On the Evolution and Interaction of Short and Long Baroclinic Waves of the Eady Type

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ABSTRACT

The nonlinear evolution of unstable two-dimensional Eady waves is examined by means of a two-layer version of the Hoskins and Bretherton (1972) model. The upper layer is characterized by a higher static stability than the lower layer. Two types of unstable solutions are realized: the relatively long-wave solution has a vertical structure that extends throughout the vertical depth of the fluid and is the counterpart of the solution for a single layer system, while the shorter wave is essentially confined to the lower fluid layer. Model parameters, lower layer depth and static stability difference are chosen such that the two waves have comparable growth rates. The solution is determined by means of a Stokes expansion and terminated at second-order in the amplitude. The nonlinear interaction process between these growing baroclinic waves is then related to the wave interaction process described by the one-dimensional advection equation. Finally, an interpretation is proposed to explain disparate observations of cyclogenesis in polar air streams.

1. Introduction

Observations of the lower atmosphere by radar, instrumented towers and instrumented aircraft are becoming increasingly available of late. Moreover, objective analysis techniques have been designed to make optimal use of data in order to improve the resolution of relatively strong gradients of meteorological variables (e.g., Shapiro and Hastings, 1973). The need for accurate analyses on relatively small space and time scales is also essential for the development of improved fine-mesh numerical prediction models. Yet, Keyser et al. (1978) have shown that even the horizontal (100 km) and vertical (1.5 km) grid spacing of a typical fine-mesh model is still too coarse for adequate resolution of phenomena, such as frontal structures, that are characterized by horizontal scales of a few hundred kilometers and vertical scales $< 2$ km.

These activities in analysis and numerical prediction point up the continued need for simpler modeling efforts in order to examine significant physical processes associated with relatively smaller scales of motion, and to provide guidance in numerical model developments and interpretation of results. The insights to atmospheric and oceanic dynamics provided by the quasi-geostrophic model are well known, albeit limited. The theoretical studies of Hoskins and Bretherton (1972) and Hoskins (1976), for example, have demonstrated the value of the geostrophic momentum approximation in the depiction of relatively small-scale atmospheric phenomena that are not accessible within the limitations of quasi-geostrophic theory.

The principal intent of the present paper is to employ the baroclinic instability model proposed by Hoskins and Bretherton (hereafter the HB model), to extend the author's previous study of short-wave baroclinic instability (Blumen, 1979b). The HB model represents the counterpart of the two-dimensional Eady (1949) model: the distinction lies in the use of the geostrophic momentum approximation (Hoskins, 1975) in the model development. However, if the geostrophic coordinate transformation $X' = x' + f_0^{-1}v_y$ is made, where $x'$ is the eastward directed horizontal coordinate, $v_y$ the northward directed wind and $f_0$ the constant Coriolis parameter, then the HB model in transformed space is identical to the quasi-geostrophic Eady model in physical space. The growth rates of the unstable baroclinic disturbances are the same in each model but the coordinate transformation is such that a trough in the pressure field is accentuated while a ridge is flattened. This process is described by the one-dimensional advection equation (Blumen, 1980). Here a two-layer version of the HB model will be examined. The model's response to variations in the static stability difference between the two layers and the relative depths of these layers is significant because the characteristics of the unstable baroclinic waves that develop are relatively sensitive to these parameters. Blumen (1979b) has shown, for certain ranges of static stability differences and layer depths, that both long- and short-wave instability occur with
comparable growth rates. The short-wave instability is essentially confined to the lower layer while the long-wave instability is basically the same as that of the single-layer HB model. Particular attention will be directed to the evolution of both types of baroclinic wave disturbances. The analysis is carried out by means of a Stokes expansion in the disturbance amplitude. Since the expansion is terminated at second-order, the nonlinear terms only contribute \( \sim 20\% \) to the solution. Nonetheless, some physical insight is provided about the process of wave steepening through the interaction of two baroclinic waves.

The solutions obtained in the present study are limited by the use of static stability differences that are smaller than typical differences between the troposphere and stratosphere. Consequently, the present results should be viewed as reflecting the effect of static stability differences within the troposphere on the early stages of baroclinic wave development rather than as a model of either surface or upper level frontogenesis. Within this context, the results are perhaps more relevant, for example, to atmospheric conditions associated with the development of baroclinic disturbances in polar air streams (Harrold and Browning, 1979; Mansfield, 1974; Duncan, 1977; Reed, 1979). In fact, the present results do provide a physical interpretation of some observational characteristics of these disturbances that have not been adequately explained by previous modeling efforts. This particular application of the model is presented in the final section.

2. Model and geostrophic coordinate space solution

a. Model

The present model is a two-dimensional and two-layer version of the HB model with uniform static stability \( N_i \) (i = 1, 2), the Brunt-Väisälä frequency, specified in each layer. The fluid is bounded top and bottom by rigid boundaries at \( z' = 0 \) and \( D \). The undisturbed interface, separating each layer, is situated at \( z' = H' \). A linear basic velocity profile \( \bar{u}'(z') = U z'/D \) extends throughout the fluid depth. The geostrophic momentum approximation is employed and the disturbance motion is described in terms of geostrophic coordinates defined by

\[
\begin{align*}
X' &= x' + u_0'/f_0 \\
Y' &= y' - u_0'/f_0 \\
Z' &= z', \quad T' = t'
\end{align*}
\]

where \((x', y')\) are horizontal coordinates directed eastward and northward, \( z' \) is directed vertically upward, \( t' \) is time, \((u_0', v_0')\) are geostrophic velocity components along the \((x', y')\) axes and \( f_0 \) denotes the Coriolis parameter.

At this point it is convenient to cast all variables into nondimensional form using \( U \) to scale velocities, \( D \) to scale the vertical coordinate, \( L = N_i D f_0 \sim 10^3 \text{ km} \) to scale the horizontal coordinates and \( L/U \) to scale the time. Then the total \((T)\) streamfunction in each layer \( \Phi_i(X, Y, Z, T) \) (i = 1, 2)

\[
\begin{align*}
\Phi_1^T &= \theta_0 Z + Z^2/2 - \text{Ro} \ YZ + \text{Ro} \ \Phi_1, \\
\Phi_2^T &= \theta_0 Z + Z^2/2 - \text{Ro} \ YZ \\
&+ (\alpha^2 - 1)(Z - H)^2/2 + \text{Ro} \ \Phi_2,
\end{align*}
\]

where unprimed variables refer to nondimensional quantities in each layer and the nondimensional parameters are the Rossby number \( \text{Ro} = U f_0 L \) and static stability measure \( \alpha = N_z^2/N_1 \). The potential temperature is given by \( \partial \Phi_i^T/\partial Z \) and \( \theta_0 \) is a nondimensional constant reference value. The disturbance streamfunction \( \Phi_i \) satisfies

\[
\begin{align*}
\frac{\partial^2 \Phi_1}{\partial X^2} + \frac{\partial^2 \Phi_1}{\partial Z^2} &= 0, \\
\frac{\partial^2 \Phi_2}{\partial X^2} + \alpha^2 \frac{\partial^2 \Phi_2}{\partial Z^2} &= 0.
\end{align*}
\]

The normal velocity vanishes at \( Z = 0, 1 \). These conditions are expressed by

\[
\begin{align*}
\frac{\partial}{\partial T} \left( \frac{\partial \Phi_1}{\partial Z} \right) - \frac{\partial \Phi_1}{\partial X} &= 0, \quad Z = 0, \\
\left( \frac{\partial}{\partial T} + \frac{\partial}{\partial X} \right) \left( \frac{\partial \Phi_2}{\partial Z} \right) - \frac{\partial \Phi_2}{\partial X} &= 0, \quad Z = 1,
\end{align*}
\]

where \( \theta_i = \partial \Phi_i/\partial Z \) denotes the disturbance potential temperature. The interface represents a material surface situated at \( Z = H + \text{Ro} \ h \), where \( h(X,T) \) represents a displacement from the undisturbed position \( Z = H \). Both the pressure and velocity are continuous across the interface. Consequently, \( \Phi_i^T \) is continuous across the interface. This latter condition may also be expressed as

\[
d(\Phi_1^T - \Phi_2^T) = \left( \frac{\partial \Phi_1^T}{\partial X} - \frac{\partial \Phi_2^T}{\partial X} \right) dX
\]

\[+ \left( \frac{\partial \Phi_1^T}{\partial Z} - \frac{\partial \Phi_2^T}{\partial Z} \right) dZ = 0, \quad Z = H + \text{Ro} \ h.
\]

Since the geostrophic velocity, \( v_{gi} = \partial \Phi_i/\partial X \), is continuous across the interface, the total potential temperature, \( \theta_i^T = \partial \Phi_i/\partial Z \), is continuous across the interface. Therefore, the continuity of potential tem-
perature, \(d(\theta_2^f - \theta_2^-) = 0\), provides the slope formula for the interface,
\[
\frac{dh}{dX} = \frac{\partial \theta_1/\partial X - \partial \theta_2/\partial X}{\alpha^2 - 1 + \frac{\partial \theta_1/\partial Z - \partial \theta_2/\partial Z}{\alpha^2 - 1 + \frac{\partial \theta_1/\partial Z - \partial \theta_2/\partial Z}}}
\]  \tag{6}

b. Solution

Hoskins and Bretherton (1972) and Hoskins (1972) used a numerical relaxation procedure to determine the position of the material interface that satisfies the matching conditions (5) and Laplace's equation (3a,b) in the interior. This procedure has the advantage that there is no restriction placed on the displacement of the free surface from its equilibrium position other than it not intersect a boundary. However, present purposes are served by retaining an analytical approach that focusses on the interactions of two-wave solutions.

In the present study the difficult analytic problem of satisfying the matching conditions at \(z = H + \) \(Ro\) \(h\) is simplified by making use of a Stokes expansion (Lamb, 1945, p. 417; Whitham, 1974, p. 471) in which
\[
\Phi_i = a\Phi_i^{(1)} + a^2\Phi_i^{(2)} + \cdots \quad \text{and} \quad h = ah^{(1)} + a^2h^{(2)} + \cdots \tag{7}
\]
where the expansion parameter satisfies \(a \ll 1\). The magnitude of this parameter will be specified since there is an undetermined amplitude in the first-order eigenvalue problem.

The streamfunctions \(\Phi_i\) that satisfy (3a,b) may be expressed as
\[
\Phi_1 = a\Phi_1^{(1)} + a^2\Phi_1^{(2)} + \cdots = a[(\cosh kZ - A_1 \sinh kZ) \sin(kX - c, T) + A_2 \sinh kZ \cos(kX - c, T)]e^{\sigma \tau}
\]
\[+ a^2[(B_1 \sinh 2kZ + B_2 \cosh 2kZ) \cos(2kX - c, T)
\]
\[- (B_2 \sinh 2kZ + B_4 \cosh 2kZ) \sin(2kX - c, T)]]e^{2\sigma \tau} + a^2B_0 e^{2\sigma \tau} + \cdots \tag{8a}
\]
\[
\Phi_2 = a\Phi_2^{(1)} + a^2\Phi_2^{(2)} + \cdots = a[(A_5 \sinh k\alpha Z + A_7 \cosh k\alpha Z) \cos(kX - c, T)
\]
\[- (A_6 \sinh k\alpha Z + A_8 \cosh k\alpha Z) \sin(kX - c, T)]e^{\sigma \tau} + a^2[(B_3 \sinh 2k\alpha Z + B_7 \cosh 2k\alpha Z) \cos(2kX - c, T)
\]
\[- (B_6 \sinh 2k\alpha Z + B_8 \cosh 2k\alpha Z) \sin(2kX - c, T)]e^{2\sigma \tau} + \cdots \tag{8b}
\]
where \(k\) denotes the \(X\) wavenumber, \(c\), the phase speed and \(\sigma\), the growth rate. The \(A\)'s and \(B\)'s are real constants to be determined by boundary and matching conditions (4) and (5); the coordinates are defined by (1), in nondimensional form and \(\alpha = N_2/N_1\).

The nondimensional interface height may be expressed similarly as
\[
h = H + Ro (ah^{(1)} + a^2h^{(2)} + \cdots)
\]
\[= H + Ro \{a[M_1 \cos(kX - c, T)
\]
\[- N_1 \sin(kX - c, T)]e^{\sigma \tau}
\]
\[+ a^2[M_2 \cos(2kX - c, T)
\]
\[- N_2 \sin(2kX - c, T)]e^{2\sigma \tau} + \cdots \}, \tag{9}
\]
where the \(M\)'s and \(N\)'s are real constants to be determined and \(Ro\) is the Rossby number.

Higher order terms in (8a,b) and (9) ultimately become important due to a faster rate of exponential growth. For example, the relative growth rate of an \(a^n\) term is \(N_0\). Consequently, some higher order terms should be retained to provide a more adequate representation of the evolution of the flow. The present calculations are only carried out through second-order in \(a\).

3. The eigenvalue problem

The eigenvalues obtained by Blumen (1979b) are relatively sensitive to differences in static stability between the layers as well as to the relative layer depths. The static stability difference is expressed by \(\alpha = N_2/N_1\) and the nondimensional depth of the lower layer is \(H\), where the parameter range examined is \(0.5 \leq \alpha \leq 1.5\) and \(0 < H < 1\). A summary of the principal results appears in Fig. 6 and Table 2 of Blumen (1979b). In this parameter range, the long waves are unstable. This instability is essentially the same as that of a one-layer fluid of uniform static stability bounded by rigid lids. These waves essentially move with the mid-level basic flow and exhibit growth rates that are characteristic of a weighted average, depending on \(H\), of the static stabilities in each layer. Otherwise, the presence of two layers has little effect on the characteristics of this response.

When \(\alpha > 1\) and \(H < 0.5\) a short-wave instability also occurs. The interface acts as a flexible lid that essentially confines the instability to the lower layer. These latter waves are slightly dispersive and the region of instability is terminated by a short-wave cutoff and, for \(H \ll 1\), a long-wave cutoff. As \(H\) increases this region of instability tends to blend into the region of long-wave instability discussed above. When \(\alpha > 1\) and \(H \gg 0.5\), the short-wave instability
is not present. The unstable growth rates of the short-wave instability are characteristic of a shallow one-layer model of uniform static stability $N_1$.

The interesting feature of these instabilities, for present purposes, is the fact that the maximum growth rates of the short waves may be comparable with, and may even exceed, the maximum growth rates of the long waves. The nature of the nonlinear wave interactions that occur when these growth rates are comparable will be examined. The region of interest in parameter space is essentially confined to $\alpha = 1.3$, $0.2 \leq H \leq 0.4$. A value of $\alpha = 1.3$, for example, may correspond to lapse rates of 7 K km$^{-1}$ in the lower layer and 5 K km$^{-1}$ in the upper layer, with the typical lower layer depths ranging from 2–4 km. These values of $\alpha$ and $H$ are not typical of some observational data of polar air streams presented by Duncan (1977).

The parameter values used in the present study are provided in Table 1. The characteristic scales are: $L = N_1 D f_0 \approx 10^3$ km, $D \approx 10$ km, $f_0 \approx 10^{-4}$ s$^{-1}$, $U \approx 20$ m s$^{-1}$ and $T = L/U \approx 5 \times 10^4$ s$^{-1}$ $\approx 14$ h. The Rossby number was set equal to $Ro = 0.2$ in all computations. The wavenumbers and growth rates associated with the short-wave instability (cases IB and IIB) do not represent the exact values associated with the most unstable waves. In each case the wavenumber was chosen to be a multiple of the wavenumber associated with the most unstable long wave. This adjustment was made for computational simplicity since cyclic boundary conditions have been imposed. However, the growth rates and corresponding wavenumbers of the short waves only differed by $\sim 2\%$ from their respective optimal values. Finally, the arbitrary amplitude $a$ was specified by setting the initial first-order geostrophic velocity amplitude $ak$ equal to $10^{-4}$ (~2 m s$^{-1}$) in each case.

4. Physical space solutions

The determination of the eigenfunction coefficients for $\Phi(X,Z,T)$ and interface displacement $h(X,T)$, given by (8a,b) and (9), is presented in the Appendix. A procedure for transforming these solutions from geostrophic coordinate space to physical space has been presented by Blumen (1979a). This procedure, outlined in the Appendix, has been adopted. The geostrophic velocity becomes

$$v_\theta = 2v\{1 - Ro \frac{\partial v}{\partial x} + [(1 - Ro \frac{\partial v}{\partial x})^2 - 2 Ro^2 \frac{\partial^2 v}{\partial x^2}]^{1/2}\}^{-1},$$

Table 1. Parameter values associated with long- and short-wave instabilities. The values for a single-layer (SL) fluid are also provided for comparison.

<table>
<thead>
<tr>
<th>Cases</th>
<th>IA</th>
<th>IB</th>
<th>IIA</th>
<th>IIB</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavenumber $k$</td>
<td>1.230</td>
<td>3.690</td>
<td>1.272</td>
<td>2.544</td>
<td>1.606</td>
</tr>
<tr>
<td>Growth rate $\sigma$</td>
<td>0.240</td>
<td>0.225</td>
<td>0.248</td>
<td>0.232</td>
<td>0.310</td>
</tr>
<tr>
<td>Phase speed $c_r$</td>
<td>0.494</td>
<td>0.254</td>
<td>0.485</td>
<td>0.351</td>
<td>0.500</td>
</tr>
<tr>
<td>Static stability ratio $\alpha$</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
<td>—</td>
</tr>
<tr>
<td>Lower layer depth $H$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.4</td>
<td>0.4</td>
<td>—</td>
</tr>
<tr>
<td>Amplitude $a$</td>
<td>0.081</td>
<td>0.027</td>
<td>0.079</td>
<td>0.039</td>
<td>—</td>
</tr>
</tbody>
</table>

where $(x,z,t)$ represent the nondimensional physical space coordinates, defined by (1),

$$v = ak[(\cosh kz - A_1 \sinh kz) \cosk(x - c_r t)$$

$$- A_2 \sinh kz \sink(x - c_r t)]e^{\sigma t}$$

$$- 2a^2k[(B_1 \sinh 2kz + B_3 \cosh 2kz)$$

$$\times \sin 2k(x - c_r t) + (B_2 \sinh 2kz + B_4 \cosh 2kz)$$

$$\times \cos 2k(x - c_r t)]e^{2\sigma t},$$

$$0 \leq z \leq H + Ro ah^{(3)},$$

(11)

$$v = -ak[(A_3 \sinhk az + A_7 \coshk az) \sink(x - c_r t)$$

$$+ (A_6 \sinhk az + A_6 \coshk az) \cosk(x - c_r t)]e^{\sigma t}$$

$$- 2a^2k[(B_5 \sinh 2kaz + B_7 \cosh 2kaz) \sin 2k(x - c_r t)$$

$$+ (B_6 \sinh 2kaz + B_8 \cosh 2kaz) \cos 2k(x - c_r t)]e^{2\sigma t},$$

$$H + Ro ah^{(3)} \leq z \leq 1.$$  

(12)

The height of the interface is given by

$$h = H + Ro \{a[M_1 \cosk(x - c_r t)$$

$$- N_1 \sink(x - c_r t)]e^{\sigma t}$$

$$- ak Ro [M_1 \sink(x - c_r t)$$

$$+ N_1 \cosk(x - c_r t)]v_v e^{\sigma t}$$

$$+ a^2[M_2 \cos 2k(x - c_r t)$$

$$- N_2 \sin 2k(x - c_r t)]e^{2\sigma t}\}.$$  

(13)

where $v_v = v_v(H)$ is evaluated to first-order by means of (10) and (11). Expressions for the potential temperature and ageostrophic velocities have also been determined by the same procedure.

These solutions have been evaluated for the range of parameters presented in Table 1. However, some physical aspects of the solutions will be examined before these evaluations are displayed. Principal attention will be given to the evolution and interaction of both long and short waves that are characterized by comparable growth rates.

These waves, corresponding to either case I or case II in Table 1, are characterized by different phase speeds $c$ and $c'$ and different wavenumbers.

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The growth rates of the long- and short-wave instabilities also become comparable when $\alpha < 1$, but the short-wave instabilities are confined to shallow layers below the upper lid. This does not represent a typical atmospheric situation, so will not be considered.
To simplify the present development, it will be assumed that \( \sigma_1 \), the growth rate, is the same for both waves. The second-order terms in (8a,b) only make a small contribution over the time period that is considered. Moreover, the nonlinear features of the solution are most prominent at the lower boundary. At \( Z = 0 \), each wave may be expressed approximately as

\[
\begin{align*}
v &= \frac{1}{2}(ak) e^{\sigma_1 T} \cos k(X - cT) \\
v' &= \frac{1}{2}(a'k') e^{\sigma_1 T} \cos k'(X - c'T)
\end{align*}
\]

(14)

where the total solution is \( v = v + v' \). The initial amplitude of \( v_0 \) is chosen such that \( ak = a'k' = v_0 \). Now a physical space solution will be derived, from (14), that exposes the connection between the coordinate transformation and the physical process of wave steepening in the one-dimensional advection equation.

The expressions in (14) satisfy

\[
\begin{align*}
\frac{\partial v}{\partial T} + c \frac{\partial v}{\partial X} &= \sigma_1 v \\
\frac{\partial v'}{\partial T} + c' \frac{\partial v'}{\partial X} &= \sigma_1 v'
\end{align*}
\]

(15)

Addition of these equations yields

\[
\frac{\partial v_0}{\partial T} + c \frac{\partial v_0}{\partial X} - \sigma_1 v_0 = \left( c - c' \right) \frac{\partial v'}{\partial X},
\]

(16)

Then, making use of the transformation formulas

\[
\begin{align*}
\frac{\partial v_0}{\partial T} &= \left( 1 + Ro \frac{\partial v_0}{\partial x} \right)^{-1} \frac{\partial v_0}{\partial t} \\
\frac{\partial v_0}{\partial x} &= \left( 1 + Ro \frac{\partial v_0}{\partial x} \right)^{-1} \frac{\partial v_0}{\partial x}
\end{align*}
\]

(17)

and inserting the expression for \( v' \) into (16), the equation for \( v_0 \) becomes

\[
\frac{\partial v_0}{\partial t} + c \frac{\partial v_0}{\partial x} - \sigma_1 v_0 \left( 1 + Ro \frac{\partial v_0}{\partial x} \right)
= -\frac{1}{2} k'(c - c') v_0 e^{\sigma_1 t} \times \sin k'[\lambda - c' t + Ro v_0].
\]

(18)

It is instructive to first consider the solution for a single fluid layer. In this case the right-hand side of (18) vanishes. Further, in the coordinate frame \( \chi = x - ct \), (18) reduces to

\[
\frac{\partial v_0}{\partial \tau} - \sigma_1 v_0 \left( 1 + Ro \frac{\partial v_0}{\partial \chi} \right) = 0.
\]

(19)

If we let \( v_0 = V(\chi, \tau) \exp \sigma_1 t \), then the variable \( U = -Ro V \) satisfies the one-dimensional advection equation,

\[
\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \chi} = 0,
\]

(20)

where \( \tau = e^{\sigma_1 t} \). The correspondence between the periodic solution of the single-fluid model equations, presented by Andrews and Hoskins (1978), and the solution of (20) has been discussed by Blumen (1980).

For present purposes, a suitable approximate solution of (19) may be determined by employing a
power series expansion of \( v_p \) in terms of the small initial amplitude \( v_p = \alpha k = 10^{-1} \). The first- and second-order terms satisfy respectively

\[
\begin{align*}
\frac{1}{\sigma_i} \frac{\partial v_{p_i}}{\partial t} - v_{p_i} &= 0, \quad (21a) \\
\frac{1}{\sigma_i} \frac{\partial v_{p_k}}{\partial t} - v_{p_k} &= Ro \frac{\partial v_{p_k}}{\partial x}. \quad (21b)
\end{align*}
\]

The appropriate solution of (21a,b) is

\[
v_p = V_i e^{\sigma_i t} + Ro V_1 (\partial V_1/\partial x)e^{2\sigma_i t}, \quad (22)
\]

where \( V_i = \alpha k \cosh \chi = \alpha k \cos k(x - ct) \). Note that \( v_p(x,0,t) \) given by (10) and (11), may be expressed by (22) when the second-order terms are relatively small.

The second-order term in (22) always acts to steepen the positive slope and decrease the negative slope as shown schematically in Fig. 1. This feature of the solution may be inferred directly from (20). However, the same approach will now be employed to derive a solution of (18) that describes the evolution and interaction of two unstable waves. First, the sine term on the right-hand side of (18) will be expressed as

\[
\begin{align*}
\sin k'(x - c't) + Ro v_p \\
= \left[1 - \frac{1}{2}k'(Ro v_p)^2\right] \sin k'(x - c't) \\
+ k'Ro v_p \cos k'(x - c't), \quad (23)
\end{align*}
\]

where \( Ro \ll 1 \) and \( v_p \ll O(1) \). Then the equations that determine the first- and second-order contributions to \( v_p \) are

\[
\begin{align*}
\frac{\partial v_{p_i}}{\partial t} + c \frac{\partial v_{p_i}}{\partial x} - \sigma_i v_{p_i} &\quad = -\frac{1}{2} k' v_0 (c - c') e^{\sigma_i t} \sin k'(x - c't), \quad (24a) \\
\frac{\partial v_{p_k}}{\partial t} + c \frac{\partial v_{p_k}}{\partial x} - \sigma_i v_{p_k} &\quad = Ro \left[ \frac{\partial v_{p_k}}{\partial x} - \frac{1}{2} k' v_0 (c - c') e^{\sigma_i t} \right. \\
&\quad \left. \times \left[ \frac{\partial v_{p_k}}{\partial x} \sin k'(x - c't) + k' v_p \cos k'(x - c't) \right] \right]. \quad (24b)
\end{align*}
\]

The appropriate solution of (24a) is

\[
v_{p_i} = \frac{1}{2} v_0 [\cos k(x - ct) + \cos k'(x - c't)] e^{\sigma_i t}, \quad (25)
\]

where \( v_0 = \alpha k \). This solution simply represents the sum of two growing waves that translate at different phase speeds. It may be verified by substitution that \( v_{p_i} = Ro v_p \partial v_p / \partial x \). The solution of (24a,b) is

\[
v_p = v_{p_i} + Ro v_p \partial v_p / \partial x, \quad (26)
\]

where \( v_{p_i} \) is given by (25). The solution of (18), to second-order, has the same form as (22). However, the phase difference between the two waves in (25), at time \( t \), affects the wave steepening process. An example, which illustrates the second-order effect in (26), is shown in Fig. 2. The second-order terms neglected in (14) have been retained in the computation, shown in Fig. 2. However, the principal second-order effect is provided by (26).

In summary, the above analysis and discussion provides the essential physical aspect of the present model solutions. The process of wave steepening that is inherent in the one-dimensional advection equation concentrates gradients of geostrophic velocity as well as other variables. This process is a forerunner to the ultimate formation of an infinity.
in relative vorticity, at least in the case of a single fluid layer. It is not possible to determine if the interface separating two layers actually intersects the lower boundary before the infinity in relative vorticity occurs, due to the limitation of the expansion procedure employed.

5. Discussion

The geostrophic velocity was evaluated from (10) for each of the cases represented in Table 1. Due to the constraints placed on the solution by the exponential growth of the higher order terms, particularly in the expansion of the hyperbolic functions (A3), the height displacement is limited. The maximum value of this displacement was arbitrarily taken to be equal to one-half the undisturbed layer depth $H$. The errors in the expansions of the hyperbolic functions were then limited to 10% or less. The terms proportional to $a^2$ in (11) and (12) are about two orders of magnitude smaller than the leading terms. However, the product of two first-order terms is order $Ro^2 = 0.2$ as shown, for example, in Fig. 2.

The phases of the geostrophic velocity $v_g$ and potential temperature perturbation $\theta$ and amplitude of $v_g = 0$ are shown in Figs. 3–6 for each of the cases in Table 1. Since these solutions have been carried out to second order in amplitude $a$, the matching condition has been applied at the height of the first-order displacement shown in the figures. The long-wave response (Figs. 3 and 5) differs little from the eigenfunctions associated with the linear Eady wave for a single fluid layer, at least for the moderate ratio $a = 1.3$ used here. There is a jump in the magnitude of $\theta$ but the phases appear to be essentially continuous, using an increment $\Delta x = 0.1 (=100$ km) to evaluate the solutions. Moreover, the velocity amplitude is not symmetric with respect to mid-level but the difference is small.

The greater static stability aloft essentially confines the short-wave response (Figs. 4 and 6) to the lower layer in each case. The similarity between these solutions and those found by Eady (1949, Fig. 2), for the case of an upper layer of infinite
depth, has been discussed more fully by Blumen (1979b). The phase displacement of \( \theta \) next to the upper lid is necessary to satisfy the upper boundary condition (4b), which requires the vertical velocity to vanish at \( z = 1 \).

The ageostrophic circulations associated with both the long and short waves (not shown), have the same cellular structure that characterizes the circulation of the single-layer fluid (Blumen, 1979a; Fig. 5). The cells are thermally direct circulations that maintain the unstable growth of the baroclinic waves. The circulation axes (into the paper) are situated at mid-level, \( z = 0.5 \), for the long-wave instability and the vertical velocity maxima occur at mid-level with values reaching \( |w| = 10^{-2} \text{ m s}^{-1} \). The axes of the circulations associated with the short waves are situated slightly below the level of the undisturbed interface height \( H \). Moreover, the amplitudes of the vertical velocity increase upward reaching maxima at \( z = H \) rather than at mid-level, then decay to zero at the upper lid (see Fig. 2, Eady, 1949). In all cases of both short- and long-wave instability, descending motion is associated with the troughs in the height displacements and rising motion occurs in association with the ridges. Consequently, amplification of the interface occurs in consort with the baroclinic development of the other variables.

The combined contributions of both the long and short waves corresponding to cases I and II are shown in Figs. 7 and 8. These solutions have been derived from the geostrophic coordinate space solutions for \( v_y \) and \( h \) by means of the procedure outlined in the Appendix. To first-order, each wave simply translates along the \( x \)-axis and undergoes exponential growth. The principal second-order contribution, resulting from the interaction of the two first-order waves, affects the magnitude of the gradients as discussed in Section 4. This second-order feature contributes up to, \( \sim 20\% \) to the solu-

**Fig. 7.** Isopleths of geostrophic velocity \( v_y \), corresponding to the combined solution for cases IIA and IIB at time \( t = 9.6 \). The initial amplitude is 0.1 with the maximum situated at the origin at \( z = 0 \). The dashed lines correspond to \( v_y = 0 \), the isopleths are drawn for increments of 0.25 (5 m s\(^{-1}\)) and the sign of \( v_y \) is indicated. The interface displacement is shown by the thicker solid curve.

**Fig. 8.** As Fig. 7 except cases IIA and IIB at time \( t = 9.26 \).
tions near the boundaries, as shown in Fig. 2, with a lesser contribution in the interior of the fluid.

The solutions in the lower layers of the figures reflect the contributions from both the long and short waves whose amplitudes are comparable. The upper layer solutions are dominated by the long-wave response since the short-wave amplitudes decay rapidly above the interface.

6. Concluding remarks

Reed (1979) has recently reexamined the so-called polar low phenomenon. This name refers to small-scale (1000–1500 km) cyclones associated with cold-air outbreaks in both the North Atlantic and North Pacific Oceans. Additional discussion and analysis of the phenomenon have also been provided by Mansfield (1974) and Duncan (1977). Reed concurs with the earlier evidence presented by both Mansfield and Duncan that these small-scale cyclones are a baroclinic phenomenon. There is also agreement that the short horizontal scale is related to the relatively small static stabilities in low levels. This conjecture is also supported by Blumen’s (1979b) analysis and the results presented here. However, Reed also points out that the disturbances are not always confined to the lower levels. He cites examples “in which the perturbations were as large at upper levels as near the surface.” Moreover, Harrold and Browning (1969) note that pressure perturbations associated with the passage of polar lows extended throughout the troposphere and into the stratosphere in the case study examined. On the other hand, Mansfield and Duncan provide evidence that suggest the lows are relatively shallow baroclinic waves.

The present study, together with the earlier work of Blumen (1979b), provides an explanation that will accommodate either shallow or deep polar low disturbances. As we have stressed here, the unstable growth rates associated with both long and short waves are comparable when \( \alpha = 1.3, 0.2 \leq H \leq 0.4 \) (\( \approx 2 \) to 4 km). In this circumstance, the existence of both types of baroclinic disturbance would result in geostrophic velocity fields, or equivalently pressure perturbations, that extend throughout the fluid depth. Typical velocity fields are displayed in Figs. 7 and 8. In particular, each of the three waves evident in Fig. 7 has a wavelength of \( \approx 1700 \) km (Table 1), which is relatively close to the upper value of 1500 km revealed in the observations. However, the wavelength of maximum instability of the short waves is relatively sensitive to the undisturbed layer depth \( H \) of the model. For example, this wavelength is \( \approx 1200 \) km when \( \alpha = 1.3 \) and \( H = 0.2 \) (\( \approx 2 \) km) (Blumen, 1979b; Table 2). Moreover, the present model does provide an explanation for the rapid passage of a number of troughs in the pressure field, as shown by Reed (Fig. 3).

The calculations presented by Blumen (1979b, Table 2) show that the maximum growth rate of the long waves is a decreasing function of \( \alpha \), while the maximum growth rate of the short waves is an increasing function of \( \alpha \) for \( H \leq 0.4 \). The crossover occurs when \( \alpha \approx 1.3 \), the value of the static stability ratio used in the present study.

The maximum short-wave growth rate exceeds the long-wave growth rate by \( \approx 20\sim25\% \) when \( \alpha = 1.5 \). This latter value of \( \alpha \) would represent, for example, lapse rates in the lower and upper layers of 7 and 3 K km\(^{-1}\), respectively. Other differences between the lapse rates would also yield a value of \( \alpha = 1.5 \). It is evident that even larger values of \( \alpha \) would provide the selection mechanism responsible for the prominence of only the short-wave disturbance. The static stability data presented by Duncan (Table 3) yields values of \( \alpha \) in the range \( 1 \leq \alpha \leq 1.3 \). Nonetheless, the mechanism proposed here is not necessarily excluded because these data represent areal averages. As a consequence, the values presented are probably underestimates of the actual static stability differences between the layers.

In summary, the present model offers a relatively simple physical explanation of the polar low that is able to reconcile disparate observations of the phenomenon. In view of the limitations imposed, such as two-dimensionality and uniform potential vorticity among others, it may only serve as a prototype to offer direction to further investigation.

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APPENDIX

Determination of the Eigenfunction Coefficients and Physical Space Solution

a. First-order coefficients

The coefficients \( A_1 \) and \( A_2 \) in (8a), determined by satisfying the lower boundary condition (4a), are given by

\[
A_1 = \frac{\sigma_r}{\sigma_r^2 + \sigma_i^2}, \quad A_2 = \frac{\sigma_i}{\sigma_r^2 + \sigma_i^2},
\]

where \( \sigma_r = k c_r \). The remaining \( A_j \) (\( j = 5, \ldots, 8 \)) may be determined by satisfying the condition at the upper boundary (4b) and requiring that \( \Phi \) be continuous across the undisturbed interface \( z = H \). This procedure generates an inhomogeneous set of four equations for the coefficients \( A_j \). The analysis
is straightforward and details are omitted. The determination of these coefficients completes the solution for the streamfunction in each layer.

The first-order displacement of the interface $h^{(1)}$ is determined by requiring that the potential temperature be continuous across $Z = H$. Use of (2) and (9) yields

$$h^{(1)} = (a^2 - 1)^{-1}[\theta_1^{(1)} - \theta_2^{(1)}], \quad Z = H, \tag{A2}$$

where $\theta_1^{(1)} = \partial \Phi_1^{(1)}/\partial Z$. [Note that $\partial \theta_1^{(1)}/\partial X$ is the first-order expression of the slope formula (6).] The coefficients $M_1$ and $N_1$ in (9) are found from (A2), using (8a) and (8b) to calculate the $\theta_j$.

b. Second-order coefficients

The evaluation of the second-order coefficients proceeds in an analogous manner. First, however, the hyperbolic functions are expanded to first-order as

$$\sinh[kH + Ro ah^{(1)}] 
\approx \sinh[kH] + Ro ak^{(1)} \cosh[kH], \tag{A3}$$

$$\cosh[kH + Ro ah^{(1)}] 
\approx \cosh[kH] + Ro ak^{(1)} \sinh[kH].$$

A similar expansion of the hyperbolic functions in (8b) is also carried out. Then the expressions $\Phi_1$, $\Phi_2$ and $h$ are introduced into the boundary conditions and matching conditions across the interface, $h = H + Ro ah^{(1)}$. This procedure results in an inhomogeneous set of eight equations for the coefficients $B_j (j = 1, \ldots, 8)$. The coefficient $B_0$ is given by

$$B_0 = -\frac{1}{4} Ro (a^2 - 1)(M_1^2 + N_1^2). \tag{A4}$$

However, the second-order term $a^2 B_0 \exp(2\sigma_1 T)$ does not enter into the present study, since only spatial derivatives of $\Phi_1$ are considered. The second-order displacement of the interface was then determined from (6). However, the term $Ro a^2 M_0 \times \exp(2\sigma_1 t)$ in (9) was neglected since this term is an order of magnitude smaller than the terms retained.

c. Physical space solution

A principal difference between the present solutions and those for a single fluid layer is the presence of higher order terms. They arise from the nonlinear matching condition $\Phi_1^+ = \Phi_2^-$ at $Z = H + Ro h$, where the $\Phi_j^\pm$ are given by (2). Nonetheless, the procedure used by Blumen (1979a) may be used to derive the physical space solutions. As shown by Blumen, it is more convenient to derive the physical space solution for the geostrophic velocity $v_g$ and then to use the result to determine the other variables.

In the present case, $v_g$ may be represented in geostrophic coordinate space by differentiation of the $\Phi_i$ in (8a,b). The term $\sin k(X - c_t T)$, for example, may be expressed as

$$\sin k(X - c_t T) = \sin k(x - c_t t + Ro v_g)$$

$$= \sin k(x - c_t t) \cos k Ro v_g$$

$$+ \cos k(x - c_t t) \sin k Ro v_g$$

$$= \left[1 - \frac{1}{2}(Ro k v_g)^2\right] \sin k(x - c_t t)$$

$$+ \left(k Ro v_g\right) \cos k(x - c_t t), \tag{A6}$$

where $Ro = 0.2$. The same procedure is applied to all the trigonometric terms. This straightforward procedure yields a quadratic equation for $v_g$. The solution is given by (10).

A similar path is followed to determine solutions for other variables. In these cases the known geostrophic velocity $v_g$ is used in the evaluation. In particular, the interface height is given by (13). Finally, the same procedure is followed when two waves of comparable growth rate are transformed from geostrophic to physical coordinate space.

REFERENCES


