Predictability in a Solvable Stochastic Climate Model

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ABSTRACT

We present a simple Budyko-Sellers type climate model which is forced by a heating term whose time dependence is white noise and whose space-separated autocorrelation is independent of position and orientation on the sphere (statistical homogeneity). Such models with diffusive transport are analytically soluble by expansion into spherical harmonics. The modes are dynamically and statistically independent. Each satisfies a simple Langevin equation having a scale-dependent characteristic time. Climate anomalies in these models have an interval of predictability which can be explicitly computed. The predictability interval is independent of the wavenumber spectrum of the forcing in this class of models. We present the predictability results for all scales and discuss the implications for more realistic models.

1. Introduction

One of the most important problems of modern climatology is the prediction of future climatic states as they evolve from given initial conditions with the external conditions held fixed or allowed to vary seasonally. Because of instabilities in the atmospheric equations of motion, meaningful numerical predictions of the precise state of the atmosphere are limited theoretically to about 10 days (Lorenz, 1969; Leith, 1971). Large scales of some meteorological variables seem to have characteristic times longer than this and may therefore constitute a basis for predicting time-averaged and/or space-averaged quantities for extended periods. So far no limits have been established for climate prediction.

By climate we mean the multidimensional probability distribution of states of the atmosphere-ocean system. The climate is assumed to have a probability distribution which is stationary. The usual device borrowed from statistical mechanics (Leith, 1975) of a fictitious *ensemble* of independent states (planets) can then be employed for the purpose of computing the moments of the probability distribution. If we look at a certain subset of the full ensemble (the *subensemble*) all of whose members pass through a small neighborhood of some given initial condition (*anomaly*), we can imagine that this subset will eventually fill out the ensemble probability distribution by ergodicity. But in a small initial interval the probability distribution of the subensemble will be strongly influenced by the initial conditions. For the interval during which it differs significantly from the stationary distribution we have predictability of the first kind as defined by Lorenz (1975). Later in this section the concept will be illustrated by example.

In this paper we present a highly idealized atmospheric model for the purpose of examining the limits of predictability for the large scales of the temperature field. The model is of the semi-empirical type introduced by Budyko (1968, 1969) and Sellers (1969), but forced by a white noise heating term. Such systems can be adjusted to yield fluctuations statistically similar to those observed (Robock, 1978). Other energy balance models with stochastic forcing have been studied by Lemke (1977), Fraedrich (1978), and North *et al.* (1981).

The advantage of our model is its simplicity and the fact that analytical methods can be used throughout so that each assumption and simplification can be examined explicitly. On the other hand, the model lacks many features expected to be important in the real geophysical system. Our conclusions must be considered tentative at best until similar work is done with more realistic models. Still, we suspect that our quantitative results are of practical importance.

Consider a meteorological field such as the sea level air temperature. At a given time the field may be expanded into spherical harmonics,

$$T(\hat{\mathbf{r}}, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_{lm}(t) Y_l^m(\hat{\mathbf{r}}), \tag{1}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the earth's

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center to the point in question. The largest spatial scales of the field are described by the first few spherical harmonics having the lowest values of l. The amplitudes $T_{lm}(t)$ of these large-scale features have the longest characteristic times. These slow large-scale features contribute a significant portion of the total variance at a given point $\hat{\mathbf{r}}$. Clearly, the ability to predict these low-l amplitudes would be of considerable importance. In this paper we discuss the errors in forecasting these amplitudes, starting from a well-defined initial state.

To illustrate the characteristic times involved in atmospheric thermal adjustment, consider the global mass-averaged temperature of a model atmosphere whose surfaces carry no heat capacity. The simplified general circulation model studied by Manabe and Wetherald (1975) and Wetherald and Manabe (1975) is such a model. In their numerical experiments they started with initial conditions far from equilibrium. The global temperature relaxed toward a steady state in an exponential-like fashion. The time constant for this thermal relaxation is ~50 days. This interval characteristic of the atmosphere alone is what motivated us to undertake a study of predictability of the thermal field. Note that this value seems considerably larger than the autocorrelation time of the pressure at a point (~3 days), (Leith, 1973), or the "spin-down time" of a cyclone (~4 days) (Holton, 1979).

This long characteristic time for the large-scale thermal field suggests that the collective effect of short-term weather fluctuations (eddies, cloudiness, etc.) may be treated as a random white-noise forcing. The statistical method of treating the response to such rapidly varying forces was provided more than 40 years ago in connection with Brownian motion (cf. Wax, 1954). Its relevance to climate was suggested by Mitchell (1966) in connection with sea-surface temperature anomalies. It was given a general formulation by Hasselmann (1976), who emphasized the role of negative feedback processes in limiting climatic variability. The hope is that even though we are unable to compute the evolution of individual eddies and cloud systems in detail, the thermal field may have a degree of predictability even if these less predictable components are replaced by a white noise.

In the remainder of this section we illustrate the predictability problem by considering first a simple model for the global temperature, T_{00} in Eq. (1). The characteristic time for the decay of a global temperature anomaly is determined by the ratio of the associated heat storage to the radiative loss rate. We shall see that the uncertainty due to the random forcing grows to saturation in a time less than the decay time of the anomaly. In Section 2, we consider a space dependent Budyko-Sellers model driven by a stochastic forcing term. The model is analytically soluble for all the higher modes, the T_{lm} in Eq. (1). In this case some information on the spatial properties of the white noise forcing is required, and for simplicity we assume statistical homogeneity. That is, we assume that the forcing at a given point on the sphere is correlated with the forcing at other points by an amount dependent only upon the great circle distance separating the two points. The functional form of the correlation function itself need not be specified, since our main results do not depend on it. Using a theorem originally proven by Obukhov (1947) and reviewed here in the Appendix, we show in Section 2 that the T_{lm} are statistically uncorrelated in our homogeneous model, and each one behaves in complete analogy to the global mode, but with a characteristic time decreasing rapidly as the spatial scale decreases. Hence the one-mode model serves as a prototype for the more general case to follow.

The departure from steady state in a globally averaged model is described by the equation

$$C\frac{dT_0'(t)}{dt} + BT_0'(t) = 0, \qquad (2)$$

where the 0 denotes global average, the prime denotes departure from steady state, C is the effective heat capacity per unit area of the earth-atmosphere system, and B is an empirical coefficient taken from the Budyko (1968, 1969) infrared terrestrial radiation rule

$$I = A + BT, \qquad (3)$$

where I is infrared flux (W m^{-2}) and T local sea-level temperature (°C). North and Coakley (1979) found $B = 2.09$ W m^{-2} K^{-1} from satellite data. If one takes C to be the heat capacity of a column of air at constant pressure (~10^7 J m^{-2} K^{-1}), then the relaxation time for T_0 in Eq. (2) is

$$\tau^*_{\rm air} = \frac{C_{\rm air}}{B} \approx 58 \text{ days}. \qquad (4)$$

Of course, the parameterization (3) is very crude and different authors have estimated values for B tens of percent different from our value, depending upon the data used and other assumptions employed; however, for the present purposes this estimate will be adequate (cf. North *et al.* (1981) for a discussion).

If the column of earth-atmosphere is situated over ocean, the heat capacity of the wind-driven mixed-layer will dominate over that of the column of air. In this case we must use $C_{\rm mixed} = 3.14 \times 10^8$ J m^{-2} K^{-1} for a 75 m mixed layer and we obtain for the relaxation time (Schneider and Mass, 1975)

$$\tau^*_{\rm mixed} = \frac{C_{\rm mixed}}{B} \approx 4.8 \text{ years}. \qquad (5)$$

Of course, in this case we have completely neglected the dynamics of the mixed layer, considering it to be a perfect heat conductor. These simplifications surely would not suffice in a detailed model, but seem adequate to account for the amplitude and lag of the zonally averaged seasonal cycle in energy-balance models of the planetary scale (North and Coakley, 1979). Roughly the same relaxation time (few years) was found by Manabe and Stouffer (1980) in a seasonal general circulation model of the atmosphere which is coupled to a similar non-dynamic mixed layer ocean.

Yet another characteristic time involves the relaxation time for an atmospheric column attached to an ocean with a given fixed temperature (a common GCM configuration). Heat exchange with the ocean might be schematically modeled by a Newtonian cooling law leading us again to (2) but the new effective value of B is increased due to the oceanic heat exchange. We would be lead to relaxation times considerably less than (4). We shall not explore this interesting possibility here, but we suspect that it may be closest to reality especially in view of the shorter (5–15 days) relaxation times found by Lorenz (1973) with real data.

Consider now a global model governed by the dynamical equation (2), but subject to a stochastic forcing $F(t)$, whose average is zero. The noise term is white, that is, its autocorrelation interval is very small compared to the characteristic time of the climate variable, τ^*. The model is governed by

$$\frac{dT(t)}{dt} + \frac{T(t)}{\tau^*} = F(t), \tag{6}$$

where the stochastic force satisfies

$$\langle F(t) \rangle = 0, \tag{7a}$$

$$\langle F(t)F(t') \rangle = f^2\delta(t - t'). \tag{7b}$$

Angular brackets indicate ensemble averages and δ is the Dirac delta function. Now we imagine an experiment in which T is displaced (or found to be displaced) from its steady state value (zero, since T is the departure only). This anomaly will tend to decay in a characteristic time τ^* except erratically because of the random forcing $F(t)$. Another similar experiment will lead to a similar decay, but the details will be different because $F(t)$ will present a different time series in the second realization. In order to make an optimum prediction of the evolution of T we consider a subensemble of experiments and compute the subensemble average of T denoted by $\langle T \rangle$. Since the subensemble average of $F(t)$ vanishes, we can average (6) to obtain

$$\frac{d\langle T \rangle}{dt} + \frac{\langle T \rangle}{\tau^*} = 0, \tag{8}$$

which is equivalent to (2). In other words our best prediction is that the anomaly will decay exponentially in a characteristic time τ^*.

Our prediction will only be useful, however, if the individual experiments do not differ too much from the average. Deviation from this subensemble mean can be considered noise which contaminates the prediction. If we define the noise to be the rms deviation from the subensemble mean, then a convenient measure of our prediction's value will be the signal (size of anomaly) to noise (rms deviation of individual runs) ratio. When this ratio falls below unity we shall arbitrarily say that the *predictability interval* ends. The rms deviation from the mean, defined as

$$N(t) \equiv \langle (T - \langle T \rangle)^2 \rangle^{1/2} \tag{9}$$

may be determined from (6) and (7) by standard techniques (Papoulis, 1965). The result is

$$N(t) = N_\infty[1 - \exp(-2t/\tau^*)]^{1/2}, \tag{10}$$

where

$$N_\infty \equiv f(\tau^*/2)^{1/2}. \tag{11}$$

The temperature variance, given by the square of N, is initially zero, since all members of the subensemble have the same initial condition. It grows to within e^{-1} of its asymptotic level N_∞^2 in half the decay time of $\langle T \rangle$. Taking the square root gives an even faster saturation for N itself.

Fig. 1 shows the exponential solution to Eq. (8) with error bars indicating the rms deviation of the ensemble members as calculated from Eq. (10). This particular example is for an initial value of the anomaly equal to twice the asymptotic noise level N_∞. The abscissa is in units of the system relaxation time τ^*, while the ordinate is in units of the asymptotic noise level. This choice of units is convenient since the level of the forcing noise does not need to be expressed. The predictability interval τ_P ends when the error bar just touches the horizontal axis. We call attention to the rapid growth of noise for small τ.

Fig. 2 shows 12 separate numerical solutions to Eq. (6) for the same initial condition as in Fig. 1, twice the asymptotic noise level. This figure gives an idea of how the probability distribution is filled out by the individual realizations in the subensemble. Figs. 3 and 4 also show the theoretical mean and 12 separate solutions to Eq. (6), except that the initial anomaly is $20N_\infty$. Note that in this (ridiculously) extreme case the predictability interval is only increased by a factor of order three, indicative of the logarithmic dependence of τ_P upon $T(0)$. Fig. 5 shows a graph of the predictability interval versus the initial anomaly size.

The exponential decay of the subensemble mean to the stationary ensemble mean is characteristic of a first order autoregressive statistical predictor model. It has been called damped persistence by Lorenz

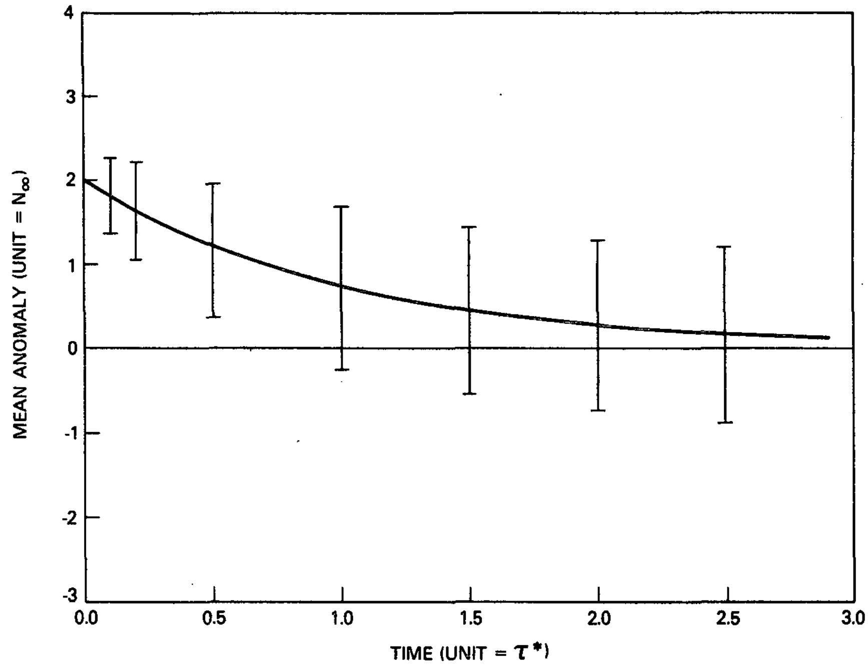

FIG. 1. The relaxation of the ensemble average of a thermal anomaly $\langle T \rangle$ described in units of the asymptotic noise level N_∞. Each ensemble member is assumed to start at $2N_\infty$ in this case. The time scale is in units of the characteristic relaxation time for the variable τ^*. The error bars are the rms deviation for an ensemble of such runs [cf. Eqs. (8)–(11)].

(1973). In that paper Lorenz showed that a small but statistically significant amount of predictability existed in certain observed large-scale thermal fields for up to 15 days. For the simple dynamical model presented in the present paper, damped persistence is the exact analytical solution to the model.

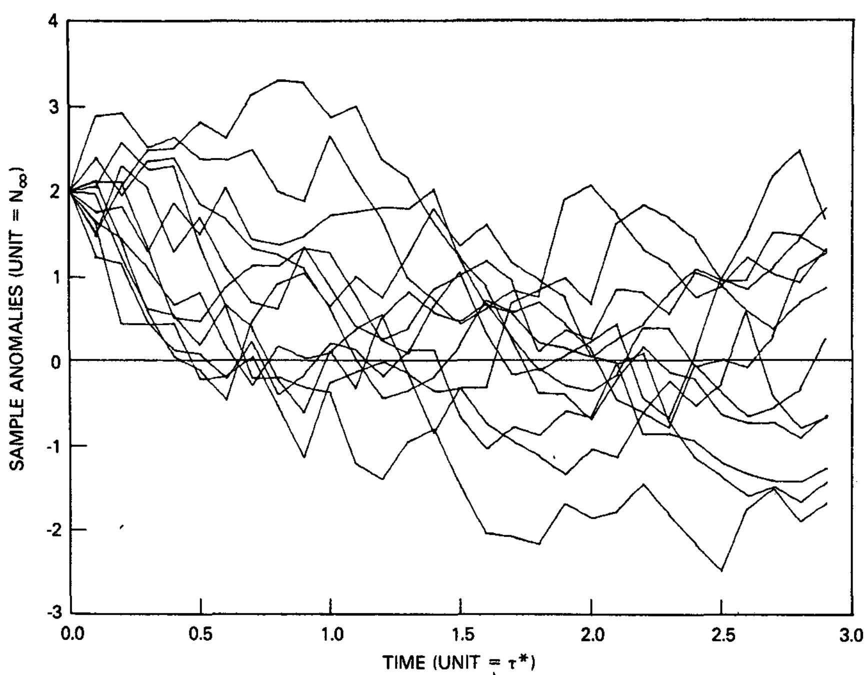

FIG. 2. Twelve separate realizations of anomalies computed numerically from Eq. (6). These graphs indicate how the (normal) probability distribution, whose instantaneous mean and width are shown in Fig. 1, is filled out by individual runs.

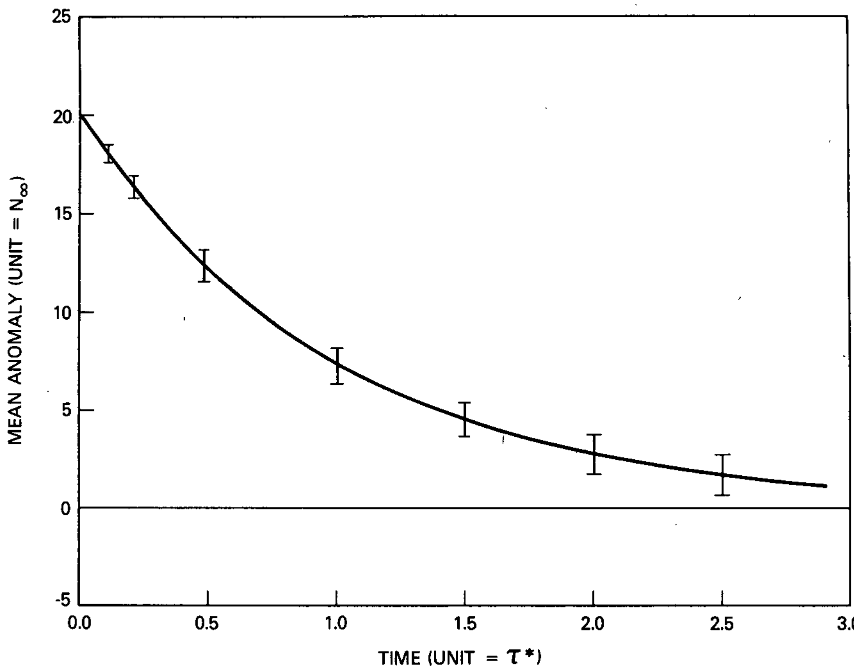

FIG. 3. As in Fig. 1 except that the initial anomaly is $20N_\infty$ (absurdly large).

The system (6) is so simple that elementary methods can be used to analyze the subensemble probability distribution, $P(T, t)$. Since $F(t)$ is a white noise input, it follows from the Central Limit Theorem that $P(T, t)$ is Gaussian in T for each t. At $t = 0$, the distribution is the Dirac delta function, $\delta(T - T_{t=0})$. As t increases the peak broadens and shifts toward $T = 0$, the standard deviation approaching N_∞. $P(T, \infty)$ is the stationary ensemble probability distribution. In more complicated cases such as that in which the left-hand side of (6) is nonlinear, one would have to solve the Fokker-

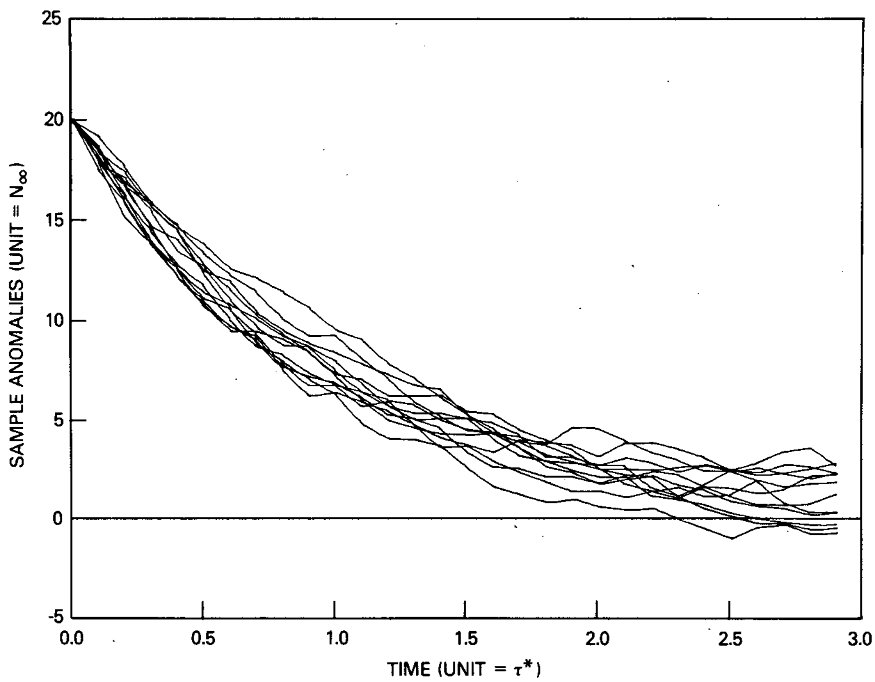

FIG. 4. As in Fig. 2 except that the initial anomaly is $20N_\infty$.

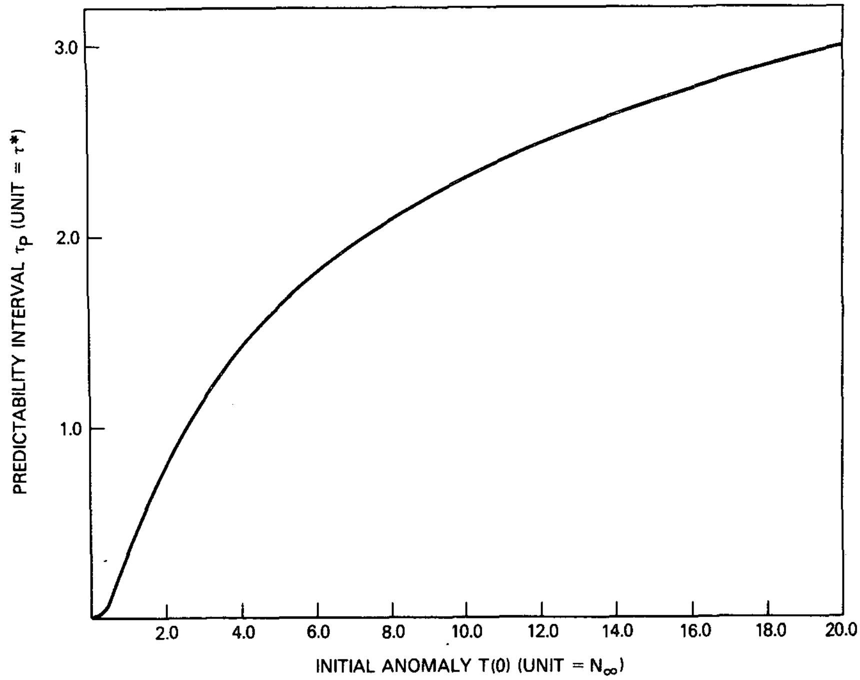

FIG. 5. The predictability interval τ_p in units of the relaxation time τ^* as a function of the size of the initial anomaly in units of the asymptotic noise level N_∞. The slow growth of τ_p for large $T(0)$ is logarithmic, i.e., ½ ln[1 + $T(0)^2$].

Planck equation for the evolution of $P(T, t)$ (cf. review by Haken, 1975; or Hasselmann, 1976). Through such a framework one might study predictability in more general cases involving multiple equilibria [multimodal $P(T)$].

In the next section we introduce the space-dependent model which can be used to study predictability as a function of scale size. The model may be considered a one- or two-dimensional stochastically driven energy-balance model (Budyko, 1968; 1969; Sellers, 1969). It is completely solvable analytically by elementary methods.

2. A soluble scale-dependent model

Consider the energy-balance climate model (North, 1975) defined by the equation

$$C \frac{\partial T}{\partial t}(\hat{\mathbf{r}}, t) - R^2 D \nabla^2 T(\hat{\mathbf{r}}, t) + A + BT(\hat{\mathbf{r}}, t) = QS(\hat{\mathbf{r}})a(\hat{\mathbf{r}}). \quad (12)$$

The terms in this equation are familiar to many readers. The first term is heat storage with $T(\hat{\mathbf{r}}, t)$ the sea level temperature at surface position $\hat{\mathbf{r}}$: C is the heat capacity per unit area; the second term represents horizontal diffusion of all forms of heat, R the earth's radius, D a constant, and the next two terms represent the Budyko infrared radiation rule; the right-hand side represents absorption of solar radiation, where Q is one-fourth of the solar constant, $S(\hat{\mathbf{r}})$ mean annual distribution of radiation at point $\hat{\mathbf{r}}$, and $a(\hat{\mathbf{r}})$ the coalbedo. The ice-cap albedo feedback mechanism is ignored in this paper.

We wish to modify (12) by adding a stochastic forcing term to its right-hand side. Such a term might arise from any of several causes such as eddy transport fluctuations, stormy bursts of latent heat, flickering cloudiness variables, etc. Most of these effects are of a shorter time scale (days) than the response time C/B of the largest scales of the temperature field (50 days). So for simplicity we shall neglect the autocorrelation time of the forcing, and take it to be represented by white noise. In addition we shall simplify the spatial dependence by assuming that the forcing is statistically homogeneous on the sphere. In what follows let $T(\hat{\mathbf{r}}, t)$ denote the deviation from the mean field $T^\circ(\hat{\mathbf{r}})$ which satisfies the time-dependent version of (12). Each realization of the stochastic field $T(\hat{\mathbf{r}}, t)$ then satisfies

$$\frac{\partial T}{\partial t}(\hat{\mathbf{r}}, t) - \frac{DR^2}{C} \nabla^2 T(\hat{\mathbf{r}}, t) + \frac{B}{C} T(\hat{\mathbf{r}}, t) = F(\hat{\mathbf{r}}, t). \quad (13)$$

$F(\hat{\mathbf{r}}, t)$ is a stochastic field whose statistics are homogeneous on the sphere and whose temporal spectrum is white. That is, the correlation of F with itself at different times t and t' and points $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ vanishes unless $t = t'$ and depends only upon the

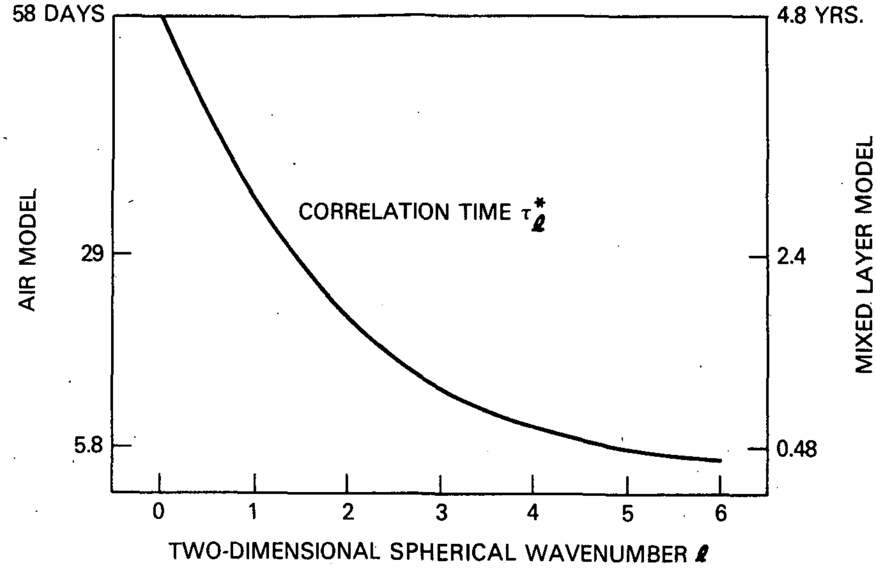

FIG. 6. Relaxation time τ_l^* for a particular mode as function of Legendre index l. Since τ_l^* is scaled by the heat capacity per unit area we show two scales for the ordinate: on the left is depicted time scales for an air only model whose surface contributes no heat capacity; on the right we show time units corresponding to a 75 m deep mixed-layer ocean. Earth is, of course, a mixture of these two extremes.

great circle distance between the points:

$$\langle F(\hat{\mathbf{r}}, t)F(\hat{\mathbf{r}}', t')\rangle = f^2\rho(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\delta(t - t'). \quad (14)$$

According to a theorem by Obukhov (1947), discussed in the Appendix, the empirical orthogonal functions (EOF's) for such a field are the spherical harmonics. That is, taking

$$\langle F(\hat{\mathbf{r}}, t)\rangle = 0, \quad (15)$$

we may expand F in terms of spherical harmonics Y_l^m as

$$F(\hat{\mathbf{r}}, t) = \sum_{l,m} F_{lm}(t)Y_l^m(\hat{\mathbf{r}}), \quad (16)$$

and the wavenumber amplitudes F_{lm} satisfy

$$\langle F_{lm}(t)F_{l'm'}(t')\rangle = f_l^2\delta_{ll'}\delta_{mm'}\delta(t - t'), \quad (17)$$

so that different wavenumbers are uncorrelated in this basis.

The wavenumber spectrum f_l^2 is the Fourier-Legendre transform of the autocovariance of F at different separations (A10). It would be nearly constant if neighboring points were uncorrelated. However, we emphasize that the results below hold irrespective of the wavenumber dependence of the forcing. By inserting the expansions (1), (16) into (13) and projecting the mode amplitudes we obtain

$$\frac{dT_{lm}(t)}{dt} + \frac{T_{lm}(t)}{\tau_l^*} = F_{lm}(t), \quad (18)$$

where

$$\tau_l^* = \frac{C}{l(l+1)D + B}. \quad (19)$$

Since the $Y_l^m(\hat{\mathbf{r}})$ are the eigenfunctions of ∇^2 and the EOF's of $F(\hat{\mathbf{r}}, t)$, we have the remarkable property that the equations (18) for each l, m are not only dynamically uncoupled but statistically uncorrelated. It follows immediately that the $T_{lm}(t)$ are uncorrelated for different l, m:

$$\langle T_{lm}(t)T_{l'm'}(t')\rangle = N_l^2 \exp(-|t - t'|/\tau_l^*)\delta_{ll'}\delta_{mm'}, \quad (20)$$

where $N_l = f_l(\tau_l^*/2)^{1/2}$ is the asymptotic noise level of mode l. From the converse of Obukhov's theorem it follows that the statistics of $T(\hat{\mathbf{r}}, t)$ are homogeneous on the sphere.

Now we note that each member of the uncoupled set (18) is formally equivalent to the zero-dimensional prototype studied earlier, (6), except that the characteristic time depends upon the scale index l through (19). Note that $\tau_0^* = \tau^* = C/B$ of (6) for the global average mode. The analysis of the introductory section suffices to understand the predictability for all scales in this model.

By taking a value of D from the energy-balance model results (North and Coakley, 1979) equal to $0.30\,B = 0.618$ W m^{-2} K^{-1}, we may estimate τ_l^* for the air-only model with $\tau_0^* = 58$ days, Eq. (4), or the mixed layer model, with $\tau_0^* = 4.8$ years, Eq. (5). Fig. (6) shows the result for each case as a function of scale index l. The real earth is somewhere between these, since 30% of the surface area is land. We wish to especially call attention to the rapid decrease of τ_l^* with scale index l. For reference it might be noted that synoptic scales (~1000 km) cannot be resolved for $l \leqslant 12$.

The above constitutes an analysis of the predictability interval for any scale l for this simple model. It is remarkable that the theory is independent of the spatial spectrum f_l^2 of the forcing noise, [cf. Appendix (A9)]. Since each mode is independent of every other dynamically (because of diffusive transport) and statistically (because of homogeneous noise forcing), the predictability interval depends only upon the autocorrelation interval τ_l^* for that mode and very weakly upon the size of the initial anomaly expressed in units of the stationary ensemble standard deviation, N_∞. The wavenumber spectrum of the forcing f_l^2 expresses the relative power (variance associated with the forcing of each mode). Since the modes are uncoupled the relative power in each mode does not affect the predictability in other modes. The forcing noise power in a given mode does, of course, determine the steady state ensemble variance, N_∞^2, in the particular mode, and the predictability in that mode does depend weakly upon the size of the initial anomaly expressed in units of N_∞ as depicted in Fig. 5. Ideally, N_∞ is a directly observable quantity for the real climate so that f_l^2 need not be known explicitly. If either the diffusive transport or the homogeneous noise forcing assumptions are relaxed the predictability would depend upon the wavenumber spectrum of the forcing; i.e., the predictability of one mode would depend upon the forcing power and time constants from other modes. In the real world, of course, both assumptions are violated so that the modes are coupled. We conjecture that our conclusions are not drastically altered by the errors introduced by these simplifying assumptions.

3. Discussion

Before drawing conclusions about this work it is well to recall the model's position in the hierarchy of climate models. The energy-balance models are among the most primitive approaches (Schneider and Dickinson, 1974; Saltzman, 1978), and many authors have cautioned that the numerical results should not be applied willy-nilly (Coakley and Wielicki, 1979; Warren and Schneider 1979; North *et al.*, 1981). Our model is even crude by *these* standards since new simplifications are introduced as mathematical or pedagogical conveniences. Little attempt has been made even to tune the model as is the accepted practice in sensitivity studies. Still the results are so striking that it seems compelling to ask about their applicability to numerical forecasting.

The interpretation to the solution of our model is simply stated: the present models under consideration which lack surface heat storage do not have predictability intervals beyond a few days for $l > 4$. A crude mixed layer may increase the interval by a factor of 10. For the mixed-layer model the predictability interval for an anomaly of twice the noise level at $l = 12$ is two weeks. ($l = 12$ corresponds to a wavelength of ~1000 km). We believe this result reinforces the broadly held belief that coupled ocean models will be essential to progress in numerical climate prediction.

But how realistic are the numerical values we obtained? We feel that some of our approximations, though seemingly unmeteorologically motivated, probably do not lead to serious errors. For example, using a more general linear operator than simple diffusion for transport should lead to a sequence of relaxation times (eigenvalues) for the new modes (eigenfunctions) not extremely different from those used here. Indeed serious is the possibility that diffusion simply is a useless parameterization on the smaller scales (Lorenz, 1979). The use of homogeneous noise forcing is not a serious drawback since more general cases can be solved (North *et al.*, 1981) with results indicating a mixing of the characteristic times rather than a tendency to lengthen them for large l. Probably more serious is application of Budyko's radiation rule (3) to the smaller scales where it has not been tested.

The most glaring shortcoming of our model is its lack of internal nonlinear dynamics. This effect has been partially taken into account by the noise forcing and the linear transport operator; however, no linear operator will have the important property of transporting error energy systematically from one scale to another as is so characteristic of fluid motions. Therefore, such forms of error growth omitted in our model may tend to limit predictability even more than in the simple linear system. On the other hand, the effects of fluctuations in higher wavenumbers are not felt strictly as white noise but rather a longer time scale forcing is felt from neighboring modes—our separation of "weather" and "climate" fluctuations being somewhat contrived. In fact, this red noise forcing (coupling to other modes with comparable time constants) may have a significant part of its variance predictable. Still some parts of the forcing to the thermodynamic equation may be taken as white noise because of the lack of predictability of such quantities as cloudiness fluctuations. One might expect that strictly deterministic nonlinear models might have longer predictability times than damped persistence models. Our goal has been to explore the possible lower limits of predictability under certain simplifying assumptions. The present analysis does suggest that some basis functions might be better than others as tools in the damped persistence prediction framework.

Finally, we should not fail to mention the interesting possibility that aspects of the atmospheric flow such as long waves interacting with geographical features may have long lifetimes comparable to the thermal relaxation time exploited in the present study. One analysis suggests that these anomalies might be associated with multiple equilibria (Charney and Devore, 1979). It will be necessary to study realistic nonlinear initial value models to explore predictability limits in these cases. Still we feel that our strictly thermal results can be used as a reference in these future efforts.

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APPENDIX

Noise on a Sphere

Consider a random scalar field $F(\hat{\mathbf{r}})$ defined on a sphere ($\hat{\mathbf{r}}$ is a unit vector directed from the center to a point on the surface). The field may be expanded into a series of orthonormal functions $g_n(\hat{\mathbf{r}})$:

$$F(\hat{\mathbf{r}}) = \sum_n F_n g_n(\hat{\mathbf{r}}), \tag{A1}$$

where the F_n are random variables, taken here to have ensemble average zero. In general the F_n are cross correlated, i.e.,

$$\langle F_n F_m \rangle \neq F_n{}^2 \delta_{nm}, \tag{A2}$$

where angle braces represent ensemble averages. However, since the covariance matrix is symmetric, there exists a linear (orthogonal) transformation U_{nm} which will render the covariance matrix diagonal:

$$F'_n = \sum_m U_{nm} F_m, \tag{A3}$$

$$g'_n(\hat{\mathbf{r}}) = \sum_m U_{nm} g_m(\hat{\mathbf{r}}), \tag{A4}$$

$$U^{-1} = U^{\mathrm{T}}, \tag{A5}$$

$$\langle F'_n F'_m \rangle = \langle F'^2_n \rangle \delta_{nm}. \tag{A6}$$

The $g'_n(\hat{\mathbf{r}})$ are called empirical orthogonal functions (EOF's); they represent spatial patterns on the sphere which are statistically independent of one another (e.g., Davis, 1976).

We now recall the theorem of Obukhov (1947) which states: If $F(\hat{\mathbf{r}})$ is a random field defined on the sphere and it is statistically *homogeneous*, that is,

$$\langle F(\hat{\mathbf{r}}) F(\hat{\mathbf{r}}') \rangle = \rho(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'), \tag{A7}$$

then the EOF's are the spherical harmonics. Furthermore, the coefficients in the expansion

$$F(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F_{lm} Y_l^m(\hat{\mathbf{r}}) \tag{A8}$$

satisfy

$$\langle F^*_{lm} F_{l'm'} \rangle = \sigma_l{}^2 \delta_{ll'} \delta_{mm'}. \tag{A9}$$

Conversely, if (A7) and (A8) are satisfied, $F(\hat{\mathbf{r}})$ is homogeneous.

Proof of the theorem is by straightforward construction of the covariance matrix (A2) and use of the addition theorem for spherical harmonics. The theorem is hardly a surprise since ordinary spectral analysis is analogously derived for systems with time translation invariance (stationarity). In that case the Fourier basis is the appropriate EOF set in time (Jenkins and Watts, 1968). Similarly in problems which are homogeneous in flat two- or three-dimensional space the appropriate EOF set is the trigonometric (Fourier) basis. Similarly, the trigonometric basis serves for homogeneous noise on a circle. An example is the wavenumber analysis familiar from homogeneous turbulence theory (Batchelor, 1959).

Still unspecified by the condition of homogeneity is the spectrum $\sigma_l{}^2$, which is given by

$$\sigma_l{}^2 = 2\pi \int_{-1}^{1} \rho(x) P_l(x) dx, \tag{A10}$$

where $\rho(x) = \rho(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')$ is the autocovariance between $F(\hat{\mathbf{r}})$ and $F(\hat{\mathbf{r}}')$, (A7), and $P_l(x)$ is the Legendre polynomial. A spatial white noise field $\rho(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \propto \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}')$ would lead to $\sigma_l{}^2$ independent of l, a flat spectrum. Whereas spatial "red noise" fields which have a finite autocorrelation length on the sphere would have a spectrum $\sigma_l{}^2$ which cuts off at a characteristic value l_0 which is in inverse proportion to the autocorrelation length.

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