Topographically Forced Wave Instability at Finite Amplitude

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ABSTRACT

The method of multiple time scales was used to study the weakly nonlinear effects on the instability of a basic state consisting of a topographically forced wave in an inviscid, barotropic beta-plane model. The results obtained differ substantially from those obtained when the basic state is a free Rossby wave. Here the basic-state wave is fixed in phase with respect to the mountain, while the amplitude of the topographic wave and perturbation evolve. The nonlinear feedback between the topographic wave and perturbation gives rise to an oscillation for a topographically subresonant zonal flow and an explosive nonlinear instability for a topographically superresonant zonal flow. In the subresonant case, the effect of the perturbation on the forced wave is a dissipative one, when averaged over the course of the nonlinear oscillation. The standing topographic wave interacts with the traveling instability on the topographic wave through the convergence of Reynolds' stresses which is suggestive of the way in which standing and traveling eddies interact in the atmosphere.

1. Introduction

The interaction between standing and transient eddies in the atmosphere is of fundamental importance to meteorologists. In an observational study of this interaction, Holopainen (1978) has shown the vertically averaged horizontal fluxes of relative vorticity by small- and large-scale transient eddies to be important in maintaining the vertically averaged, annual mean, standing eddies' vorticity balance. In this paper, as a first step in understanding the interaction between the standing and transient eddies in the atmosphere, a barotropic model will be studied in which the nonlinear evolutions of a stationary topographically forced wave and the travelling instability which develops on it are considered. The topographic wave is forced by the diversion of a uniform zonal current by sinusoidal topography. The tacit assumptions here are that the barotropic topographically forced wave will represent the vertically averaged standing wave in the atmosphere and the traveling, linear, barotropic disturbance which develops on the topographic wave will represent the large-scale vertically averaged transient atmospheric disturbances. Obviously, the cyclone-scale transient eddies cannot be properly represented in a barotropic model. Although these representations of atmospheric topographic stationary and large-scale transient eddies are crude, the interactions between them can be studied in a simple *internally consistent* manner in order to provide some insight into the interaction of their atmospheric counterparts.

To study the problem of the interaction between a forced standing eddy and traveling disturbances which develop on it as instabilities, the finite-amplitude characteristics of linearly unstable perturbations to a topographically forced wave will be analyzed. The linear stability problem has been discussed by Charney and Flierl (1981). In order to study the nonlinear effects on the evolution of the perturbation and the topographic wave, the nonlinearity is required to be weak. Then, as in Deininger (1982, hereafter referred to as DE), the method of multiple time scales can be used to close the problem for a slowly growing perturbation. In performing this analysis, it will be found that the nonlinear evolution of this topographically forced wave stability problem is markedly different from that of the free Rossby wave stability problem described in DE. This contrast will make the comparison of the nonlinear aspects of the topographically forced wave stability problem and the free Rossby wave stability problem particularly interesting.

Clearly, the atmosphere is baroclinic; however, based on some preliminary calculations, the behavior to be described in this paper using the barotropic model to be discussed in the next section, does have a counterpart in baroclinic models. Therefore, the results of this paper should be regarded as providing a foundation for the understanding of stationary and transient wave interactions in more realistic baroclinic models.

2. The model

The nondimensional vorticity equation governing the barotropic motion of a quasigeostrophic,

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inviscid, homogeneous fluid on an infinite beta-plane, bounded in the vertical direction by an upper flat horizontal plate and by a lower corrugated plate to act as topography, is

$$\frac{\partial}{\partial t}\nabla^2\Psi + \beta\frac{\partial}{\partial x}\Psi + J(\Psi, \nabla^2\Psi + \eta) = 0, \quad (2.1)$$

where $\eta = h_B/\epsilon D$ is O(1) and represents the topographic variation. h_B is the height of the topography, $\epsilon = U/f_0L$ is the Rossby number, and D is the mean depth of the fluid. All other variables have their conventional meanings and are the same as those in DE. The topography is assumed to be sinusoidal, i.e.,

$$\eta = \eta_0 \sin\theta, \quad (2.2a)$$

where

$$\theta = kx + ly. \quad (2.2b)$$

3. Formulation of the nonlinear problem

An exact solution of (2.1) for sinusoidal topography (2.2) whose stability Charney and Flierl (1981) studied, is the topographically forced wave solution

$$\Psi = -Uy + F\sin\theta, \quad (3.1)$$

where

$$F = \frac{\eta_0 U}{K^2(U-c)}, \quad (3.2)$$

and

$$c = \frac{\beta}{K^2}.$$

This solution exhibits a singularity when

$$U = c, \quad (3.3)$$

in which case the Doppler-shifted wave field becomes stationary relative to the topography. In the subsequent analysis it will be assumed $U - c = O(1)$, so the forced wave is topographically nonresonant.

If ψ is the disturbance streamfunction which is to be superimposed on topographic wave solution, i.e.,

$$\Psi = -Uy + F\sin\theta + \psi, \quad (3.4)$$

substitution of (3.4) into (2.1) using (3.2) yields the problem for the perturbation streamfunction, which can be written

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi + \frac{\eta_0}{K^2(U-c)} \times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi = -J(\psi, \nabla^2\psi). \quad (3.5)$$

If ψ is infinitesimally small, linearization of (3.5) is justified and yields

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi + \frac{\eta_0}{K^2(U-c)} \times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi = 0. \quad (3.6)$$

Eq. (3.6) is a more general form of the linear stability problem solved by Charney and Flierl (1981). The solution to (3.6) is of the form

$$\psi = e^{i\lambda t}\sum_n P_n e^{i\theta_n} + *, \quad (3.7)$$

where an asterisk denotes the complex conjugate and

$$\theta_n = k_n x + l_n y,$$

with

$$(k_n, l_n) = (k_0, l_0) + n(k, l).$$

Substitution of (3.7) into (3.6) results in a recursion relation for the P_n quite similar in form to Gill's (1974) recursion relation for the stability of a free Rossby wave. It is

$$\eta_0 d a_{n+1} Q_{n+1} + (\lambda + \delta_n)Q_n + \eta_0 d a_{n-1} Q_{n-1} = 0, \quad (3.8)$$

where

$$d = \frac{b}{K^2(U-c)}, \quad (3.9a)$$

$$b = \frac{1}{2}(kl_0 - lk_0), \quad (3.9b)$$

$$a_n = U - c_n, \quad (3.9c)$$

$$c_n = \beta/K_n^2, \quad (3.9d)$$

$$K_n^2 = k_n^2 + l_n^2, \quad (3.9e)$$

$$\delta_n = k_n a_n, \quad (3.9f)$$

$$Q_n = K_n^2 P_n. \quad (3.9g)$$

Solution of (3.8) will require the truncation of the infinite set of homogeneous equations for P_n generated by (3.8). As in the free Rossby wave analysis of DE, the qualitative nature of the weakly nonlinear dynamics can be reasonably represented by the two mode approximation, i.e.,

$$\psi = e^{i\lambda t}(P_0 e^{i\theta_0} + P_1 e^{i\theta_1}) + *. \quad (3.10)$$

The analysis to follow can be carried out for *any* finite truncation and yield the same qualitative results, so the truncation of (3.10) is not as restrictive as it first may seem. The results of a three mode truncation of (3.7) are given in Appendix A.

After truncating the sum (3.7) to (3.10) the infinite set of equations reduces to

$$(\lambda + \delta_1)Q_1 + \eta_0 d a_0 Q_0 = 0,$$
$$\eta_0 d a_1 Q_1 + (\lambda + \delta_0)Q_0 = 0. \quad (3.11)$$

For the nontrivial solution of (3.11) it is required that

$$\lambda = -\left(\frac{\delta_1 + \delta_0}{2}\right) \pm \frac{1}{2}[(\delta_1 - \delta_0)^2 + 4\eta_0^2 d^2 a_1 a_0]^{1/2}. \quad (3.12)$$

Then for instability to occur,

$$a_1 a_0 < 0 \quad (3.13a)$$

is required and the mountain height must exceed a critical value which is

$$\eta_c^2 = -\frac{(\delta_1 - \delta_0)^2}{4d^2 a_1 a_0}. \quad (3.13b)$$

Since the perturbation was chosen such that $K_0^2 < K^2 < K_1^2$, it follows that $\beta/K_1^2 < \beta/K^2 < \beta/K_0^2$. Then, from (3.13a) along with (3.9c,d) it can be seen that both subresonant and superresonant flow is possible within the parameter range corresponding to linear instability.

If the topography is just slightly higher than η_c by a small amount Δ ($\Delta \ll \eta_c$) such that

$$\eta_0 = \eta_c + \Delta, \quad (3.14)$$

the growth rate of the perturbation is proportional to $|\Delta|^{1/2}$, i.e.,

$$\lambda_i^2 = -2d^2\eta_c^2 a_1 a_0 \Delta. \quad (3.15)$$

As in the free Rossby wave problem, (3.15) suggests the long time scale

$$T = |\Delta|^{1/2} t$$

as the one over which the nonlinear evolution will occur when the perturbation is allowed to be finite but small. Now the time operator in (3.5) is replaced by

$$\frac{\partial}{\partial t} + |\Delta|^{1/2}\frac{\partial}{\partial T}$$

so (3.5) can be rewritten as

$$\left(\frac{\partial}{\partial t} + |\Delta|^{1/2}\frac{\partial}{\partial T} + U\frac{\partial}{\partial x}\right)\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi + \frac{\eta_c + \Delta}{K^2(U-c)}\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi = -J(\psi, \nabla^2\psi), \quad (3.16)$$

where (3.14) was also used. It now remains to choose the appropriate expansion for ψ so the weak effects of nonlinearity and instability can be balanced. As in DE, the appropriate choice is

$$\psi = |\Delta|^{1/2}\psi^{(1)} + |\Delta|\psi^{(2)} + |\Delta|^{3/2}\psi^{(3)} + \cdots. \quad (3.17)$$

Substitution of (3.17) into (3.16) results in a series of problems for the successive powers of $|\Delta|^{1/2}$. The $O(|\Delta|^{1/2})$ problem leads to the specification of the neutral perturbation already calculated. The neutral solution is

$$\psi^{(1)} = \sum_{n=0}^{1} L_n P_0 e^{i\hat{\theta}_n} + *, \quad (3.18)$$

where

$$\hat{\theta}_n = \theta_n - \left(\frac{\delta_1 + \delta_0}{2}\right)t, \quad (3.19a)$$

$$L_1 = \frac{(\delta_1 - \delta_0)K_0^2}{2d\eta_c a_1 K_1^2}, \quad (3.19b)$$

$$L_0 = 1. \quad (3.19c)$$

So far the analysis is essentially the same as in DE except the time stretching necessary in the free Rossby wave analysis has not been done here. Doing so would later prove unnecessary. This is due to the fact that the basic-state wave is forced, causing the basic state wave structure, produced by the self-interaction of the perturbation at the next order, to be nonresonant. As a result, the self-interaction of the perturbation produces a forced solution instead of producing a phase change of the basic state wave which in DE required the use of a stretched time coordinate. This foreshadows the major difference between the free Rossby and topographically forced wave analyses. This difference and the effects of nonlinearity will begin to be apparent at the next order.

4. The nonlinear analysis

The nonlinearity at $O(|\Delta|)$ is due to the self-interaction of the perturbation field. Using (3.18) to calculate the inhomogeneous terms at $O(|\Delta|)$, and retaining only those terms consistent with the original truncation, results in the equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi^{(2)} + \beta\frac{\partial}{\partial x}\psi^{(2)} + \frac{\eta_c}{K^2(U-c)} \times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(2)}$$
$$= K_0^2\frac{\partial P_0}{\partial T}e^{i\hat{\theta}_0} + K_1^2 L_1\frac{\partial P_0}{\partial T}e^{i\hat{\theta}_1} + 2b(K_1^2 - K_0^2)L_1 P_0^2 e^{i\bar{\theta}_1} - 2b(K_1^2 - K_0^2)L_1|P_0|^2 e^{i\theta} + *, \quad (4.1)$$

where

$$\theta_1 = \bar{k}_1 x + \bar{l}_1 y - (\delta_1 - \delta_0)t,$$
$$(\bar{k}_n, \bar{l}_n) = (2k_0 + nk, 2l_0 + nl).$$

The first three inhomogeneous terms in (4.1) are the same as in the free Rossby wave analysis and therefore give rise to the particular solutions

$$\psi_P^{(2)} = \sum_{n=0}^{1} P_n^{(2)} e^{i\hat{\theta}_n} + *, \quad (4.2)$$

where

$$P_1^{(2)} = \frac{2iL_1}{\delta_1 - \delta_0} \frac{\partial P_0}{\partial T},$$

$$P_0^{(2)} = 0$$

and

$$\psi_f^{(2)} = \sum_{n=0}^{1} F_n^{(2)} e^{i\hat{\theta}_n} + *, \quad (4.3)$$

where

$$F_n^{(2)} = if_n(P_0)^2.$$

The f_n are given in Appendix B. The remaining inhomogeneous term is analogous to the one which in the free Rossby wave analysis was resonant and therefore produced a phase correction of the Rossby wave. However, here it is nonresonant for $U \neq c$ and produces a forced solution in the basic state structure which represents an amplitude correction to the topographically forced basic state wave. This solution is

$$\psi_B^{(2)} = F_B e^{i\theta} + *, \quad (4.4a)$$

where

$$F_B = if_B |P_0|^2, \quad (4.4b)$$

and

$$f_B = -\frac{(K_1^2 - K_0^2)(\delta_1 - \delta_0)K_0^2}{2k\eta_c a_1 K_1^2}. \quad (4.4c)$$

This difference is due to the basic state being forced as opposed to being free. The solution of $O(|\Delta|)$ in the basic state structure using (3.1), (3.2) and (4.4), is

$$\Psi_B = -Uy + \left[\frac{(\eta_c + \Delta)}{K^2(U - c)} U + |\Delta| \frac{(K_1^2 - K_0^2)(\delta_1 - \delta_0)K_0^2}{k\eta_c a_1 K_1^2} |P_0|^2\right] \sin\theta, \quad (4.5)$$

and has an interesting property. Note the $O(|\Delta|)$ correction to the basic state wave is always a positive contribution to the effective basic wave amplitude which is defined as the coefficient of $\sin\theta$ in (4.5). This is so since the conditions $a_1 > 0$ and $a_0 < 0$ must be true for instability to occur. For subresonant flow ($U < c$) the absolute value of the effective basic state wave amplitude decreases as the perturbation grows, but for superresonant flow ($U > c$) the effective basic-state wave amplitude increases. Thus, for superresonant flow there seems to be a further destabilization which should be apparent when the behavior of P_0 is determined at the next order.

The $O(|\Delta|^{3/2})$ problem can be written

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi^{(3)} + \beta\frac{\partial}{\partial x}\psi^{(3)} + \frac{\eta_c}{K^2(U - c)} \times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(3)}$$
$$= -\frac{\partial}{\partial T}\nabla^2\psi^{(2)} - \frac{\Delta}{|\Delta|}\frac{1}{K^2(U - c)} \times \cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(1)}$$
$$- J(\psi^{(1)}, \nabla^2\psi^{(2)}) - J(\psi^{(2)}, \nabla^2\psi^{(1)}). \quad (4.6)$$

Using (3.18), (4.2), (4.3) and (4.4), the inhomogeneities of (4.6) can be evaluated. At this order only terms proportional to $e^{i\hat{\theta}_0}$ and $e^{i\hat{\theta}_1}$ need be retained, as the other terms only produce forced solutions which are not needed to determine the behavior of P_0 on the long time scale T. After evaluating the relevant inhomogeneous terms, (4.6) becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi^{(3)} + \beta\frac{\partial}{\partial x}\psi^{(3)} + \frac{\eta_c}{K^2(U - c)}\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(U\nabla^2 + \beta)\psi^{(3)}$$
$$= \left[K_1^2\frac{\partial P_1^{(2)}}{\partial T} + \frac{\Delta}{|\Delta|} ida_0 K_0^2 P_0 - 2ib(K_0^2 - K^2)f_B P_0|P_0|^2 + 2ib(K_0^2 - \overline{K_1^2})f_1 P_0|P_0|^2\right]e^{i\hat{\theta}_1}$$
$$+ \left[\frac{\Delta}{|\Delta|} ida_1 K_1^2 L_1 P_0 - 2ib(K_1^2 - K^2)L_1 f_B P_0|P_0|^2 - 2ib(K_1^2 - \overline{K_1^2})L_1 f_1 P_0|P_0|^2\right]e^{i\hat{\theta}_0} + *. \quad (4.7)$$

Removal of the secularities from (4.7) results in the evolution equation for P_0 which is

$$\frac{d^2P_0}{dT^2} - \frac{\Delta}{|\Delta|}\sigma_i^2 P_0 + NP_0|P_0|^2 = 0, \quad (4.8)$$

where the growth rate

$$\sigma_i^2 = \frac{\lambda_i^2}{\Delta} = -2d^2\eta_c a_1 a_0,$$

agrees with the linear result (3.15). The nonlinear coefficient

$$N = N_B + N_H,$$

is comprised of a part due to the feedback with the basic state, i.e.,

$$N_B = \frac{b^2K_0^2}{kK^2K_1^2}\frac{(K_1^2 - K_0^2)(\delta_1 - \delta_0)}{a_1(U - c)} \times \left[\left(\frac{K_0^2 - K^2}{K_0^2}\right)a_1 + \left(\frac{K_1^2 - K^2}{K_1^2}\right)a_0\right], \quad (4.9)$$

and a part due to the feedback with higher harmonics, i.e.,

$$N_H = \frac{bf_1(\delta_1 - \delta_0)}{L_1} \frac{(K_0^2 - K_1^2)(\overline{K_1^2} - K^2)}{(K_1^2 - K^2)} . \quad (4.10)$$

The latter feedback is essentially the same as discussed in the free Rossby wave analysis. However, the former feedback mechanism is unique to this paper. Upon examination of N_B it can be seen for subresonant flow ($U < c$) that

$$N_B > 0,$$

which says the nonlinear feedback is stabilizing and therefore gives rise to an oscillatory exchange of energy between the effective basic state wave amplitude and perturbation. When this oscillation is averaged over one cycle, the effect of the perturbation on the topographically forced wave is a dissipative one, although during part of the cycle the perturbation reinforces the topographic wave. This dissipative effect on the topographic wave is in qualitative agreement with the observational and numerical model results of Youngblut and Sasamori (1980). But for superresonant flow $N_B < 0$, so the nonlinear feedback is destabilizing and results in an explosive instability in which P_0 blows up in a finite time. This was to be expected from the solution at $O(|\Delta|)$, which showed the effective amplitude of the basic state to be increased as the perturbation grew, causing it to be more unstable. This explosive instability is somewhat reminiscent of the form-drag instability discussed by Charney and Devore (1979) in that it occurs for superresonant flow. However, the explosive instability here is actually a distinct phenomenon. It arises by the interaction of the topographically forced wave with its perturbation via the convergence of Reynolds' stresses. Thus, this instability is independent of form drag (note N_B is independent of η_c), whereas the form-drag instability is due to the interaction of the topographically forced wave with a zonal flow via the form drag.

As previously mentioned, the flow can be barotropically resonant only for unrealistically large values of zonal velocity for planetary-scale waves in this model, making the explosive instability unlikely. However, in a baroclinic system, resonances corresponding to free baroclinic modes may be more easily realized from the point of view of the zonal velocity required, which may allow the explosive instability to be more easily excited provided the topographically forced wave and linear instability are vertically trapped. Also, in a baroclinic model the baroclinic nature of the topographically forced basic state (e.g., Holopainen, 1970) as well as short wavelength transient eddies and some large-scale transient eddies (e.g., Bottger and Fraedrich, 1980) can be accounted for. Therefore, extending this analyses to a baroclinic model is worth further consideration. Such problems are being pursued. The neglect of baroclinic and frictional effects in this study precludes any comparison of the present results with the atmosphere.

A few remarks concerning truncation which apply to this analysis as well as that given in DE are in order. In the course of this analysis it has been convenient to truncate the perturbation (3.7) to (3.18); however, it is possible to carry out this analysis for any finite truncation. For less restrictive truncations the growth rate, σ_i^2, and the nonlinear coefficient N of (4.8) are, of course, different from those given here and depend on parameters such as the phase speed and the critical amplitude, which are determined in the course of the linear analysis. It is known that successive estimates of the eigenvalue do converge to a value (although somewhat slowly in some parameter regimes) as the truncation is made less severe (Tung, 1976). Thus, we expect σ_i^2 will converge to a finite value. It is not clear, however, that the nonlinear coefficient N will converge to a finite value. What would a diverging nonlinear coefficient imply? It would suggest that the nonlinearity is felt immediately and would then imply that the weakly nonlinear analysis as well as the linear stability analysis is meaningless in the region of parameter space in which N diverged. This is because the time scale of the nonlinearity is much faster than that of the linear instability when N diverges, which implies that the separation of the linear instability and nonlinear time scales used to close the problem is no longer valid. In this case we might expect turbulent behavior to be present immediately as opposed to the case of finite N where turbulent behavior may develop more slowly as the wavenumber spectrum fills out and stronger nonlinearity develops. Presumably, this occurs over time scales longer than the time $|\Delta|^{-1/2}$ after which the theory in this study becomes no longer valid. Hence, the question of truncation is a serious one and should be borne in mind in interpreting any study which employs truncation, such as the present work, or numerical simulations where truncation is implied.

5. Concluding remarks

In this paper the nonlinear evolution equations were obtained which govern the interaction between a topographically forced wave and its weakly unstable perturbation. This analysis, valid away from topographic resonance, showed that the topographically forced wave remained in a fixed position relative to the mountain while its amplitude was altered as the perturbation evolved. An explosive instability occurs for superresonant flow for which the analysis is not valid after a short time and a smooth oscillatory exchange of energy takes place betwen the perturbation and topographic wave for subresonant flow. When this oscillation is averaged over one cycle, the effect of the perturbation on the topographically forced wave is a dissipative one. This behavior is markedly different from that of the free-wave analysis by Deininger (1982) in which the phase of the basic-state Rossby wave was altered as the oscillatory exchange took place between the Rossby wave amplitude and perturbation.

The mechanism behind the nonlinear feedback in this study is the convergence of vorticity flux, i.e., essentially the tilted trough mechanism. This mechanism has been shown by Holopainen (1978) to be important in the maintenance of the standing eddies' vorticity balance in the atmosphere. The preceeding analysis, then, is suggestive of a type of interaction which may occur between the topographically forced standing eddies and transient eddies in the atmosphere.

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APPENDIX A

Results for the Three-Term Truncation

When (3.7) is truncated to

$$\psi^{(1)} = L_1 P_0 e^{i\hat{\theta}_1} + P_0 e^{i\hat{\theta}_0} + L_{-1} P_0 e^{i\hat{\theta}_{-1}}, \quad \text{(A1)}$$

where

$$\hat{\theta}_n = \theta_n + \hat{\lambda}_r t, \quad \text{(A2a)}$$

$$L_n = -\frac{\eta_c d a_0 K_0^2}{(\hat{\lambda}_r + \delta_n) K_n^2}, \quad \text{for} \quad n = \pm 1 \quad \text{(A2b)}$$

and

$$\hat{\lambda}_r = \lambda_r + \delta_0. \quad \text{(A2c)}$$

the coefficients of (4.8) become

$$\sigma_i^2 = \frac{2}{H\eta_c}\left(\frac{a_1}{\lambda_r + \delta_1} + \frac{a_{-1}}{\lambda_r + \delta_{-1}}\right), \quad \text{(A3a)}$$

$$\begin{aligned} N = &-\frac{8b^2(K_0^2 - K^2)}{HkK^2(U-c)}\left[\frac{K_0^2 - K_1^2}{K_1^2(\hat{\lambda}_r + \delta_1)} + \frac{K_{-1}^2 - K_0^2}{K_{-1}^2(\hat{\lambda}_r + \delta_{-1})}\right]\left[\frac{a_1}{\lambda_r + \delta_1} + \frac{a_{-1}}{\lambda_r + \delta_{-1}}\right] \\ &+ \frac{8\eta_c^2 b^2 d^2 (K_0^2 - K^2)^2 (K_{-1}^2 - K_1^2)[(K_1^2 - K^2)(4K^2 - K_{-1}^2) + (K_{-1}^2 - K^2)(4K^2 - K_1^2)]}{HK_1^4 K_{-1}^4 (\hat{\lambda}_r + \delta_1)^2 (\hat{\lambda}_r + \delta_{-1})^2 kK^2 (4U - c)} \\ &+ \frac{1}{H}\left\{\frac{2b}{d\eta_c}\frac{(\overline{K_1^2} - K^2)(K_1^2 - K_0^2) f_1}{K_1^2 (K_0^2 - K^2)(\hat{\lambda}_r + \delta_1)} + \frac{4b(K_1^2 - K_{-1}^2)(K^2 - \overline{K_0^2}) f_0}{K_1^2 K_{-1}^2 (\hat{\lambda}_r + \delta_1)(\hat{\lambda}_r + \delta_{-1})}\right. \\ &\left. - \frac{2b}{d\eta_c}\frac{(\overline{K_{-1}^2} - K^2)(K_{-1}^2 - K_0^2) f_{-1}}{K_{-1}^2 (K_0^2 - K^2)(\hat{\lambda}_r + \delta_{-1})}\right\} \end{aligned} \quad \text{(A3b)}$$

where

$$H = \frac{a_1}{(\lambda_r + \delta_1)^3} + \frac{a_{-1}}{(\lambda_r + \delta_{-1})^3}, \quad \text{(A4a)}$$

$$f_1 = \frac{2}{\overline{K_1^2}(\bar{\delta}_1 + 2\hat{\lambda}_r)}[-\eta_c d(\overline{K_0^2} - K^2) f_0 + b(K_1^2 - K_0^2)], \quad \text{(A4b)}$$

$$f_{-1} = \frac{-2}{\overline{K_{-1}^2}(\delta_{-1} + 2\hat{\lambda}_r)}[\eta_c d(\overline{K_0^2} - K^2) f_0 + b(K_{-1}^2 - K_0^2)], \quad \text{(A4c)}$$

$$\begin{aligned} f_0 = \frac{1}{\overline{K_0^2} D}[&4b(K_1^2 - K_{-1}^2) L_1 L_{-1} (\bar{\delta}_1 + 2\hat{\lambda}_r)(\bar{\delta}_{-1} + 2\hat{\lambda}_r) - 4bd\eta_c \bar{a}_1 (K_1^2 - K_0^2)(\bar{\delta}_{-1} + 2\hat{\lambda}_r) L_1 \\ &- 4bd\eta_c \bar{a}_{-1}(K_0^2 - \overline{K_{-1}^2})(\bar{\delta}_1 + 2\hat{\lambda}_r) L_{-1}] \end{aligned} \quad \text{(A4d)}$$

$$D = -4d^2\eta_c^2 \bar{a}_0 [\bar{a}_1(\bar{\delta}_{-1} + 2\hat{\lambda}_r) + \bar{a}_{-1}(\bar{\delta}_1 + 2\hat{\lambda}_r)] + (\bar{\delta}_1 + 2\hat{\lambda}_r)(\bar{\delta}_0 + 2\hat{\lambda}_r)(\bar{\delta}_{-1} + 2\hat{\lambda}_r), \quad \text{(A4e)}$$

and η_c and $\hat{\lambda}_r$ can be determined from the two equations

$$\eta_c^2 = \frac{3\hat{\lambda}_r^2 + 2(\delta_1 + \delta_{-1})\hat{\lambda}_r + \delta_1\delta_{-1}}{d^2 a_0 (a_1 + a_{-1})}, \quad \text{(A5a)}$$

$$\hat{\lambda}_r^3 + (\delta_1 + \delta_{-1})\hat{\lambda}_r^2 + [\delta_1\delta_{-1} - \eta_c^2 d^2 a_0 (a_1 + a_{-1})]\hat{\lambda}_r - \eta_c^2 d^2 a_0 (a_1\delta_{-1} + a_{-1}\delta_1) = 0. \quad \text{(A5b)}$$

The singularity in N when $U = \frac{1}{4}c$ corresponds to a resonance with the second harmonic of the basic-state wave and the analysis is not valid there. For the special case in which $l = k_0 = 0$, one must go to the $O(|\Delta|^2)$ to obtain closure as done in Deininger and Loesch (1982).

APPENDIX B

Forced Solution Coefficients

The f_n are

$$f_1 = \frac{2b(2\lambda_r + \bar{\delta}_0)(K_1^2 - K_0^2)L_1}{\overline{K_1^2}D}, \quad \text{(B1a)}$$

$$f_0 = \frac{-4\eta_c b d \bar{a}_1(K_1^2 - K_0^2)L_1}{\overline{K_0^2}D}, \quad \text{(B1b)}$$

where

$$\lambda_r = -\left(\frac{\bar{\delta}_1 + \bar{\delta}_0}{2}\right), \quad \text{(B2a)}$$

and

$$D = (2\lambda_r + \bar{\delta}_0)(2\lambda_r + \bar{\delta}_1) - 4\eta_c^2 d^2 \bar{a}_1 \bar{a}_0. \quad \text{(B2b)}$$

The barred quantities are

$$\left.\begin{aligned} \overline{K_n^2} &= \overline{k_n^2} + \overline{l_n^2} \\ \bar{c}_n &= \beta/\overline{K_n^2} \\ \bar{a}_n &= U - \bar{c}_n \\ \bar{\delta}_n &= \bar{k}_n(U - \bar{c}_n) \end{aligned}\right\} . \quad \text{(B3)}$$

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