The Secondary Flow near a Baroclinic Planetary Wave Critical Line

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(Manuscript received 13 March 1980, in final form 18 November 1980)

ABSTRACT

The wave-mean flow interaction has been computed near an energy-absorbing, baroclinic, planetary wave critical line tilted at an arbitrary angle from the vertical. This problem is a generalization of the critical line interaction problems studied by Matsuno and Nakamura (1979) and Schoeberl (1980).

A tilted critical line can directly tap the eddy heat transport of a Rossby wave and produce a singular rate of change in the zonally averaged temperature at the critical line. This implies that sudden stratospheric warmings may not always require an induced Eulerian-mean secondary circulation to create significant temperature changes in the zonally averaged flow as suggested by Matsuno (1971). Strong Lagrangian-mean motion also exists along the critical line if it is not perfectly vertical. These results are discussed with application to the 1976/77 sudden warming.

1. Introduction

A critical line (hereafter abbreviated CL) is the surface where the phase speed of a wave in a fluid is equal to the speed of the background flow. The linearization assumptions which describe wave propagation far from the CL breakdown at the CL where the flow can be highly nonlinear. In the case of stationary Rossby waves, wave propagation and eddy transport of heat and momentum occur only on the side of the CL where westerlies are present (Charney and Drazin, 1961). Three conditions might occur as a Rossby wave encounters a CL. First, the wave could be absorbed at the CL; second, it could be partially or totally reflected; and third, it could be overreflected. During the evolution of the flow near a CL most of these properties exist at one time or another, and their appearance is to some extent a function of basic flow parameters (Belánd, 1976; Geisler and Dickinson, 1974). Steady-state models of the CL predict partial reflection (Tung, 1979) in agreement with long time integrations of numerical models (Belánd, 1976). However, after its initial formation, the CL probably absorbs most of the wave energy until the nonlinear flow dominates the damping processes. Observations of the 1976/77 sudden stratospheric warming reported by O'Neill and Taylor (1979) show the formation of a CL near the pole associated with poleward wave energy flux. Since the existence of this CL was relatively brief, current theory would support the inference of wave-energy absorption by the CL.

In this paper we will look at one aspect of the simplest model of a CL in which the CL is assumed to totally absorb energy from steady, stationary, planetary waves. The aspect we are interested in is the secondary mean circulation near a CL in a baroclinic atmosphere. The motivation for this study is the observation of the nearly vertical CL by O'Neill and Taylor (1979) which appeared during the sudden warming of 1976/77. We will also look at the Lagrangian-mean properties of an idealized baroclinic CL.

This problem was first examined by Matsuno (1971) and later by Matsuno and Nakamura (1979) in the context of sudden warmings for a CL aligned horizontally and confined to a β channel. Schoeberl (1980) used Matsuno and Nakamura's results to compute the secondary circulation around a vertically aligned CL. In the former case Matsuno and Nakamura found the zonal mean flow is decelerated in a 10 km region around the CL; the rate of change in the temperature is discontinuous at the CL, and the zonally averaged flow is such that air rises in the north and flows equatorward in the vicinity of the CL. The zonally averaged mass streamfunction computed by Matsuno and Nakamura is illustrated in Fig. 1.

The Lagrangian-mean flow, which is the zonally averaged flow plus the Stokes drift due to the waves, is quite different for the horizontally aligned CL. Matsuno and Nakamura (1979) found that a singular meridional velocity in the Stokes flow, and hence the Lagrangian-mean flow, at the CL. Weak return flow was present both above and below the CL. The Lagrangian-mean streamfunction they computed is shown in Fig. 2. Since the Lagrangian-mean flow gives the center of mass motion experienced by conservative tracers (Andrews and McIntyre, 1978), this result suggests that the average position of a material string of particles extending in the zonal direction near the CL would be transported rapidly poleward as in a jet, while the average position of particles above and below the CL moves equatorward slowly. In the case of the vertically aligned CL, Schoeberl (1980) found that the zonally averaged flow was identical to the Lagrangian-mean flow since the Stokes flow vanished. As a consequence no strong Lagrangian jets appeared.

In what follows we will unify the results of Matsuno and Nakamura (1979) and Schoeberl (1980) into a single formulation. The procedure will be to rotate the CL to an arbitrary angle with the vertical and then use the analytical solutions obtained by Schoeberl. We will then examine both the Eulerian and Lagrangian mean flow induced by CL.

2. Basic equations

We shall consider a stationary, steady, conservative, planetary wave in the presence of a background flow. The mean flow forced by the stationary waves is given by

$$\left[\frac{\partial^2}{\partial y^2} + \frac{f^2}{N^2}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{1}{H}\frac{\partial}{\partial z^*}\right)\right]\frac{\partial \bar{\phi}}{\partial t} = -\frac{1}{f}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\overline{\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial x}}\right) + \frac{f^2}{N^2}\left(\frac{\partial}{\partial z^*} - \frac{1}{H}\right)\left(\overline{\frac{\partial \phi}{\partial z^*}\frac{\partial \phi}{\partial x}}\right)\right], \quad (1)$$

where $\bar{\phi}$ is the geopotential of the zonally averaged flow. The wave equation is

$$im\bar{u}_0\left[\frac{\partial^2 \phi}{\partial y^2} - m^2\phi + \frac{f^2}{N^2}\left(\frac{\partial^2 \phi}{\partial z^{*2}} - \frac{1}{H}\frac{\partial \phi}{\partial z^*}\right)\right] + \bar{q}_y im\phi = 0, \quad (2)$$

where ϕ is the wave geopotential where $\phi = \phi_0 e^{imx}$. These equations are described by Holton (1975), and the following notation has been used:

- N Brunt-Väisälä frequency
- H atmospheric scale heights (~7 km)
- z^* $H \ln(p_0/p)$, where p_0 is a reference pressure and p is the pressure.
- f the Coriolis parameter
- y the meridional (northward) coordinate
- x the zonal (eastward) coordinate
- m zonal wavenumber
- $\bar{q}_y$ meridional gradient of the potential vorticity of the mean flow, u_0.

In this problem the CL absorbs the wave energy but remains stationary since $\bar{u}_0$ in Eq. (2) is assumed fixed. A useful viewpoint is that this model is a kind of snapshot of the CL. However, this view also is

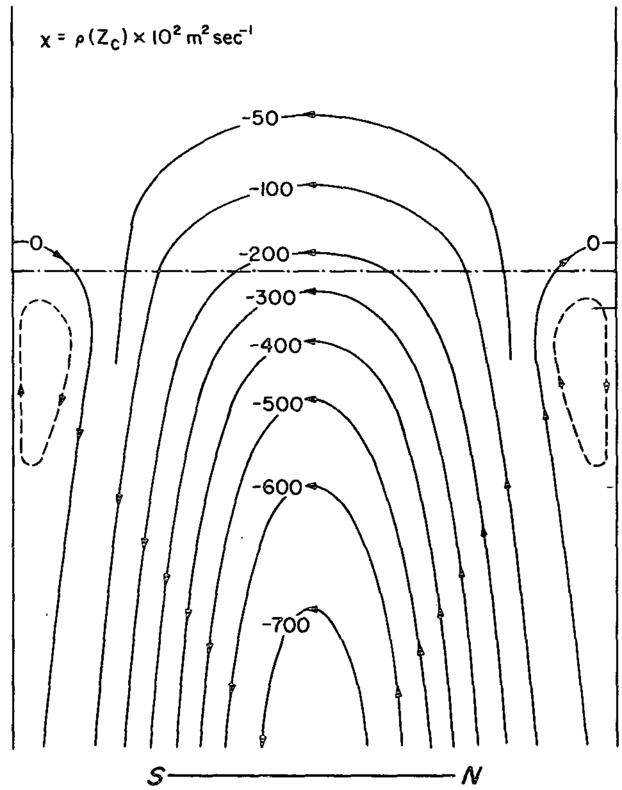

FIG. 1. Zonally averaged mass streamfunction computed near a horizontal CL after Matsuno and Nakamura (1979).

not quite accurate since the singular components in expressions for $\partial\bar{u}/\partial t$ and $\partial\bar{T}/\partial t$ occur as a result of our assumptions about the stationary mean flow in Eq. (2). Despite these drawbacks, this model exposes many of the fundamentals of wave-mean flow interaction along the CL.

We wish to compute the secondary flow about a CL at an angle, θ, from the vertical (Fig. 3). We assume in this calculation that the Eliassen-Palm flux (hereafter EP flux, ϵ) is always normal to the CL as would be the case for a CL which totally absorbs wave energy. The EP flux source is distant from the CL lying in the $-y'$ region where y' and z' are coordinates fixed to the CL (Fig. 3). The domain of the problem is assumed to be unbounded.

The following transformations simplify and rotate Eqs. (1) and (2) into a coordinate system oriented along the CL:

$$\left.\begin{aligned} y' &= y\cos\theta + \frac{Nz^*}{f}\sin\theta, \quad z = \frac{N}{f}z^*, \\ z' &= -y\sin\theta + \frac{Nz^*}{f}\cos\theta, \\ \epsilon &= \hat{j}\left(\overline{\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x}}\right) + \hat{k}\left(\overline{\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial x}}\right) \\ \psi &= e^{-z^*/2H}\phi, \quad \bar{\psi} = e^{-z^*/2H}\bar{\phi} \end{aligned}\right\} . \quad (3)$$

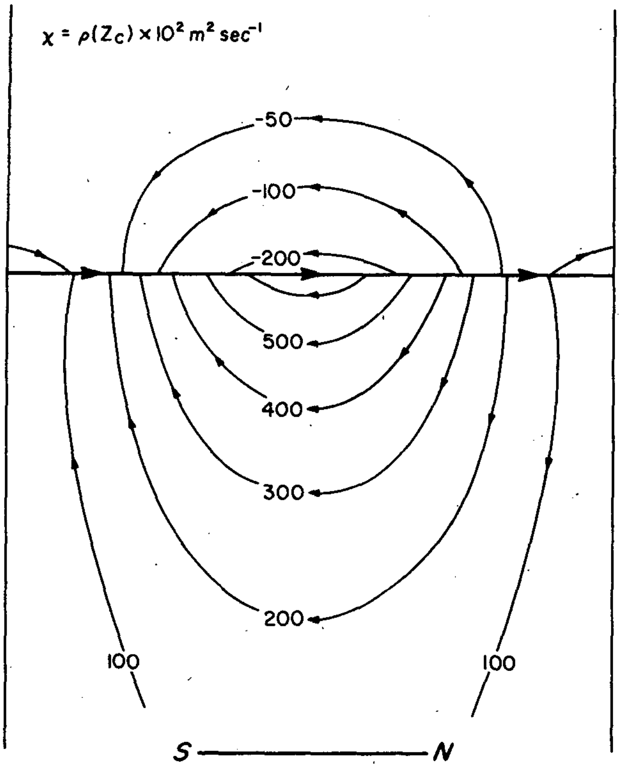

FIG. 2. Lagrangian-mean mass streamfunction computed for a horizontal CL after Matsuno and Nakamura (1979).

Eq. (1) becomes

$$\left(\frac{\partial^2}{\partial y'} + \frac{\partial^2}{\partial z'^2} - \frac{L_R{}^2}{4}\right)\frac{\partial\bar{\psi}}{\partial t}$$
$$= -\exp[\tfrac{1}{2}(y' \sin\theta + z' \cos\theta)L_R]$$
$$\times \left[\left(\cos\theta \frac{\partial^2}{\partial y'^2} - \sin\theta \frac{\partial^2}{\partial y'\partial z'}\right)\left(\overline{\frac{\partial\psi}{\partial y'}\frac{\partial\psi}{\partial x}}\right) + \left(\cos\theta \frac{\partial^2}{\partial y'\partial z'} - \sin\theta \frac{\partial'^2}{\partial z'^2}\right)\left(\overline{\frac{\partial\psi}{\partial z'}\frac{\partial\psi}{\partial x}}\right)\right]$$
$$= e^{z^*/2H}\frac{\partial}{\partial y}[\nabla\cdot\epsilon], \quad (4)$$

while Eq. (2) becomes

$$im\bar{u}_0\left[\frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + (-m^2 + L_R{}^2/4)\right]\psi + \bar{q}_y im\psi = 0, \quad (5)$$

where $L_R = f/NH$ and L_R is the inverse Rossby radius.

Assume that the structure of the wave solution along the CL is of the form

$$\psi = g(y')\exp(\alpha z' L_R), \quad (6)$$

where α is some dimensionless constant. $g(y')$ describes the meridional wave structure. For our purposes g is only assumed to be continuous. Applying the Cauchy integral theorem to Eq. (5) and using Eq. (6) we obtain

$$\left(\overline{\frac{\partial\psi}{\partial z'}\frac{\partial\psi}{\partial x}}\right) = 0,$$
$$\left(\overline{\frac{\partial\psi}{\partial y'}\frac{\partial\psi}{\partial x}}\right) = \xi H(-y')\exp(2\alpha z' L_R), \quad (7)$$

where

$$\xi = \left(\frac{m\pi\bar{q}_y|g(y')|^2}{\partial\bar{u}/\partial y'}\right)_{y'=0}.$$

Substituting these quantities into Eq. (4), we obtain

$$\left(\frac{\partial^2}{\partial y'^2} + \chi^2\right)\frac{\partial\bar{\psi}_{y'}}{\partial t} = -\frac{\xi}{f}\exp\left[\frac{y' L_R}{2}\sin\theta\right] \times \left[2\alpha L_R \sin\theta\delta(y') - \cos\theta\frac{\partial}{\partial y'}\delta(y')\right], \quad (8)$$

where

$$\frac{\partial\bar{\psi}}{\partial t} = \frac{\partial\bar{\psi}_{y'}}{\partial t}\exp[z' L_R(\tfrac{1}{2}\cos\theta + 2\alpha)],$$

$$\chi^2 = L_R{}^2(\tfrac{1}{2}\cos\theta + 2\alpha)^2 - \tfrac{1}{4}L_R{}^2.$$

The sign of χ^2 may change depending on the values of α and θ, and as a result, the solution character to Eq. (8) changes. This fact was pointed out by Schoeberl (1980) who noted that positive χ^2 values resulted in spatially oscillating mean flow solutions on the y' side of the CL. Negative χ^2 values produce exponential mean flow solutions on both sides of the CL. In general, exponential solutions occur

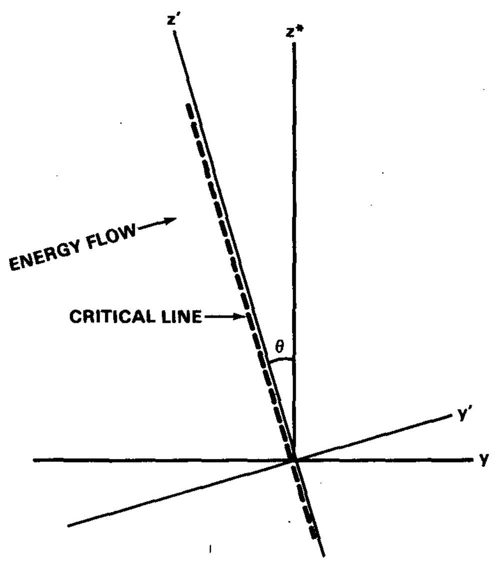

FIG. 3. The alignment of the (y', z') coordinate system relative to the CL.

for $-(1 + \cos\theta)/4 < \alpha < (1 - \cos\theta)/4$ with oscillating solutions elsewhere. In the special cases for

$$\theta = \begin{cases} 0, & -1/2 < \alpha < 0 \\ \pi/2, & -1/4 < \alpha < 1/4 \text{ for the exponential solutions.} \end{cases}$$

The oscillatory solutions have no physical analogs and result from boundary effects. That is, when the solution in the direction normal to the CL does not decay to zero at infinity the forcing effects at infinity determine the local solution. We shall just consider the exponential solutions here.

3. Solution to the critical line problem

The critical line can be thought of as a boundary separating two regions. The mean flow is forced only at the CL. In order to compute the solution we must derive interface conditions between the two regions. These conditions may be derived from Eq. (8).

First we integrate Eq. (8) across the CL denoting the integral $\int_-^+ dy'$. This gives the jump condition for the slope, $\partial\psi_{y'}/\partial y'$. That is

$$\left.\frac{\partial\psi_{y'}}{\partial y}\right]_-^+ = -\xi f^{-1}(\tfrac{1}{2}\cos\theta L_R + 2L_R\alpha)\sin\theta. \quad (9)$$

The jump in $\psi_{y'}$ across the CL can be derived by integrating Eq. (8) with $\int_-^+ \int_{-\infty}^{y'} dy''dy'$. We obtain

$$\psi_{y'}]_-^+ = \xi f^{-1}\cos\theta. \quad (10)$$

In evaluating the last integral we have used the fact that $\int_{-\infty}^{\infty} \psi(y')dy'$ is bounded which is consistent with our rejection of the oscillatory solutions. The solution form of $\psi_{y'}$ is

$$\psi_{y'} = Ae^{\chi y'}H(-y') + Be^{-\chi y'}H(y'). \quad (11)$$

The constants A and B can be evaluated using Eqs. (9) and (10). However, before writing down the total solution it is interesting to note that the form of Eq. (11) can also be written as

$$\psi_{y'} = R_1T_1 + R_2T_2,$$

where $T_1 = e^{\chi y'}H(-y') + e^{-\chi y'}H(y')$ and $T_2 = e^{-\chi y'} \times H(-y') - e^{-\chi y'}H(y')$. These functions have the following recursive properties:

$$\frac{\partial T_1}{\partial y'} = \chi T_2,$$

$$\frac{\partial T_2}{\partial y'} = \chi T_1 - 2\delta(y').$$

Using the interface conditions we obtain the solution.

$$\frac{\partial\bar{\phi}}{\partial t} = \frac{\xi}{f}\exp(z^*/2H + z'l)(R_1T_1 + R_2T_2), \quad (12)$$

where

$$R_1 = \frac{l}{2\chi}\sin\theta, \quad l = (\tfrac{1}{2}\cos\theta + 2\alpha)L_R,$$

$$R_2 = -\tfrac{1}{2}\cos\theta.$$

4. Discussion

a. Observed angles

We might ask at this point how the angle θ relates to the observed angle (θ_{ob}) the CL makes with the vertical. Since the transformations described by Eq. (3) expands the vertical coordinate by the Rossby radius divided by a scale height, the angle θ_{ob} is magnified. The relation between θ_{ob} and θ is given by

$$\theta_{ob} = \tan^{-1}\left[\frac{N}{f}\tan\theta\right].$$

Taking the values: $f = 10^{-4}$ s^{-1} and $N = 2 \times 10^{-2}$ s^{-1}, then if $\theta = 30°$ $\theta_{ob} = 89.5$. Thus tilted CL's in this coordinate system (y', z') would appear nearly horizontal to the observer. And, a CL which is observed to be almost vertical (ex. $\theta_{ob} \approx 10°$) appears as an extremely steep line in this system ($\theta \approx 0.005°$).

b. Eulerian fields

The other Eulerian quantities associated with the change in $\bar{\phi}$ with time are described by

$$\frac{\partial\bar{u}}{\partial t} = -\frac{1}{f}\frac{\partial}{\partial y}\frac{\partial\bar{\phi}}{\partial t}, \quad (13)$$

$$\frac{\partial\bar{T}}{\partial t} = \frac{H}{R}\frac{\partial}{\partial z^*}\frac{\partial\bar{\phi}}{\partial t}, \quad (14)$$

$$-\frac{\partial\bar{u}}{\partial t} + f\bar{v} = -\frac{1}{f^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}}\right), \quad (15)$$

$$\frac{\partial\bar{v}}{\partial y} = -\frac{\partial\bar{w}}{\partial z^*} + \frac{1}{H}\bar{w}, \quad (16)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\bar{\phi}}{\partial z^*}\right) + N^2\bar{w} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\overline{\frac{\partial\phi}{\partial z^*}\frac{\partial\phi}{\partial x}}\right), \quad (17)$$

where $\bar{T}$ is the zonally averaged temperature, $\bar{u}$ is the eastward mean zonal wind, $\bar{v}$ is the meridional mean zonal wind and $\bar{w}$ is the vertical velocity, $\bar{w} = dz^*/dt$.

We shall focus attention on the fields which can be derived from Eq. (12). The following relations and the recursion properties are useful in simplifying the algebra involved

$$\frac{\partial}{\partial y}\eta(y') = \frac{\partial\eta}{\partial y}\cos\theta, \quad (18a)$$

$$\frac{\partial}{\partial z^*}\eta(y') = \frac{\partial\eta}{\partial y'}\frac{N}{f}\sin\theta, \quad (18b)$$

$$\frac{\partial}{\partial y}\eta(z') = -\frac{\partial\eta}{\partial z'}\sin\theta, \tag{18c}$$

$$\frac{\partial}{\partial z^*}\eta(z') = \frac{\partial\eta}{\partial z'}\frac{N}{f}\cos\theta, \tag{18d}$$

where η is any function. Since the eddy fluxes are originally given in the (y', z') system they must be rotated for use in Eqs. (15) and (17). Thus

$$\left(\overline{\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial y}}\right) = \frac{\zeta}{2}m\ \mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial y'}\right)\cos\theta = \zeta\xi H(-y')\cos\theta, \tag{19}$$

$$\left(\overline{\frac{\partial\phi}{\partial z^*}\frac{\partial\phi}{\partial x}}\right) = \frac{\zeta}{2}m\ \mathrm{Im}\left(\psi^*\frac{\partial\psi}{\partial z'}\right)\frac{N}{f}\sin\theta = \zeta\xi H(-y')\sin\theta\frac{N}{f}, \tag{20}$$

where $\zeta = \exp[L_R(y'\sin\theta + z'\cos\theta) + 2L_R\alpha z']$.

Using relations given by Eq. (18) and Eqs. (13) and (14) we have

$$\frac{\partial\bar{u}}{\partial t} = -\frac{\xi}{f^2}[-l\sin\theta G_1 + \delta(y')\cos^2\theta + \cos\theta G_2\chi] \times \exp(z^*/2H + lz'), \tag{21}$$

where

$$z_1 = R_1 + R_2,$$

$$G_1 = z_1H(-y')\exp(\chi'y') + z_2H(y')\exp(-\chi'y'),$$

$$z_2 = R_1 - R_2,$$

$$G_2 = z_1H(-y')\exp(\chi'y') - z_2H(y')\exp(-\chi'y'),$$

$$\frac{\partial\bar{T}}{\partial t} = \frac{H}{R}\frac{\xi N}{f^2}[(\tfrac{1}{2}L_R + l\cos\theta)G_1 + \delta(y')\cos\theta\sin\theta + G_2\sin\theta]\exp(z^*/2H + lz'). \tag{22}$$

The most interesting part of these two equations is the $\delta(y')$ term and its angular dependence. The $\delta(y')$ term in Eq. (21) represents the divergence of eddy momentum flux over an infinitely thin layer. This produces a δ function in the deceleration of the mean flow at the CL. As the angle the CL makes with the vertical increases, this deceleration decreases as the CL intercepts less horizontal momentum flow per unit length of CL. Further, our assumptions that the EP flux flow is always normal to the CL reduces the horizontal momentum flow as the CL is tilted. Each factor produces a $\cos\theta$ dependence in the divergence of momentum flux at the CL; thus the term is multiplied by $\cos^2\theta$.

The time rate of change in $\bar{T}$ also has a $\delta(y')$ component which was not evident in the studies by Matsuno and Nakamura (1969) and Schoeberl (1980) since it vanishes for the cases they studied. This term corresponds to the interception of the horizontal heat flow by the CL and the convergence of heat over the infinitely thin layer which produces the $\delta(y')$. This term vanishes if $\theta = 0$ because the energy flow is perfectly horizontal so no horizontal heat transport exists. When the CL is horizontal, $\theta = \pi/2$, all the heat flow slips under the CL so the time rate of change in $\bar{T}$ is entirely that due to the induced secondary circulation.

The $\bar{v}$ and $\bar{w}$ fields are obtained from Eqs. (15) and (17):

$$\bar{v} = -\frac{\xi}{f^3}(-l\sin\theta G_1 + G_2\chi\cos\theta)\exp(z^*/2H + lz') \times \exp(z^*/2H + lz') + \frac{2\alpha L_R}{f^3}\xi\sin\theta \times \cos\theta H(-y')\exp(z^*/H + 2\alpha z'L_R) \tag{23}$$

and

$$\bar{w} = \frac{-\xi}{f^2N}[(\tfrac{1}{2}L_R + l\cos\theta)G_1 + \chi\sin\theta G_2]\exp(z^*/2H + lz') + \frac{2\alpha L_R\xi}{f^2N}\sin^2\theta H(-y')\exp(z^*/H + 2\alpha z'L_R). \tag{24}$$

Figs. 4a–4d show the flow field and time rate of change for the variables for $\theta = \pi/4$ and $\alpha = -0.181$ near the CL. Note that the solution is exponentially decaying in the $-y'$ and $+y'$ directions. The tilt of the CL in Fig. 4 in the (y, z^*) coordinates appears the same as the (y', z') system due to the choice of scales for y and z^*.

The fields described by Eqs. (23) and (24) are not, in general, continuous across the CL for any value of θ but their components normal to the CL are continuous. The flow tangential to the CL is discontinuous (except for $\theta = \pi/2$) so $\bar{v}$ and $\bar{w}$ are seen to be discontinuous in Figs. 4b and 4d. The fact that the mean field profiles do not smoothly match across the CL case results from the geometry of the problem which allows the existence of *both* heat and momentum fluxes at the CL. Because the heating or deceleration of the wind along the tilted CL varies with height, a thermal wind stress is set up across the CL. This produces a secondary circulation ($\bar{v}$, $\bar{w}$). The discontinuity in the tangential component of the secondary circulation is due to the jump in the eddy fluxes at the CL. If the flux divergence were spread out over a larger region, the mean field profiles would smoothly match over that region.

The eddy convergence of momentum and heat is no longer totally confined to the CL region when the CL is tilted. This can be seen from Eqs. (19) and (20) where if the fluxes are differentiated, terms appear that have definite value outside the CL. These terms occur because of the exponential character of the wave structure along the CL which gives the horizontal and vertical projections of the EP

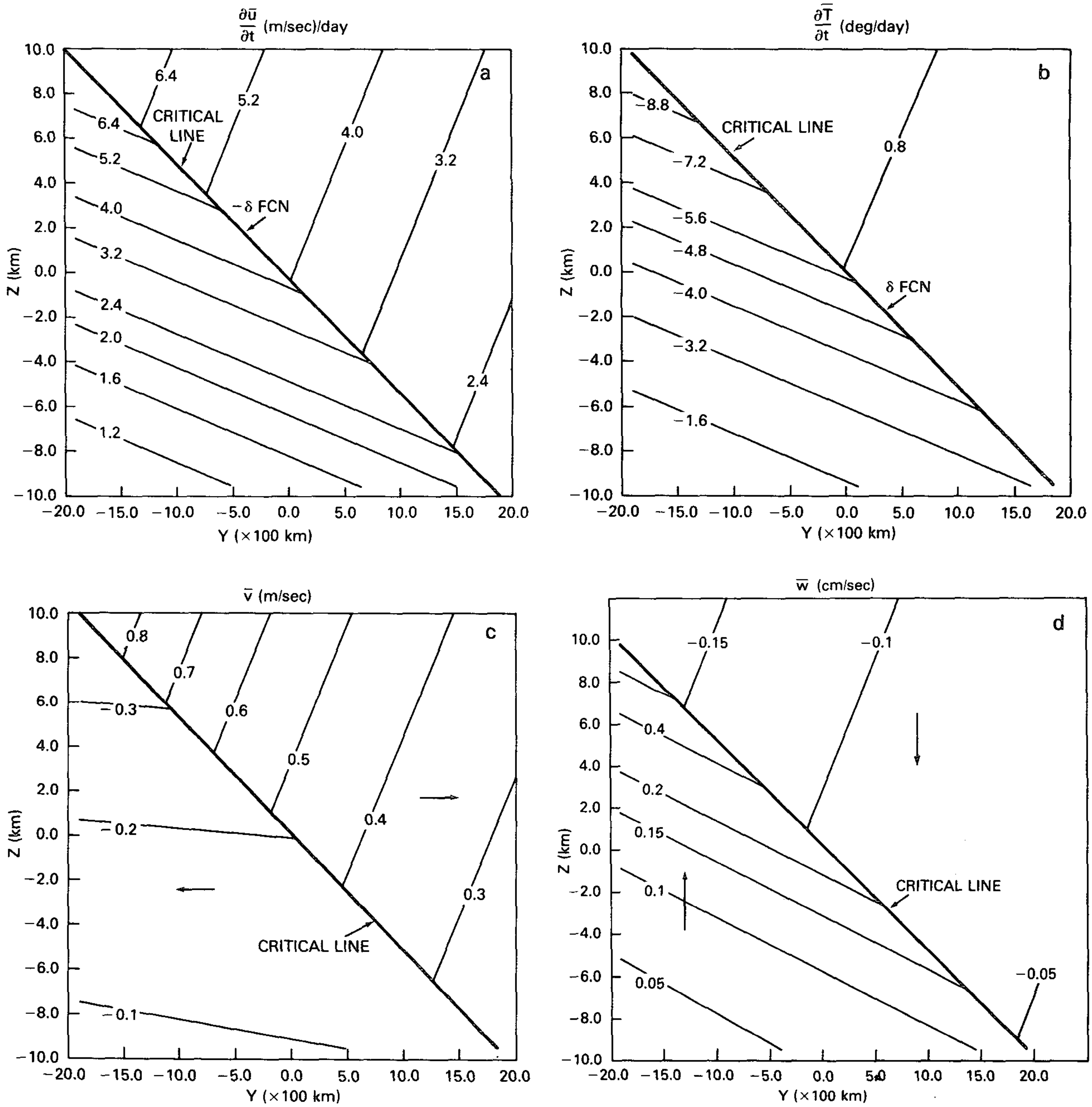

FIG. 4. The variables $\partial\bar{u}/\partial t$ (a), $\partial\bar{T}/\partial t$ (b), $\bar{v}$ (c) and $\bar{w}$ (d) for a baroclinic CL, $\theta = 45°$, $\alpha = -0.181$.

flux an exponential character outside the CL. Both the heat and momentum fluxes are divergent approaching the CL.

c. Physical range for α

Since we are dealing with an unbounded region, we might ask if all values of α are physical. In the stationary critical line assumption there exists the implication that the wave is $O(a)$, where $\bar{u} \sim O(1)$ and $a < 1$. The induced circulations $(\bar{v}, \bar{w})$ are $O(a^2)$. Now if the wave has an exponential behavior such that wave kinetic energy is bounded at infinity then a^2 must be bounded at infinity. For the mean flow kinetic energy induced by the wave to be bounded then it would appear that a^4 must be bounded. This restriction is more severe than bounding a^2, and implies that physically reasonable wave amplitudes induce unphysical secondary flows. A simple resolution to this problem is to require that only the time rate of change of the mean flow kinetic energy remain bounded, i.e., $\frac{1}{2}\rho(\partial\bar{u}^2/\partial t)$ (Matsuno, private communication). This quantity is $O(a^2)$ like the wave amplitude and no inconsistency results. The physical range for α is just those values for which χ^2 is negative and bounding a^2 at infinity implies $\alpha \leq 0$.

d. Lagrangian-mean flow

We may write the Lagrangian mean flow after Matsuno and Nakamura (1979) (see also Andrews and McIntyre, 1978)

$$\bar{w}_L = \bar{w} + \bar{w}_s, \quad \bar{v}_L = \bar{v} + \bar{v}_s,$$

where $\bar{w}_s$ is the Stokes drift due to the presence of waves. $\bar{w}_L$ and $\bar{v}_L$ may be interpreted as the motion of the center of mass of a tube of fluid extending in the x direction which undulates, thickens and stretches due to the presence of finite amplitude waves. Since

$$\bar{w}_s = -\frac{f}{N^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial \phi}{\partial z^*}\frac{\partial \phi}{\partial x}}\right),$$

then using Eq. (17)

$$\bar{w}_L = -\frac{R}{HN^2}\left(\frac{\partial \bar{T}}{\partial t}\right).$$

If diabatic heating is present we also must include the term, Q_L discussed by Dunkerton (1978):

$$\bar{w}_L = -\frac{R}{N^2H}\left(\frac{\partial \bar{T}}{\partial t}\right) + \frac{R}{N^2H}Q_L,$$

where Q_L is the Lagrangian-mean diabatic heating. The quantity $\bar{v}_L$ from Eq. (25) is given by

$$\bar{v}_L = \frac{1}{f}\frac{\partial \bar{u}}{\partial t} - \frac{1}{f^2}\frac{\partial}{\partial y}\left(\overline{\frac{\partial \phi}{\partial y} im\phi}\right) - \frac{1}{N^2}\frac{\xi}{f}\left\{\frac{\partial}{\partial z^*}\left[\frac{\partial \phi}{\partial z^*} im\phi \exp(-z^*/H)\right]\right\}\exp(z^*/H).$$

With a little manipulation, we find that

$$\bar{v}_L = \frac{1}{f}\frac{\partial \bar{u}}{\partial t} + \frac{1}{f^3}[\xi\delta(y')\exp(z^*/H + 2\alpha z' L_R)].$$

We now divide the expression for $\bar{v}_L$ into two parts:

$$\bar{v}_L = A + B\delta(y').$$

The term A represents weak Lagrangian-mean flow along and outside the CL and $B(y')$ represents the Lagrangian-mean jet located at the CL. In a similar fashion we divide $\bar{w}_L$ into two parts

$$\bar{w}_L = C + D\delta(y')$$

and find

$$\left.\begin{aligned} B &= \frac{\xi}{f}(1 - \cos^2\theta)\exp[z'(\cos\theta L_R + 2\alpha L_R)] \\ D &= \frac{\xi}{f^2N}\sin\theta\cos\theta\exp[z'(\cos\theta L_R + 2\alpha L_R)] \end{aligned}\right\}.$$

Since $\bar{w}_L$ and $\bar{v}_L$ are in the non-stretched reference frame we multiply $\bar{w}_L$ by f/N to put these variables into the (y', z') coordinate system.

$$\left.\begin{aligned} \bar{v}_L^\delta &= \frac{\xi}{f^3}\sin^2\theta\delta(-y') \times \exp[z'(L_R\cos\theta + 2\alpha L_R)] \\ \bar{w}_L^\delta &= \frac{\xi}{f^3}\sin\theta\cos\theta\delta(-y') \times \exp[z'(L_R\cos\theta + 2\alpha L_R)] \end{aligned}\right\}, \quad (26)$$

or we may write

$$V^\delta = (\bar{v}_L^{\delta 2} + \bar{w}_L^{\delta 2})^{1/2} = \frac{\xi}{f^3}\sin\theta \exp z'(L_R\cos\theta + \alpha L_R),$$

where V^δ is the magnitude of the Lagrangian jet and this jet is perfectly aligned with the CL.

The fact that Lagrangian jet vanishes as the critical line moves to a vertical position is consistent with the solutions found by Schoeberl (1980). The $\sin\theta$ variation in the jet magnitude is due to the contribution of the vertical wave energy flow, which is directly proportional to the horizontal components of the Stokes drift which varies as $\sin\theta$ as the CL is tilted. The δ function in $\partial\bar{T}/\partial t$ at the CL found in Eq. (22) can be understood in the Lagrangian-mean framework as the direct manifestation of the tilted Lagrangian jet. The Lagrangian jet moves the air parcels downward along the CL infinitely fast which produces infinite heating. When the jet is horizontal no Lagrangian-mean vertical motion occurs at the CL so no singular times rate of change in the temperature occurs.

The transport of a conservative tracer in the Lagrangian mean formulation is given by

$$\frac{\partial \mu_L}{\partial t} + \mathbf{V}_L\cdot\nabla\mu_L = 0,$$

where $\bar{\mu}_L \approx \bar{\mu} + (\overline{\boldsymbol{\zeta}\cdot\nabla\mu}) + \frac{1}{2}(\overline{\zeta_i\zeta_j\partial_i\partial_j\bar{\mu}})$ where i, j represent the components of $\boldsymbol{\zeta}$ (Andrews and McIntyre, 1978). Outside the CL

$$\nabla\cdot\mathbf{V}_L = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{\zeta} = 0,$$

where $\boldsymbol{\zeta}$ is the position vector indicating the displacement of the air parcels from their Lagrangian mean position by the planetary wave. Since

$$\bar{u}_0\frac{\partial\boldsymbol{\zeta}}{\partial x} = \mathbf{V}' + (\boldsymbol{\zeta}\cdot\nabla\bar{u}_0)_x,$$

(Matsuno and Nakamura, 1979) even though the wave amplitude ϕ is defined at the CL, $\boldsymbol{\zeta}$ may become singular. Thus, the actual location of an air parcel is unknown at the CL and there may be some question as to how the parcel moves. If damping is introduced in the problem then some of the difficulties described above are avoided. The displacement vectors are still singular at the CL but the Lagrangian mean velocity becomes finite and diffused around the CL.

The incompressibility of the Lagrangian-mean flow is a direct result of the stationary CL assumption and the steadiness of the planetary waves. In reality, the CL would move rapidly with the velocity roughly given by

$$\left(\frac{\partial \bar{u}}{\partial t} \bigg/ \frac{\partial \bar{u}_0}{\partial y'}\right)_{y'=0}.$$

The divergence of the Lagrangian-mean flow could not be neglected in this case. Therefore care must be taken in applying the results of this calculation to real critical line motion.

The phenomenon of Lagrangian jets provides a possible mechanism for rapid transport of constituents from stratosphere to the troposphere and within the stratosphere. For example, consider a stationary CL as pictured in Fig. 5 which twists around at the tropopause. The average position of a material string of particles imbedded in the CL described above will move with the jet. If the CL changes its angle with the vertical as in Fig. 5 then the Lagrangian mean positions of particles will converge along the CL if the angle decreases and diverge if the angle increases. Thus the CL shown in Fig. 5 would transport particles from the stratosphere and deposit them in the troposphere. More likely though, during a sudden warming, some air parcels will be accelerated along the CL for a short period as the CL passes and convergence or divergence will occur where the CL changes its angle (the cusps). The remarks above rest on the assumption that a curved CL can be treated as a piecewise fashion connecting the solutions for a CL at an arbitrary angle. The EP flux is always assumed normal to the curved CL, and the divergent Lagrangian-mean circulation is small compared to the solenoidal flow computed here.

e. The 1976/77 sudden stratospheric warming

In the sudden warming event of 1976/77 discussed by O'Neill and Taylor (1979), the polar stratospheric temperature increase occurred in the presence of a nearly vertical CL. About 7 January easterlies in the mean zonal flow appear at 20 mb and advance southward to 58°N in the period 7–12 January. During this period the divergence southward momentum flux associated with the $m = 1$ planetary wave is strong, highly organized and localized north of 60° (O'Neill and Taylor, Fig. 7a). The magnitude of the momentum flux divergence is about 5 m s^{-1} day^{-1}. The net change in the mean zonal wind is ~30 m s^{-1} from 5–10 January in this region.

The temperature change in the mean zonal flow over the same period is ~2–3 K day^{-1} at 20 mb from O'Neill and Taylor's (1979) Fig. 3c while the northward heat flux convergence for $m = 1$ from their Fig. 7c would produce ~2 K day^{-1}. The heat flux convergence for $m = 2$ is negligible. Thus if all the polarward heat flux by the planetary $m = 1$ is converted into a local temperature increase and all the momentum flux convergence is used to decelerate the mean zonal flow the dynamics of this warming event can be explained. The properties of a tilted CL provide exactly such a local mechanism by which heat and

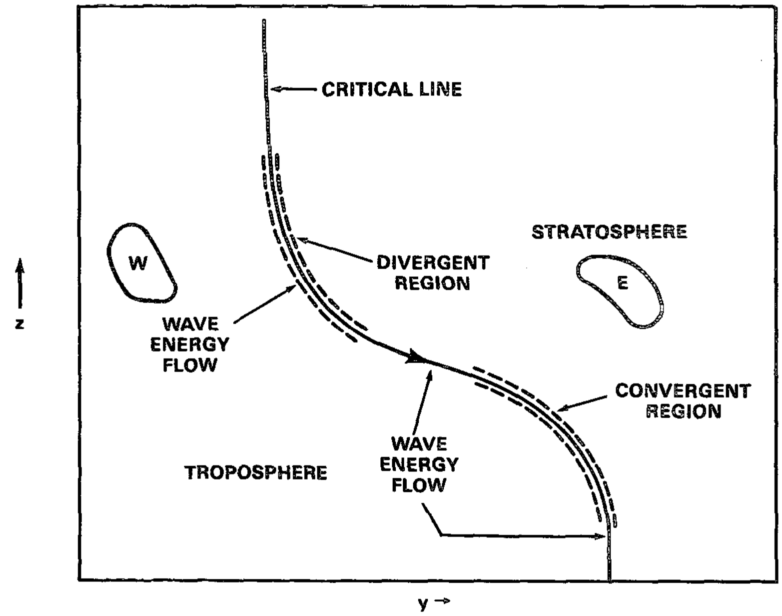

FIG. 5. Undulating CL and a diagram of associated Lagrangian-mean flow.

momentum flux are converted directly into acceleration and heating of the mean zonal flow from Eq. (21) and Eq. (22). A CL which is either vertical or horizontal would also produce local changes in the mean flow but not changes consistent with the 1976/77 warming event. For example, the perfectly vertical CL produces only broad (~1000 km) patterns of heating and cooling outside the CL region while the eddy momentum flux divergence decelerates the flow at the CL (Schoeberl, 1980). Likewise, in the horizontal CL, deceleration of the mean zonal flow in the vicinity of the CL and the regions of heating and cooling are spread out over a scale height centered on the CL (Matsuno and Nakamura, 1969). Thus, the vertically and horizontally aligned CL's do not appear to have properties consistent with observations of the 1976/77 sudden warming. Further, O'Neill and Taylor's Fig. 4a indicates on 10 January at 20 mb the CL was observed to have a tilt from vertical.

5. Conclusions

This paper has examined a few aspects of CL which is tilted from the vertical. Two important properties have been emphasized. First, the deceleration of the zonal mean flow along the CL is singular except when the CL is horizontal and maximum when the CL is vertical. Second, the tilted CL can directly tap the eddy heat transport by the wave producing a singular time rate of change in the zonally averaged temperature along the CL. The fact that neither Matsuno and Nakamura (1979) nor Schoeberl (1980) noted this thermal property of the CL is due to their choice of alignment. By definition, waves interacting with a vertically aligned CL transport no heat, and a horizontally aligned CL is parallel to the heat flow. As discovered by Schoeberl (1980), two classes of zonally averaged flow exist outside the CL no matter what its orientation. The mean flow can have a sinusoidal spatial behavior with no flow on the opposite side or the mean flow may exist on both sides of the CL with an exponential spatial character. The former flow field, however, is due to boundary effects and is not realistic. These flow regimes are controlled by the assumed wave structure along the CL which produces a thermal wind stress in a baroclinic atmosphere.

Even though the treatment of the CL is highly idealized in this paper, there is evidence which indicates a very large rate of change in the zonally averaged temperature along a CL may occur. In the sudden warming event of 1976/77 discussed by O'Neill and Taylor (1979), the sudden increase in temperature in the mean flow occurred in the presence of a nearly vertical CL. Schoeberl (1980) discussed a perfectly vertical CL showing that the secondary circulation around the CL can produce warming and cooling regions, but the large variation in size of these regions with α does not lend clear support for this description of the 1976/77 warming. A better explanation is provided if the CL is tilted slightly and the rate of change in $\bar{T}$ is directly provided by the convergence of the eddy heat flow in the highly localized region around the CL.

The Lagrangian mean flow also is singular along the critical line and parallel to it. However, the strength of the flow is proportional to the sine of the angle the critical line makes with the vertical and vanishes when the line is perfectly vertical. The Lagrangian mean position particles imbedded in the CL thus moves rapidly along the surface collecting or dispersing as the CL curves and recurves. The Lagrangian jets may provide an important transport process for exchange of stratospheric and tropospheric air.

Acknowledgments. This research was supported by NASA through Ames Research Center Contract A-47997B(DA) and the Office of Naval Research. The author would also like to thank Dr. D. F. Strobel for comments and H. M. Mitchell and R. S. Lindzen for helpful discussions. Special thanks go to Taroh Matsuno and Michael McIntyre for an exhaustive review of the manuscript.

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