A Comparison of a Numerical Model and an Approximate Analytical Model of the Growth of Snowflakes

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ABSTRACT

A comparison is made between the evolution of initially exponential snowflake size distributions growing by deposition and aggregation as calculated with a numerical model and with the approximate analytical model of Passarelli. The agreement between the two models implies that to a good approximation the assumptions of the analytical model are consistent with the additional assumption that the distribution maintains an exponential shape.

1. Introduction

The exponential nature of raindrop size distributions was first demonstrated by Marshall and Palmer (1948) and since then by many others. Young (1975) and Gillespie and List (1976) were able to show, on the basis of numerical models of the evolution of raindrop distributions by coalescence and droplet breakup, that exponential distributions are to be expected. These results led Srivastava (1978) to propose a new parameterization of raindrop size distributions.

The sequence of events in the study of snowflake size distributions has not been quite the same. Measurements have shown that they also tend to have a negative exponential form (Gunn and Marshall, 1958). Based on such observations Passarelli (1978) formulated an approximate analytical model of snowflake size distribution evolution by vapor deposition and aggregation. By assuming that the distribution is always exponential, and by also making a number of other simplifying assumptions, Passarelli was able to show analytically that exponential snow size distributions in a steady-state heterogeneous cloud tend to an equilibrium that is independent of the parameters of the initial exponential distribution. The conclusions drawn from the analytical model would be placed on a firmer basis if it were demonstrated that the basic assumptions of the model were, in fact, consistent with the assumption that the exponential shape is maintained. One step in this direction was made by comparing the results of the approximate analytic technique for raindrops growing by coalescence using Srivastava's (1971) numerical integration scheme (Passarelli, 1978). The purpose of this note is to present comparisons of the results of the analytical model with those from a numerical model that imposes the same physical restrictions but allows the shape of the distribution to evolve without constraint. In this way the consistency, or lack of consistency, of the analytical model can be demonstrated explicitly.

2. Theory

The numerical model is a modification of an earlier model of droplet growth by condensation and coalescence (Leighton and Rogers, 1974). The modifications are those necessary to make the model compatible with the steady-state analytical model of Passarelli.

If $N(x,h)$ is the number density of particles with mass x at height h, then the change of $N(x,h)$, assuming a steady state and an updraft speed small compared to the particle fallspeeds, is given by

$$v_T(x)\frac{\partial N(x,h)}{\partial h} = \frac{\partial}{\partial x}[\dot{x}N(x,h)] - \frac{1}{2}\int_0^x dx' N(x_c,h)K(x_c,x')N(x',h) + \int_0^\infty dx' N(x,h)K(x,x')N(x',h), \quad (1)$$

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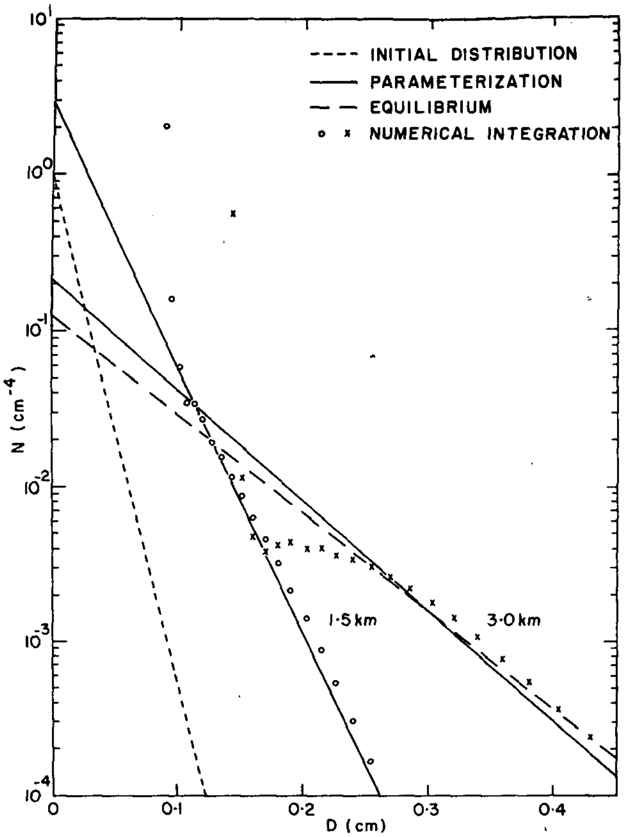

FIG. 1. A comparison of the results of the numerical and analytical models at 1500 and 3000 m below the reference level. Also shown is the equilibrium spectrum at 3000 m.

where $K(x,x')$ is the collection kernel, $x_c = x - x'$, and $v_T(x)$ is the particle fallspeed. The first term on the right describes the change due to deposition and the last two terms the change in N due to aggregation.

Neglecting for the moment growth by deposition, Eq. (1) is written in a form suitable for computation by expressing x in the form suggested by Berry (1967)

$$x(J) = x_0 \exp[3(J - 1)/J_0].$$

Here J assumes integer values from 1 to 81, $J_0 = 12/\ln 2$ and $x_0 = 2.62 \times 10^{-8}$ g. Assuming spherical ice particles of density 0.05 g cm^{-3}, the diameters of the particles considered range from 0.01 to 1.02 cm. The collection kernels are calculated with the same assumptions as in the analytical model, namely gravitational, geometric sweepout with unit collection efficiency due to differential fallspeeds given by $v_T(x) = aD(x)^b$, where D is the snowflake diameter and a and b are constants. The resulting equation is integrated using the methods described by Berry and Reinhardt (1974).

The diffusional growth rate is expressed in the form given by Passarelli

$$\dot{x} = f(h)D^{\delta},$$

where in the present calculations we set $\delta = 1$. If we also assume that $f(h)$ is a slowly varying function, then the change in the particle distribution due to deposition follows the parabolic growth law. The magnitude of $f(h)$ at any level is fixed by the requirement that the rate of increase in the particle mass flux is equal to the rate of water vapor flux convergence. The vapor density is assumed to decrease exponentially with height at a rate given by

$$\rho_{vs} = \rho_{vs,0}e^{-Ah}$$

and the updraft speed is assumed to be uniform. The details of the evaluation of the deposition growth are similar to those of the condensation growth calculations in the model of Leighton and Rogers (1974). Deposition does not occur at every time step (1 s) of the aggregation calculation but is restrained so that the lower boundary of the particle size distribution always coincides with a radius corresponding to an integral value of J. Nevertheless, the effective time step for deposition is such that the precipitation rate is always close to the value required by the steady-state assumptions.

3. Results

Fig. 1 shows a comparison of the results from the numerical integration with those from the analytical model at 1.5 and 3.0 km below the initial

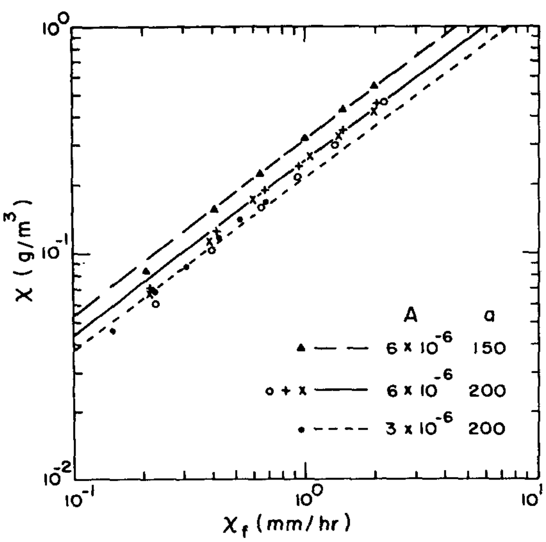

FIG. 2. A comparison of the equilibrium mass-flux relationship from the analytical model (lines) with the results from the numerical model. The symbols ○, + and × refer to three different initial distributions defined by $N_0 = 0.1$, $\lambda = 29.31$; $N_0 = 1$, $\lambda = 50$; $N_0 = 5$, $\lambda = 72.6$, respectively. The symbols ▲ and ● show results for an initial distribution defined by $N_0 = 1$, $\lambda = 50$ but for different values of A and a.

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level. The initial distribution had a shape given by

$$N(D) = N_0 e^{-\lambda D},$$

with $N_0 = 1$ cm^{-4} and $\lambda = 50$ cm^{-1}. The vapor density profile is defined by $\rho_{vs,0} = 1.1 \times 10^{-6}$ g cm^{-3}, $A = 6 \times 10^{-6}$ cm^{-1} and $w = 10$ cm s^{-1}. The fallspeed constants a and b are 200 cm s^{-1} and 0.31, respectively. Also included in the figure is the line representing the equilibrium shape predicted from the analytical model. The large-particle portions of the distributions are seen to maintain exponential shapes that are in good agreement with the predictions of the analytical model. Similar agreement was found for other initial distributions and for $A = 0.3 \times 10^{-5}$ cm^{-1} and $a = 150$ cm s^{-1}. The disagreement at small sizes is not surprising since by using mass and reflectivity moments, the analytical model is designed to model the large end of the spectrum.

Passarelli also derived equilibrium relationships between the particle mass concentration χ and the mass flux χ_F. These expressions were compared to measured values and found to be in fair agreement. The equilibrium relations are reproduced in Fig. 2 together with the results of numerical integrations for different initial distributions or different values of a or A. The results of each integration are plotted for height intervals of 500 m with the first point indicating the results at a distance 500 m below the starting level. In all cases the numerical results are seen to converge closely to the corresponding equilibrium line.

4. Conclusions

The numerical integrations have demonstrated that the assumptions of the analytical model are to a good approximation consistent with the preservation of an exponential size distribution. This, in turn, places the analytical model on a firmer foundation and gives more confidence in the validity of the formulas for the equilibrium shape and the mass-flux relationships.

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