time according to the equation

\[ \frac{d}{dt} \frac{1}{2} (\psi_L^2 + (n^2 + 1)(\psi_K^2 + \psi_L^2)) = \frac{h}{2} \psi_L \hat{\Lambda} \hat{\psi}_L, \]  

where primed quantities refer to the deviations from the steady state.

This relationship indicates that for a positive perturbation of the mean flow (\( \psi_L > 0 \)), \( \psi_L \) should be positive for the perturbation kinetic energy to increase with time for positive \( \hat{\Lambda} \). Thus an anticyclone should develop downstream of the mountain ridge and upstream of the valley, while for the case of a negatively perturbed zonal flow a low should develop at this location.

The anticyclone that develops downstream of the mountain ridge for the positively perturbed zonal flow induces southerlies over the mountain ridge, rising up the mountain on the southern half of the ridge and sinking on the northern half, producing anticyclonic and cyclonic vorticity in these two regions, respectively. Similarly, the northerlies that are induced over the valley tend to produce cyclonic and anticyclonic vorticity in the northern and southern halves of the channel, respectively, as the air that moves southward now goes downhill in the northern half and uphill in the southern half. This process gives a net contribution of cyclonic and anticyclonic vorticity in the zonal average for the northern and southern latitudes, respectively. Such a configuration leads to a further increase in the amplitude of the zonally averaged mode. Also, the \( \psi_L \) ridge downstream from the mountain ridge (or the \( \psi_L \) low upstream from it) tends to intensify the high pressure over the mountain (of the mode \( \psi_K \) parallel to the mountain contours) through the beta effect, while the ridge is weakened by the advection of vorticity by the zonal flow. Therefore, if the beta effect is stronger than the advection of vorticity by the zonal flow the amplitude of the \( \psi_K \) mode will increase. This condition is reflected in the instability criterion given by (19).

4. Conclusions

We have shown that when a Rossby wave interacts with a latitudinally variable mean flow in the presence of an inhomogeneous lower boundary a periodic oscillation of the mean flow results. This oscillation may partially account for observed oscillations of the zonal index in the atmosphere.

A linearized analysis shows that the requirement for instability is linked to the phasing of the waves with respect to the topographic relief.

Two shortcomings of the present study are, first, the high degree of truncation of the system and, second, the assumption of a barotropic quasi-geostrophic model to simulate topographical structures in global scales. Paege et al. (1979) have shown that the topographically induced ultralong waves of the Northern Hemisphere do possess a nearly barotropic structure, but the essential balances on these scales are significantly distorted with the quasi-geostrophic model used here. Both of these questions should be addressed in future studies.

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REFERENCES


Comments on “Some Factors Governing Ice Particle Multiplication in Cumulus Clouds”

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Mossop (1978) undertook the difficult task of assembling in-cloud measurements from a variety of field programs that were widely dispersed in time and location. In his paper Mossop differentiates between clouds that have an active ice multiplication mechanism and those that do not, on the basis of cloud base temperature and cloud base droplet concentrations. This leads him to postulate the existence of a “multiplication boundary” which should be of benefit to cloud seeding operations. There are two serious drawbacks to this paper. First, as presented, the data are of limited value in establishing the existence of a multiplication boundary. Second, the data sets probably contain numerous clouds that were precipitating through a warm rain process before a riming-splinter (Hallett–Mossop) ice multiplication mechanism developed.

Sixteen data sets, eight of which are unpublished, are used by Mossop (1978) in his Table 1. A data set can consist of a number of clouds on a given day, month, year or years for which a few characteristic numbers are abstracted. As an example, the ice crystal concentrations listed in Table 1 are obtained from the referenced data sets in many different ways. For the Australian clouds, Mossop uses “the highest value of the average ice crystal concentration and graupel concentration found in any cloud on that day.” For the clouds in Arizona, the ice crystal concentration is a generalization from the literature, i.e., “the crystal concentration commonly exceeding 100 ℓ⁻¹.” In Florida during July 1975 “some clouds had regions where columnar ice crystals reached concentrations of 60 ℓ⁻¹.” For ice crystals in the clouds of northeast Colorado, the comment “frequently absent at −10°C” is used.

Other parameters in Table 1 are also obtained by combinations of measurements and estimates which leaves one with reservations about the compatibility or comparability of the data sets. For example, without knowing the appropriate size ranges for the droplet concentrations, it is very difficult to intercompare the data. The author also neglects to mention the volume sampled for these and other measurements and he does not indicate whether maximum or average values are quoted. The graupel data of Table 1 are predominantly obtained from replication and impaction devices which leaves a certain residual doubt in the concentrations quoted. The highest values are from Hallett et al. (1978) where “any large ice particles which had no well-defined habit and/or showed evidence of fragmenting upon impact were regarded as graupel.” Riming was not even a criterion in this graupel classification! At best, this technique leads to an overestimate of graupel concentrations.

If one examines the cloud base and top temperatures of Mossop’s Table 1, it is evident that the clouds in the data sets where ice multiplication is postulated are much thicker than the clouds without a multiplication mechanism. In fact, of the six data sets in the non-multiplication category, two do not meet Mossop’s own minimum criteria and for another case only partial data are listed. It is not surprising

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