

Radar Backscattering by Inhomogeneous Precipitation Particles¹

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ABSTRACT

Calculations of radar backscattering by inhomogeneous precipitation particles require values of the dielectric function of two-component mixtures. Four such dielectric functions are critically examined and their relative merits are weighed. Although apparently different, two are shown to be equivalent: the effective-medium and Polder-van Santen theories. All the dielectric functions agree when the two components are dielectrically similar. All except the Maxwell-Garnet dielectric function are symmetric with respect to interchange of the components. When compared with measurements on ice-air mixtures, the effective-medium and Maxwell-Garnet dielectric functions are marginally better than the Debye function, which has previously been used in backscattering calculations. When the fraction of water is high, the effective-medium function gives calculated values of radar backscattering that are in good agreement with measurements on ice spheres coated with a mixture of ice and water. The Maxwell-Garnet theory, with ice inclusions embedded in a water matrix, is also in good agreement with these measurements, and is so over a wider range of water-volume fractions than the effective-medium theory. Although there are no compelling reasons for preferring one above the other, on the basis of the evidence presented, we would be inclined to use the Maxwell-Garnet dielectric function in radar backscattering calculations.

1. Introduction

Many precipitation particles in the atmosphere are inhomogeneous: dry snowflakes and snow pellets are mixtures of ice and air in such proportions as to have densities much below 920 kg m^{-3} ; ice pellets and hailstones sometimes are mixtures of ice and water, with the fractions approaching 50% in some cases. In order to calculate the radar cross sections of these particles, it is necessary to know the dielectric functions³ of mixtures of ice with air and water.

2. Dielectric function of a mixture

No material medium is homogeneous in the absolute sense but is composed of a large number of individual subunits or heterogeneities. The description of the interaction of an electromagnetic wave with such a medium by means of a spatially uniform dielectric function is, therefore, ultimately statistical, and its validity requires that the heterogeneities be small compared with the wavelength. It is for this reason that the optics of a medium usually considered to be homogeneous, such as pure water, is adequately described to a first approximation by a dielectric function (or, equivalently,

refractive index). There is no sharp line of demarcation between the optics of such homogeneous media and media composed of particles each of which contains sufficiently many atoms or molecules such that they can be individually assigned a bulk dielectric function.

It is not trivial to determine the dielectric function of a mixture from a knowledge of the properties of its components; the large number of interactions among them leads to problems that are insoluble except by approximate methods. Each type of approximation leads to a different dielectric function. As a consequence, there is a bewildering array of choices in the scientific literature, all of which are at least superficially different. Without attempting to be exhaustive, we critically examine and compare four dielectric functions that are frequently used: Maxwell Garnet, effective medium, Polder and van Santen, and Debye.

The dielectric function of a particulate medium is clearly a function of the shape distribution, orientation, and separation of the particles, as well as the medium in which they are embedded. In the absence of detailed knowledge about a particulate medium, the simplest assumption is that the particles are spheres. Thus, we regard a mixture as being composed of spherical inclusions embedded in a homogeneous matrix, where ϵ and ϵ_m are their respective dielectric functions and f is the volume fraction of the inclusions. In some circumstances, the matrix may be nonabsorbing (i.e., ϵ_m is real), but there appears to be nothing in the derivations

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³ The ratio of the permittivity of a substance to the permittivity of free space is usually called the dielectric constant, but since it may vary with wavelength and the properties of the substance, we prefer the term dielectric function.

that requires this, and we shall assume that there are no restrictions on ϵ and ϵ_m .

The dielectric function

$$\epsilon_{\text{MG}} = \epsilon_m \left[1 + \frac{3f \left(\frac{\epsilon - \epsilon_m}{\epsilon + 2\epsilon_m} \right)}{1 - f \left(\frac{\epsilon - \epsilon_m}{\epsilon + 2\epsilon_m} \right)} \right] \quad (1)$$

was first derived by Maxwell Garnet (1904). Subsequently, the Maxwell-Garnet (MG) dielectric function has been rederived under various sets of assumptions (see, for example, Genzel and Martin, 1972, 1973; Barker, 1973; Bohren and Wickramasinghe, 1977).

The effective-medium (EM) dielectric function ϵ_{EM} is the solution to the quadratic equation

$$f \left(\frac{\epsilon - \epsilon_{\text{EM}}}{\epsilon + 2\epsilon_{\text{EM}}} \right) + (1 - f) \left(\frac{\epsilon_m - \epsilon_{\text{EM}}}{\epsilon_m + 2\epsilon_{\text{EM}}} \right) = 0. \quad (2)$$

Eq. (2) was first obtained by Bruggeman (1935), who also gives a similar expression for cylindrical inclusions, and later was rederived by Landauer (1952). Both the Maxwell-Garnet and effective-medium dielectric functions have been shown by Stroud (1975) to follow from the same integral equation: either one or the other is obtained depending upon the approximations that are made. So we may consider (1) and (2) to be related: they issue from the same parent equation.

It is by no means obvious at a glance that the dielectric function ϵ_{PS} of Polder and van Santen (1946) that satisfies

$$\frac{\epsilon_{\text{PS}} - 1}{3\epsilon_{\text{PS}}} = f \frac{\epsilon - 1}{\epsilon + 2\epsilon_{\text{PS}}} + (1 - f) \frac{\epsilon_m - 1}{\epsilon_m + 2\epsilon_{\text{PS}}} \quad (3)$$

is the same as ϵ_{EM} . The latter satisfies a *quadratic* equation whereas the former satisfies a *cubic*. And yet they are indeed identical! The proof is tedious, and we shall omit it here, but the key step is the relation

$$\frac{f\epsilon}{\epsilon + 2\epsilon_{\text{EM}}} + \frac{(1-f)\epsilon_m}{\epsilon_m + 2\epsilon_{\text{EM}}} = \frac{1}{3}. \quad (4)$$

Debye (1929, pp. 44-47) gives the dielectric function

$$\frac{\epsilon_{\text{D}} - 1}{\epsilon_{\text{D}} + 2} = f \left(\frac{\epsilon - 1}{\epsilon + 2} \right) + (1 - f) \left(\frac{\epsilon_m - 1}{\epsilon_m + 2} \right), \quad (5)$$

which is a generalization of the Clausius-Mosotti relation. It is clear from reading Debye that he had in mind mixtures of homogeneous substances such as aqueous solutions, rather than particles large enough to be assigned a bulk dielectric function—a molecule doesn't have a dielectric function. Nevertheless, ϵ_{D} is included in our list because it has been extensively—and uncritically—used in radar backscattering calculations.

With the exception of the Maxwell-Garnet dielectric function, all the dielectric functions above have the property of invariance with respect to a change in the rôles of matrix and inclusions. If we make the substitutions $\epsilon \rightarrow \epsilon_m$, $\epsilon_m \rightarrow \epsilon$ and $f \rightarrow (1 - f)$, then ϵ_{MG} is, in general, not the same, whereas ϵ_{EM} and ϵ_{D} are unchanged. Thus, the effective-medium and Debye dielectric functions apply to two-component mixtures that are random in the sense that both components are treated symmetrically. If a two-component mixture is to be described by the MG dielectric function, a choice must be made as to which component is the matrix and which the inclusions. There may be physical reasons to guide this choice. As will be shown, such a choice markedly affects calculated backscattering cross sections of heterogeneous spheres of water and ice.

All the dielectric functions above agree at the end points $f = 0$ and $f = 1$, which is comforting. But the similarities go deeper. To see this, we expand the various dielectric functions in powers of the variable $\Delta = \epsilon - \epsilon_m$:

$$\epsilon_{\text{MG}} = \epsilon_m + f\Delta - \frac{f(1-f)\Delta^2}{3\epsilon_m} + \frac{f(1-f)^2\Delta^3}{9\epsilon_m^2} + \dots, \quad (6)$$

$$\epsilon_{\text{EM}} = \epsilon_m + f\Delta - \frac{f(1-f)\Delta^2}{3\epsilon_m} + \frac{f(1-f^2)\Delta^3}{9\epsilon_m^2} + \dots, \quad (7)$$

$$\epsilon_{\text{D}} = \epsilon_m + f\Delta - \frac{f(1-f)\Delta^2}{\epsilon_m + 2} + \frac{f(1-f)^2\Delta^3}{(\epsilon_m + 2)^2} + \dots. \quad (8)$$

If $|\Delta/3\epsilon_m| \ll 1$, then all three dielectric functions give essentially the same results. Also, to terms of order Δ , the MG dielectric function is symmetric in matrix and inclusions. Therefore, if we are dealing with mixtures the components of which are dielectrically similar, then the choice of dielectric function is easy: any of the above will do! If the inclusions are merely perturbations, then there is no way to distinguish between dielectric functions. A more stringent test of their relative merits is how well they predict the behavior of mixtures of very dissimilar substances.

Abeles and Gittleman (1976) measured the optical properties of the composite metal-insulator system Ag-SiO₂ and the composite semiconductor-insulator systems Si-SiC and Ge-Al₂O₃ and compared their results with calculations based on the Maxwell-Garnet and effective-medium dielectric functions. They concluded that only the Maxwell-Garnet

theory was compatible with their observations. On the other hand, Landauer (1952) found good agreement between conductivities⁴ calculated from effective-medium theory and measured values, even for mixtures with components having greatly different conductivities. So it is certainly not true that the Maxwell-Garnet theory is superior to the effective-medium theory in every instance; both theories have their successes. Part of the ambiguity surely arises from the difference between a random mixture as defined above and a mixture in which there are distinguishable inclusions embedded in a definite matrix. In succeeding sections, we shall bring more experimental evidence to bear on the question of which dielectric function is appropriate to radar backscattering calculations for inhomogeneous precipitation particles.

3. Experimental results for ice-air mixtures

The Debye function (5) has been used to obtain the radar cross sections of ice-air Rayleigh scatterers (Marshall and Gunn, 1952; Gunn and East, 1954). It has been the basis for the widely accepted practice of writing the backscattering cross section of an ice sphere in terms of the diameter and dielectric function of a hypothetical sphere of equivalent ice mass with a density of 1000 kg m⁻³. This practice follows from (5) by taking the dielectric function of air, $\epsilon = 1.0$, and finding that

$$\frac{K_1}{\rho_1} = \frac{K_i}{\rho_i} = \text{constant}, \quad (9)$$

where $K = (\epsilon - 1)(\epsilon + 2)$, and the subscripts 1 and i refer to ice having a density ρ of 1000 kg m⁻³ and the density of the real ice, respectively.

Cumming (1952) measured the dielectric function at 9.375 GHz (wavelength of 3.2 cm) of ice-air mixtures with densities between 230 and 920 kg m⁻³. His data show that (9) is a fair approximation: K/ρ ranges from about 5.0×10^{-4} to 4.4×10^{-4} m³ kg⁻¹ at densities of 500 and 1000 kg m⁻³, respectively.

Various calculations and measurements of dielectric function versus density of ice are given in Fig. 1. The diagram has three sets of points and a smooth, solid curve that was obtained from an expression attributed by Cumming to Polder and van Santen (1946):

$$\frac{\epsilon' - 1}{3\epsilon'} = f \frac{\epsilon'_i - 1}{\epsilon'_i + 2\epsilon'}, \quad (10)$$

where ϵ'_i is the dielectric function of ice, ϵ' the dielectric function of the mixture of ice and air, and f the volume of ice in the mixture. We showed above that (10) is identical with the effective-medium ex-

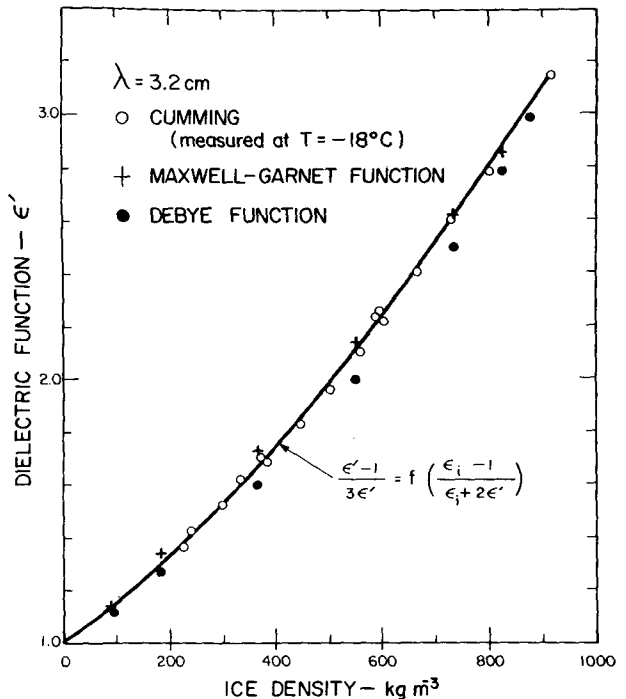


FIG. 1. Measured (open circles) and calculated values of dielectric functions versus ice densities.

pression. The crosses are values of ϵ_{MG} calculated from (1) taking ice as the matrix and air as the inclusions. The solid circles are the Debye dielectric function (5); it gives values that are less than the experimental values. It is evident that (1) and (10) fit the experimental data quite well. Cumming indicated that the measurements were made at -18°C and that errors were small. For example, in measuring the dielectric function of the same sample of ice at temperatures between 0 and -20°C , random errors were observed to be of the order of ± 0.5 percent. At densities between 200 and 500 kg m⁻³, the experimental points fall just below the Maxwell-Garnet values; the maximum discrepancy at a density of about 300 kg m⁻³ is about 2 percent. We conclude, therefore, that the Maxwell-Garnet and effective-medium theories agree about equally well for ice-air mixtures and that both are marginally better than the Debye theory. Of course, the real test of a dielectric function is how well it predicts the behavior of mixtures of greatly dissimilar substances. The difference between the dielectric functions of ice and air is not small, but it is much smaller than the difference between the dielectric functions of ice and liquid water. We now turn to this last composite system.

4. Ice-water mixtures

Calculations of radar backscattering by particles composed of mixtures of ice and water have been relatively few, but those that have been published

⁴ Dielectric functions, conductivities and magnetic permeabilities of composite media are described by the same functional relations; this follows from the form of the Maxwell equations.

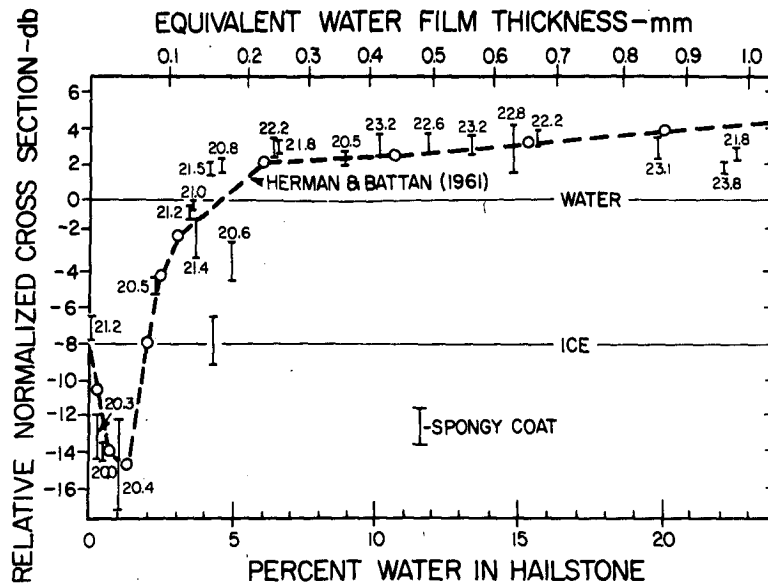


FIG. 2. Normalized backscattering cross sections (dB) of ice spheres (diameters in millimeters given by numerals) obtained at 5.05 cm wavelength. The spheres were composed of solid ice cores with outer shells of spongy ice between 0.4 and 2 mm thick and various water contents. The upper abscissa is the thickness of a water shell of the same water mass as in the ice-water shell. Each vertical line represents the range of measurements as each sphere was rotated. The all-water and all-ice values are shown as horizontal lines, taking the former at 0 dB. From Atlas *et al.* (1964).

have used the Debye equation (5) to obtain the dielectric functions (Ryde, 1946; Austin and Bemis, 1950; and Langleben and Gunn, 1952⁵). Battan and Herman (1962) calculated the backscattering cross sections of large spheres composed of ice-water mixtures.

Joss and List (1963), Joss (1964), and Joss and Aufdermaur (1965) measured the backscattering cross sections, at various wavelengths between 3.21 and 10 cm, of individual, large wet and dry ice and snow spheres. The measurements at a wavelength of 5.05 cm were analyzed by Atlas *et al.* (1964), who compared them with calculations of the backscattering of corresponding ice spheres. Specifically, the study dealt with solid ice spheres coated with layers of "spongy ice" (i.e., a mixture of water and ice). The calculations of backscattering by such spheres made by Battan and Herman (1962) used values of the dielectric function obtained from the Debye equation (5); at the time, doubts were expressed about the validity of using this expression when one or both of the substances in the mixture has appreciable absorption.

The Atlas *et al.* (1964) analysis showed that radar cross sections calculated by Battan and Herman (1962) departed appreciably from the meas-

ured ones. On the other hand, the latter corresponded well to cross sections calculated by Herman and Battan (1961) from models where solid ice spheres were coated with thin shells of water with masses equal to the masses of water in the spongy ice (Fig. 2). This result appeared to confirm the conclusion that when absorption was appreciable, the Debye dielectric function was inappropriate. As noted earlier, it is now clear that the Debye function is restricted to mixtures with a high degree of homogeneity, such as aqueous solutions.

The validity of the Maxwell-Garnet and effective-medium dielectric functions was tested by calculating refractive indices of ice-water mixtures at a wavelength of 5.05 cm for various values of f and using the results to calculate backscattering cross sections that could be compared with the measurements reported by Atlas *et al.* (1964). Fig. 3 shows, as expected, that the refractive index (m) calculated from the Maxwell-Garnet function depends to an important extent on which substance is the inclusion and which one the matrix. The effective-medium dielectric function yields values of the refractive index that are between those given by the Maxwell-Garnet function. When the fraction of water is small, m_{EM} is close to the value of m_{MG} for a mixture of water in an ice matrix; when the fraction of water in the mixture is high, m_{EM} approaches the values of m_{MG} for a mixture of ice in a water matrix.

⁵ Langleben, M. P., and K. L. S. Gunn, 1952: Scattering and Absorption of Microwaves by a Melting Ice Sphere. Sci. Rep. No. MW-5, Stormy Weather Research Group, McGill University, Montreal.

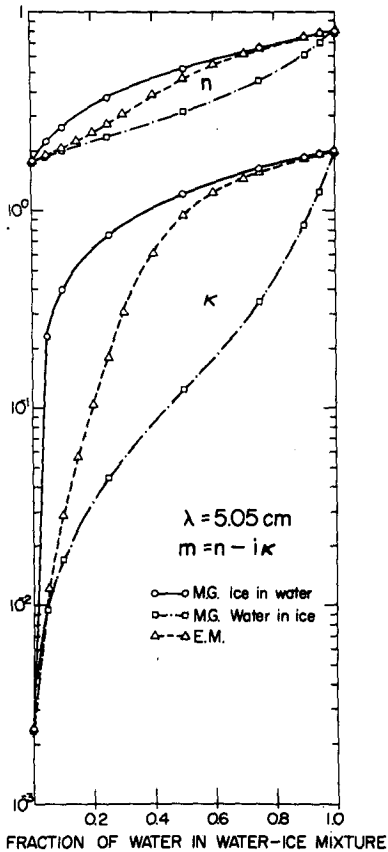


FIG. 3. Calculations of refractive index $m = n - i\kappa$ from the Maxwell-Garnet (MG) and the effective-medium (EM) dielectric functions.

Figs. 4, 5 and 6 show calculations of backscattering cross sections of spheres composed of ice cores covered with shells of ice-water mixtures. The overall diameter of all spheres was fixed at 2 cm, while the thickness of the shell and the fraction f of water in the water-ice mixture were varied. The radar cross sections have been plotted as a function of the equivalent water thickness, the same coordinates used by Atlas *et al.* (1964). Note that they measured the backscattering by spheres ranging in diameter from 2.0 to 2.4 cm and having shell thicknesses between 0.4 and 2 mm. Most of the data points of spheres having equivalent water thicknesses less than 0.4 mm had diameters between 2.0 and 2.2 mm.

Intuition suggested to us that a mixture of ice as the matrix and water as the inclusion was physically more plausible than the reverse arrangement. But when calculations were made of backscattering by spheres having shells composed of ice-water mixtures with refractive indices given by the Maxwell-Garnet function for such a mixture, the results (Fig. 4) bore little relationship to the empirical evidence in Fig. 2.

On the other hand, calculated radar cross sections where the shells had values of m_{MG} for mixtures of ice in a water matrix (Fig. 5) are in excellent agree-

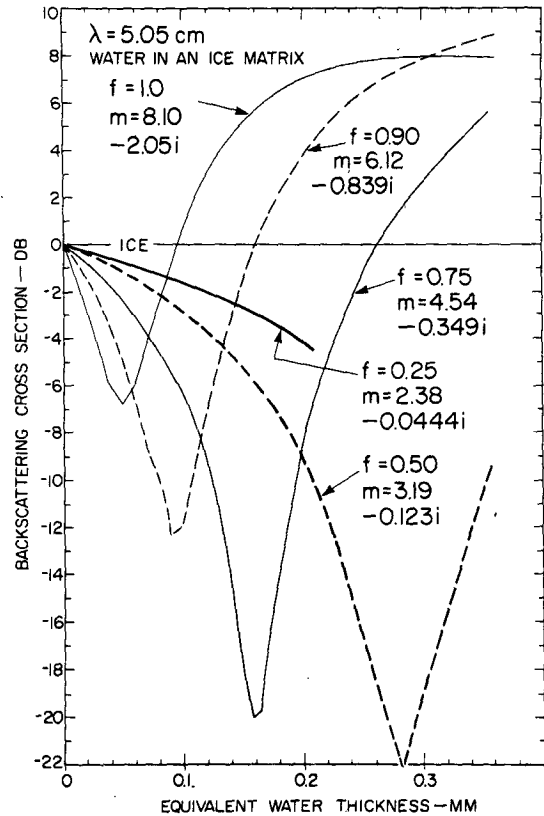


FIG. 4. Calculated normalized backscattering cross sections of spheres (in decibels with respect to the cross sections of all-ice spheres) composed of solid ice cores with shells of ice-water mixtures. The fraction of water in the mixture is given by f . The refractive indices were obtained by means of the Maxwell-Garnet function for water included in an ice matrix. Note that the 0 dB level in this diagram, and in those that follow, corresponds to the -8 dB (ice) level in Fig. 2.

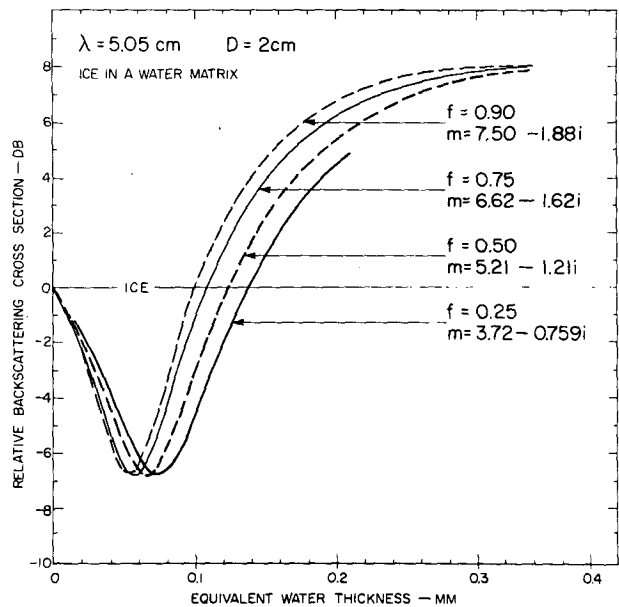


FIG. 5. As in Fig. 4 except that the refractive indices were obtained by means of the Maxwell-Garnet function for ice included in a water matrix.

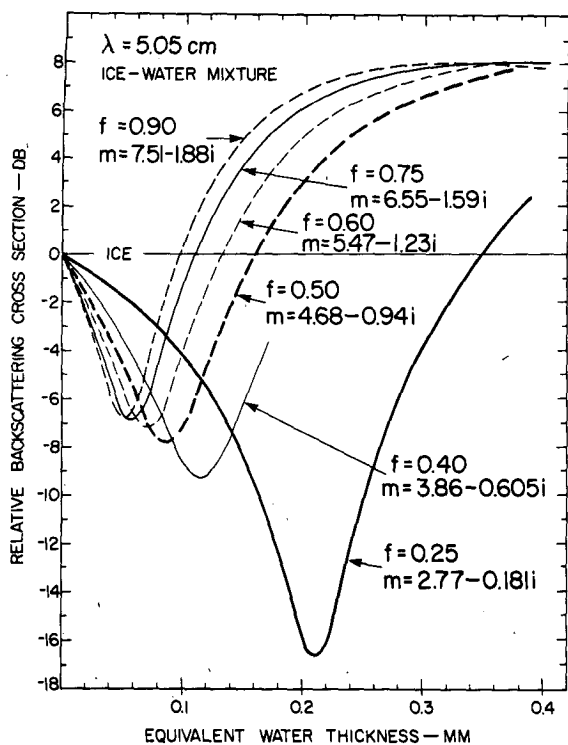


FIG. 6. As in Fig. 4 except that the refractive indices were obtained by means of the effective-medium function.

ment with the measurements in Fig. 2. All four of the curves fit the measurements reasonably well, showing that the backscattering is not sensitive to the fraction of water in the mixture when it ranges from 0.25 to 0.90.

Finally, calculations were made of backscattering by spheres having shells composed of ice-water mixtures with refractive indices given by the effective-medium function (Fig. 6). The theoretical values are in good agreement with measurements, providing the fraction of water in the mixture is high, certainly greater than 50 percent. Recall that the refractive indices yielded by the effective-medium function are independent of which medium is the inclusion and which the matrix.

These results can be summarized as follows. When the fraction of water in an ice-water mixture is high, the effective-medium function and the Maxwell-Garnet function (with ice included in a water matrix) give values of radar backscattering that are in good agreement with measurements reported by Atlas *et al.* (1964). On the other hand, the effective-medium function leads to calculated backscattering cross sections in good agreement with measurements only where the mixture is fairly rich in water. Early work by List (1959) led him to conclude that, in some cases, spongy ice spheres might have water fractions as high as 0.5.

It is not unreasonable to visualize that, at an early stage in the freezing of supercooled water, the substance resembles a mixture of ice in a water matrix, but it is more difficult to expect such a state in the spheres used by Joss and List (1963) to make the measurements reported by Atlas *et al.* (1964). Nevertheless, on the basis of the evidence presented here, we would be inclined to use the Maxwell-Garnet function for ice in a water matrix to calculate the refractive index of an ice-water mixture.

We intend to test the effective-medium and Maxwell-Garnet dielectric functions against other measurements published by Joss (1964) and Joss and Aufdermaur (1965) and whatever other empirical evidence we can find.

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