Perturbation Pressure and Cumulus Convection

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ABSTRACT

A simple model of perturbation pressure in cumulus convection is presented. The results show that the buoyancy- and drag-induced perturbation pressures act against the prescribed forcing. The dynamic pressure is found to be a consequence of the Bernoulli effect and a balance for the centrifugal force which arises from the curved motion of the air. Quantitative calculations reveal that the pressure force is of the same order of magnitude as the buoyancy and drag and offers a plausible explanation for the acceleration of negatively buoyant air near the base of convective updrafts.

1. Introduction

In recent years, the importance of perturbation pressure on cumulus convection has been realized through evidence from observations and numerical experiments.

On the basis of a balloon sounding taken inside the updraft of an active severe storm, Barnes (1970) reported an excess hydrostatic pressure of 3 mb at a level of 6 km and a pressure deficit of 1 mb near the cloud base. A number of other observations (e.g., Marwitz, 1973; Davies-Jones, 1974; Davies-Jones and Henderson, 1975) often indicate upward acceleration of negatively buoyant air near the cloud base in severe storms. These observations have led to conjectures that perturbation pressure forces may provide the answer to such phenomena (Marwitz, 1973).

The significance of the role of perturbation pressure is further borne out by the results of numerical experiments. Holton (1973) demonstrated that perturbation pressure can affect the cloud growth rate. Arnason et al. (1968), Soong and Ogura (1973), Soong (1974) and Wilhelmson (1974) showed that pressure forces oppose the buoyancy in the upper region of the cloud. Schlesinger (1973, 1975, 1978) presented evidence that perturbation pressure forces can accelerate negatively buoyant air at and near the cloud base.

Here a simple model of perturbation pressure is presented. The aims are to gain insight into the physical mechanisms involved and to confirm in a quantitative manner the importance of perturbation pressure in cumulus convection. The anelastic pressure equation is derived and a physical interpretation of the dynamic pressure is advanced. Simple forcing functions are then specified and the pressure responses are obtained by the method of Green’s functions.

2. Derivation of the anelastic pressure equation

Ogura and Phillips (1962) obtained by scale analysis a set of equations for inviscid flow suitable for the study of small-scale convection in a non-rotating atmosphere. The equation of motion modified to include water vapor buoyancy and the drag force of hydrometeors is

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -C_p g_\theta \nabla \theta + \left( g_\theta^b + 0.61 g_\theta^v \right) \nabla \theta_0 - gQk, \tag{1}
\]

where

- \( C_p \) specific heat of dry air at constant pressure
- \( g_\theta \) acceleration due to gravity
- \( k \) unit vertical vector
- \( g_\theta^b \) deviation of water vapor mixing ratio from base-state value
- \( Q \) mixing ratio of hydrometeors
- \( \mathbf{u} \) three-dimensional velocity vector
- \( \theta_0 \) constant base-state potential temperature
- \( g_\theta^v \) deviation of potential temperature from \( \theta_0 \)

For simplicity, the continuity equation is assumed to have the form

\[
\nabla \cdot \mathbf{u} = 0. \tag{2}
\]

This assumption restricts the study to the shallow convection case, but it is believed that the results obtained are at least qualitatively applicable to deep convection also. By taking the divergence of (1) and utilizing (2) the anelastic pressure equation is obtained
as
\[ C_p \nabla \cdot (\theta \nabla \pi') = -\nabla \cdot (u \cdot \nabla u) + \varepsilon \left( \nabla \cdot (\theta \nabla \phi') \right) \frac{\partial Q}{\partial z} - \varepsilon \frac{\partial Q}{\partial z}. \] (3)

Following Williamson and Ogura (1972), the first term on the right of (3) is termed dynamic, the second, buoyancy, and the third, drag. The corresponding pressure responses are denoted respectively by \( \pi_m \), \( \pi_0 \), and \( \pi_d' \). These three pressures sum to \( \pi' \) by the principle of superposition.

3. Interpretation of the physics of the dynamic pressure

From (3) the equation governing the dynamic pressure is
\[ C_p \nabla \cdot (\theta \nabla \pi'_m) = -\nabla \cdot (u \cdot \nabla u). \] (4)

By means of vector identity (4) can be written as
\[ C_p \theta \nabla^2 \pi'_m = -\nabla \cdot \left( \frac{1}{2} (u \cdot u) + \omega \times u \right) + \omega \cdot (\nabla \times \omega), \] (5)

where \( \omega \) is the vorticity.

Eq. (5) indicates that the dynamic pressure represents the response to the kinetic energy and to the rotational part of the flow. The underlying mechanism can be clarified by considering the equation of motion.

If a steady-state solution is assumed and the buoyancy and drag forces are neglected, Eq. (1) reduces to
\[ u \cdot \nabla u = -C_p \theta \nabla \pi'_m. \] (6)

The left-hand side can be further expanded to give
\[ \nabla \cdot \left( \frac{1}{2} u \cdot u \right) + \omega \times u = -C_p \theta \nabla \pi'_m. \] (7)

Taking the dot product of \( u \) or \( w \) with (7) yields the Bernoulli equation, i.e.,
\[ \frac{1}{2} u \cdot u + C_p \theta \omega'_m = \text{constant}. \] (8)

along a streamline or vortex line.

Eq. (8) represents the conversion of kinetic to potential energy and vice versa (in this case by perturbation pressure). Thus along a streamline or a vortex line, low-pressure areas will be found in regions of high kinetic energy and high-pressure areas in regions of low kinetic energy.

Further insight can be gained by taking the dot product of the unit vector \( \hat{n} \) normal to the streamline with (6). The result is
\[ -C_p \frac{\partial \pi'_m}{\partial n} = V \left( \hat{n} \cdot \frac{du}{ds} \right) = -\frac{V^2}{R_s}, \] (9)

where \( s \) is the distance along the streamline, \( V \) the air velocity in the direction of \( s \) and \( R_s \) the radius of curvature.

<table>
<thead>
<tr>
<th>Type of forcing</th>
<th>Geometry of cloud</th>
<th>Slab-symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buoyancy ( \theta' )</td>
<td>( \theta e^{-(r/a)^2} ) ( \sin(nz) )</td>
<td>( \theta e^{-(r/a)^2} ) ( \sin(nz) )</td>
</tr>
<tr>
<td>Drag ( Q )</td>
<td>( Q \sin(nz), \quad 0 &lt; r &lt; a )</td>
<td>( Q \sin(nz), \quad 0 &lt; r &lt; a )</td>
</tr>
<tr>
<td>Vertical velocity ( w )</td>
<td>( \frac{a}{2} \left[ 1 - (r/a)^2 \right] e^{-r/a} ) ( \sin(nz) )</td>
<td>( \frac{a}{2} \left[ 1 - (r/a)^2 \right] e^{-r/a} ) ( \sin(nz) )</td>
</tr>
<tr>
<td>Horizontal velocity ( u )</td>
<td>( -\frac{V}{2} e^{-r/a} ) ( \cos(nz) )</td>
<td>( -\frac{V}{2} e^{-r/a} ) ( \cos(nz) )</td>
</tr>
</tbody>
</table>

Eq. (9) states the balance of the pressure force and centrifugal force and shows that a low-pressure area is located to the right (left) of the direction of motion for clockwise (counterclockwise) rotation. Furthermore, the perturbation pressure force normal to the direction of the flow increases with a decrease in the radius of curvature. Therefore in a cumulus cloud where the circulation resembles that of a ring vortex, a low center would be expected near the cloud edge at the level of maximum vertical velocity. The strong rotation of the air in the vertical plane at that level gives rise to a large centrifugal force which balances the perturbation pressure gradient.

It is also of interest to note that \( \pi'_m \) in (8) and \( \partial \pi'_m / \partial n \) in (9) are proportional to the square of the updraft velocity. Hence similar pressure responses will occur for updrafts and downdrafts having similar configurations and the effect of the dynamic pressure will be most important for vigorous storms.

4. Quantitative assessment of the effect of perturbation pressure forces

a. The forcing functions

The effect of perturbation pressure is assessed from analytic solution of (3) with specified forcing functions as given in Table 1. Solutions are obtained for a non-precipitating cloud in the slab and axisymmetric geometries.

The vertical profiles are simple sinusoidal functions. Specification of the horizontal variations is guided by observations. Warner (1955) found that the liquid water concentration in moderate-sized cumulus often rises sharply from zero at the edge of the cloud, indicating that a top-hat profile might be a good first approximation for the liquid water content. On the basis of in-cloud temperature measurements and the concept of nonhomogeneous turbulent mixing, McCarthy (1974) suggested that a Gaussian profile might be more plausible than that of a top-hat in describing temperature variations. In accordance with these findings, a Gaussian profile is used for the buoyancy and a top-hat profile for the drag. The amplitudes \( \theta, Q \) and \( \psi \) are \( 1^\circ C, 1 \text{ g kg}^{-1} \) and \( 5 \text{ m s}^{-1} \), respectively,
in agreement with observations by Malkus (1954) and Warner (1970).

b. **Boundary conditions and method of solution**

The case for the axisymmetric model is described, with the same procedure applying to the slab case.

The top and bottom boundary conditions are \( \partial \sigma'/\partial z = 0 \) at \( z = 0 \) and \( z = H \), where \( H \) represents the top of the domain and is 4 km. For the lateral boundary conditions, it is assumed that \( \partial \sigma'/\partial r = 0 \) at \( r = 0 \) and \( \pi' = 0 \) as \( r \to \infty \).

These boundary conditions, together with the specified forcing functions, assure a unique solution so that additional assumptions, such as those in Árnason (1974), are not required.

The vertical boundary conditions can be satisfied by a series solution of form

\[
\pi' = \sum_k \pi_k(r) \cos(kn),
\]

(10)

where \( n = \pi^*/H = 3.14159/H \). By substituting (10) into (3), equations for the radial modes \( \pi_k(r) \) are obtained. The solutions for \( \pi_k(r) \) can be found by the method of Green's functions. The details are in Yau (1977).

c. **Pressure responses**

Profiles of the vertical pressure gradient forces at the central axis are depicted in Figs. 1–3. The peak

**Table 2. Peak values of forcing and induced pressure force.**

<table>
<thead>
<tr>
<th>Type of force</th>
<th>Peak value of force per unit mass (cm ( \text{s}^{-2} ))</th>
<th>Peak value of perturbation pressure force per unit mass (cm ( \text{s}^{-2} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buoyancy ( \theta'/\theta_b )</td>
<td>+3.3</td>
<td>-0.88</td>
</tr>
<tr>
<td>Drag (-gQ)</td>
<td>-0.98</td>
<td>+0.30</td>
</tr>
<tr>
<td>Velocity advection</td>
<td>+0.98 (max) \text{,}</td>
<td>-0.32</td>
</tr>
<tr>
<td>( \text{u.u} )</td>
<td>-0.98 (min) \text{,}</td>
<td>+0.32</td>
</tr>
</tbody>
</table>
values of the forcing and the induced pressure forces are in Table 2.

The results indicate that the pressure forces induced by drag and buoyancy account for 30–50% of the magnitude of the forcing. In agreement with the finding of Soong and Ogura (1973), the pressure forces are larger in the slab geometry; they are twice as large as the value in the axisymmetric case. The dynamic pressure force almost cancels the effect of vertical advection in the slab model but is much smaller for the axisymmetric cloud. The vertical advection tends to place the maximum velocity near the top of the cloud as is generally indicated in one-dimensional models where the effect of perturbation pressure is ignored. It is shown that the dynamic pressure force acts to augment the buoyancy in the lower half of the cloud and opposes motion in the upper half of the updraft (Fig. 3). This force contributes to a lessening of the strong velocity gradient near the cloud top and tends to lower the level of maximum velocity as has recently been documented (e.g., Marwitz, 1973; Holton, 1973).

d. Negative buoyancy and perturbation pressure

To determine the role of perturbation pressure in supporting a negatively buoyant updraft, the following form of the buoyancy perturbation is used:

\[ \theta' = [\tilde{\theta}_1 \sin(az) + \tilde{\theta}_2 \sin(2az)] e^{-(r/a)^2}. \]  \hspace{1cm} (11)

The buoyancy force at the central axis for \( \tilde{\theta}_1 = 0.54 \) and \( \tilde{\theta}_2 = -0.6 \) is shown by the dashed curve in Fig. 4. A minimum negative acceleration of \(-0.8\) cm s\(^{-2}\) occurs at a height of 700 m.

The effect of the variation of cloud radius \( a \) on the sum of the buoyancy and buoyancy pressure force is illustrated by the solid curves in Fig. 4. It can be seen that as the cloud radius increases, the acceleration below 1.4 km becomes progressively less downward. Finally, the acceleration turns completely upward at a cloud radius of 2.5 km.

The buoyancy pressure force is therefore capable of countering the negatively buoyant updraft near the cloud base. It has also been demonstrated that in general the drag and dynamic pressure forces act upward near the ground. It appears likely that in situations where the drag force of the hydrometeors does not interfere directly with the updraft near the surface, the acceleration of negatively buoyant air in the lower portion of the cloud by the induced perturbation forces becomes a distinct possibility.

5. Summary

Simple analytic models based on the shallow-convection set of equations have been used to examine the role and importance of perturbation pressure in cumulus convection. Analytic solutions for the pressure are obtained with simple forcings for the buoyancy, drag and updraft velocities. In general, the results indicate that the buoyancy and drag pressure forces act against the prescribed forcings. The role of the dynamic pressures can be explained in terms of energy conversion as the Bernoulli effect and the tendency to balance the centrifugal force associated with curved air motion. Quantitative analysis of the pressure forces shows that they are of the same order of magnitude as the buoyancy and drag, and may be important in supporting the negatively buoyant updraft found in observations.

The use of the shallow-convection set of equations has limited the present analysis to a shallow cloud which is 4 km deep. Observations of pressure deviation are generally made in severe storms which extend to considerable heights. The role of perturbation pressure, however, is expected to be even more important in large convective clouds where the amplitudes of the forcings are much more pronounced. In view of this fact and the anticipation that the vertical variation of air density does not affect significantly the calculation of perturbation pressure, the findings presented here would remain valid for deep convective systems.

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