

N-Stream Approximations to Radiative Transfer

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ABSTRACT

Schuster's two-stream approximation is first derived from Chandrasekhar's radiative transfer equation, and then generalized to an arbitrary number of streams. The resulting technique for solving the transfer equation that is similar to the discrete ordinate and spherical harmonic methods, is found to be especially useful for modeling atmospheres with complicated phase functions and moderate optical depths. To illustrate the method, a four-stream approximation is evaluated for a Henyey-Greenstein phase function with asymmetry factor $g = 0.5$.

1. Introduction

Two-stream approximations provide fast, conceptually simple methods for handling many radiative transfer problems. These approximations, which stem from the early work of Schuster (1905), have been applied to problems in atmospheric science such as radiation budget calculations (Rasool and Schneider, 1971; Lacis and Hansen, 1974), and to problems in space science such as planetary albedo calculations (Sagan and Pollack, 1967). Since accurate solutions to the radiative transfer equations (Chandrasekhar, 1950) were developed much later than the first two-stream approximation, such applications of two-stream techniques have led to questions concerning the soundness of these methods (Lyzena, 1973), and to suggestions for improving the approximations (Coakley and Chylek, 1975; Wiscombe and Grams, 1976; Meador and Weaver, 1980).

Two-stream approximations are more precisely two *parameter* approximations to the radiative transfer equation in which the intensity is assumed to be azimuthally independent and to have a polar angle dependence specified by two parameters. There are three types of two-stream approximations, each with a characteristic restriction on the variation of the intensity with $\mu = \cos\theta$:

(i) Schuster approximations where the two parameters are $I^+(\tau)$ and $I^-(\tau)$, with the angular dependence given by $I(\tau, \pm\mu) = I^\pm(\tau)$, $\mu > 0$.

(ii) Eddington approximations where the two parameters are $I_0(\tau)$ and $I_1(\tau)$, with the angular dependence given by $I(\tau, \mu) = I_0(\tau) + \mu I_1(\tau)$.

(iii) Quadrature approximations where the two parameters are $I(\tau, \mu_1)$ and $I(\tau, \mu_2)$, μ_1 and μ_2 being preselected points according to some quadrature rule. [The angular dependence of I is obtained by first determining the source function $J(\tau)$ by quad-

rate and then integrating over this source function (see Chandrasekhar, 1950, Sections 9–13).] In addition to restricting the μ -dependence of the intensity, further assumptions are usually made concerning the behavior of the phase function. For example, in dealing with clouds it is convenient to approximate the phase function by a delta function in the forward direction superimposed upon an isotropic background. (This approximation and others are discussed by van de Hulst, 1980.)

Each type of two-stream approximation can be viewed as a lowest order solution to the transfer equation. In order to gauge the accuracy of each approximation, or to obtain more detailed information (e.g., the azimuthal dependence of the intensity), it is necessary to go to higher order solutions.

Consider the generalization of each of the above types of approximations to more than two parameters. The generalization of (iii), the quadrature approximation, to more than two values of μ is immediate and yields the discrete ordinate method (Chandrasekhar, 1950) if the values μ_i are selected according to the Gaussian quadrature rule. The generalization of (ii) is less well known. However, if the intensity is expanded in terms of Legendre polynomials

$$I(\tau, \mu) = \sum_{l=0}^{\infty} I_l(\tau) P_l(\mu),$$

then the Eddington approximation can be seen to be a special case of the spherical harmonic method (Kourganoff, 1952). This paper develops the generalization of (i), Schuster's approximation, to n streams, providing a method that offers significant advantages over the other generalizations for problems involving complicated phase functions, such as might arise in Mie theory.

To generalize Schuster's two-stream approxi-

mation to n streams, it is important to understand the relationship between this approximation, which was first used in 1905, and the exact transfer equation (see Chandrasekhar, 1950). In fact, it is not widely recognized that Schuster's approximation can be derived from the exact transfer equation in a mathematically rigorous fashion. This paper presents such a derivation, and then generalizes the approximation to n streams. To illustrate the use of our n -stream approximation, a particular four-stream approximation is evaluated and compared with the two-stream approximation, demonstrating the dependence of the intensity on azimuthal angle to lowest order.

2. Derivation of Schuster's two-stream approximation

In Schuster's approximation, the intensity is assumed to be constant over each hemisphere:

$$I(\tau, \mu, \phi) = \begin{cases} I^+(\tau), & \text{if } \mu > 0 \\ I^-(\tau), & \text{if } \mu < 0. \end{cases} \quad (1)$$

The image evoked by the name *two-stream* is misleading in this case. Usually, a *stream* is pictured as a pencil of radiation travelling in a particular direction, and for an n -stream approximation one might expect the intensity to be written as $I(\tau, \mu, \phi) = \sum I^i(\tau) \delta(\mu - \mu_i) \delta(\phi - \phi_i)$. That image does not apply to Schuster's two-stream approximation where the intensity as defined by (1) is constant over an entire hemisphere and involves no preferred directions. Schuster's differential equations follow from the exact integrodifferential equation that describes the transport of the diffuse specific intensity through a plane parallel atmosphere (Chandrasekhar, 1950, p. 22):

$$\begin{aligned} \mu \frac{dI(\tau, \mu, \phi)}{d\tau} &= I(\tau, \mu, \phi) - (1/4\pi) \int_{-1}^1 \int_0^{2\pi} p(\mu', \phi' \rightarrow \mu, \phi) \\ &\times I(\tau, \mu', \phi') d\mu' d\phi' \\ &\quad - 1/4 Fe^{-\tau/\mu_0} p(\mu_0, \phi_0 \rightarrow \mu, \phi). \end{aligned} \quad (2)$$

This equation is used to solve the *standard problem* in radiative transfer shown in Fig. 1. Application of (1) yields two coupled linear differential equations

$$\begin{aligned} \mu \frac{dI^+(\tau)}{d\tau} &= I^+(\tau) - \frac{I^+(\tau)}{4\pi} \int_0^1 \int_0^{2\pi} p(\mu', \phi' \rightarrow \mu, \phi) d\mu' d\phi' \\ &\quad - \frac{I^-(\tau)}{4\pi} \int_{-1}^0 \int_0^{2\pi} p(\mu', \phi' \rightarrow \mu, \phi) d\mu' d\phi' \\ &\quad - 1/4 Fe^{-\tau/\mu_0} p(\mu_0, \phi_0 \rightarrow \mu, \phi), \end{aligned} \quad (3a)$$

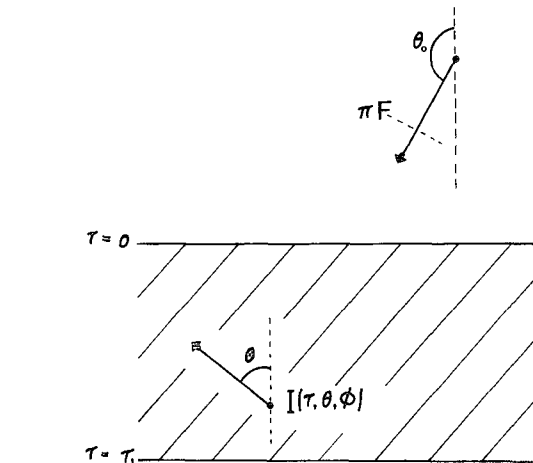


FIG. 1. The standard problem in radiative transfer. A parallel beam of radiation with net flux πF per unit area (normal to itself) is incident on top of a plane parallel atmosphere of optical depth τ_1 in a direction with polar angle θ_0 and azimuthal angle ϕ_0 . We must determine the intensity $I(\tau, \mu, \phi)$ at any optical depth τ , $0 \leq \tau \leq \tau_1$, and in any direction specified by θ, ϕ where $\mu = \cos\theta$. The azimuthal angles ϕ and ϕ_0 are not shown in this diagram.

$$\begin{aligned} \mu \frac{dI^-(\tau)}{d\tau} &= I^-(\tau) - \frac{I^+(\tau)}{4\pi} \int_0^1 \int_0^{2\pi} p(\mu', \phi' \rightarrow \mu, \phi) d\mu' d\phi' \\ &\quad - \frac{I^-(\tau)}{4\pi} \int_{-1}^0 \int_0^{2\pi} p(\mu', \phi' \rightarrow \mu, \phi) d\mu' d\phi' \\ &\quad - 1/4 Fe^{-\tau/\mu_0} p(\mu_0, \phi_0 \rightarrow \mu, \phi), \end{aligned} \quad (3b)$$

where $\mu > 0$ for (3a) and $\mu < 0$ for (3b). The integrals above that give the average value of the phase function over each hemisphere are closely related to the backscattered and forward-scattered fractions defined by

$$\beta(\mu, \phi) = \frac{1}{4\pi\bar{\omega}_0} \int_{-1}^0 \int_0^{2\pi} p(\mu', \phi' \rightarrow |\mu|, \phi) d\mu' d\phi', \quad (4)$$

$$f(\mu, \phi) = \frac{1}{4\pi\bar{\omega}_0} \int_0^1 \int_0^{2\pi} p(\mu', \phi' \rightarrow |\mu|, \phi) d\mu' d\phi', \quad (5)$$

where $\bar{\omega}_0$ is the albedo for single scattering, and $\beta + f = 1$. Using these definitions, (3a) and (3b) become

$$\begin{aligned} \mu \frac{dI^\pm(\tau)}{d\tau} &= I^\pm(\tau) - \bar{\omega}_0 f(\mu, \phi) I^\pm(\tau) - \bar{\omega}_0 \beta(\mu, \phi) I^\mp(\tau) \\ &\quad - 1/4 Fe^{-\tau/\mu_0} p(\mu_0, \phi_0 \rightarrow \mu, \phi). \end{aligned} \quad (6)$$

In (6), the upper sign applies when $\mu > 0$ and the lower sign applies when $\mu < 0$. We can eliminate

the dependence on the angles θ and ϕ by averaging over each hemisphere (i.e., integrating over a hemisphere and dividing by 2π):

$$\pm \frac{1}{2} \frac{dI^\pm(\tau)}{d\tau} = I^\pm(\tau) - \bar{\omega}_0 \langle f \rangle I^\pm(\tau) - \bar{\omega}_0 \langle \beta \rangle I^\pm(\tau) - \frac{1}{2} \bar{\omega}_0 F e^{-\tau/\mu_0} \left\{ \begin{matrix} \beta(\mu_0, \phi_0) \\ f(\mu_0, \phi_0) \end{matrix} \right\}, \quad (7)$$

where $\langle \beta \rangle$ and $\langle f \rangle$ are the average backscatter and forward-scatter fractions

$$\langle \beta \rangle = (1/2\pi) \int_0^1 \int_0^{2\pi} \beta(\mu, \phi) d\mu d\phi, \quad (8)$$

$$\langle f \rangle = (1/2\pi) \int_0^1 \int_0^{2\pi} f(\mu, \phi) d\mu d\phi. \quad (9)$$

The pair of equations given by (7) is valid for arbitrary phase functions. For isotropic scattering, it is equivalent to the pair used by Schuster. Our derivation, which closely parallels that of Coakly and Chýlek (1975), is rigorous. Furthermore, the factor of $1/2$ on the left-hand side of (7) comes from the definite integration $\int_0^1 \mu d\mu$; it is thus more than a mean value of μ selected "to allow for the mean obliquity of the rays to the outward (inward) direction" (Chandrasekhar, 1950, pg. 55). This last point must be emphasized. Following Chandrasekhar's suggestion, Sagan and Pollack (1967), Lyzenga (1973), Lacis and Hansen (1974) and others picked another mean value for μ , viz., $\langle \mu \rangle = 1/\sqrt{3}$ and simply replaced factors of $1/2$ in (7) with $1/\sqrt{3}$. While these adjusted two-stream approximations may be fairly accurate for certain calculations, it should be realized that in this development, the choice $\langle \mu \rangle = 1/2$ is not an arbitrary one. Moreover, in our approximation scheme energy is conserved, and this is not the case for quadrature methods.

3. Calculation of the scatter fractions

In most cases, the phase function depends only on the cosine of the angle Θ between the incident direction (θ', ϕ') and the scattering direction (θ, ϕ) :

$$\cos\Theta = \mu\mu' + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\phi' - \phi). \quad (10)$$

On expanding the phase function in Legendre polynomials

$$p(\cos\Theta) = \sum_{l=0}^{\infty} \bar{\omega}_l P_l(\cos\Theta), \quad (11)$$

the integrations in (4) and (5) appear rather complicated since the integrands depend only implicitly on the integration variables μ' and ϕ' . Chandrasekhar (1950, p. 150) has shown that this difficulty can be

overcome by applying the addition theorem for spherical harmonics which leads to the result

$$p(\cos\Theta) = \sum_{l=0}^{\infty} \frac{4\pi\bar{\omega}_l}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \quad (12)$$

The integrations in (4) and (5) are performed with the aid of the stepfunctions $H^\pm(\mu)$ which mask the upper and lower hemispheres:

$$H^+(\mu) = \begin{cases} 1, & \text{if } \mu > 0 \\ 0, & \text{if } \mu < 0 \end{cases} \quad (13a)$$

$$H^-(\mu) = \begin{cases} 0, & \text{if } \mu > 0 \\ 1, & \text{if } \mu < 0. \end{cases} \quad (13b)$$

Concentrating on the evaluation of β with $\mu > 0$, (4) is rewritten as

$$\beta(\mu, \phi) = (1/4\pi\bar{\omega}_0) \times \int_{-1}^1 \int_0^{2\pi} H^-(\mu') p(\cos\Theta) d\mu' d\phi'. \quad (14)$$

Expressing $H^-(\mu)$ in terms of Legendre polynomials

$$H^-(\mu) = \sum_{l=0}^{\infty} h_l^- P_l(\mu), \quad (15)$$

where the coefficients are given by

$$\begin{aligned} h_0^- &= 1/2, \quad h_{2n}^- = 0, \\ h_{2n-1}^- &= (-1)^n (2n - 1/2) \frac{(2n - 3)!!}{(2n)!!}; \\ n &= 1, 2, 3, \dots, \end{aligned} \quad (16)$$

Eqs. (15) and (12) are substituted into (14), and the integration is performed by using the orthogonality of the Legendre polynomials yielding

$$\begin{aligned} \beta(\mu, \phi) = \beta(\mu) &= \frac{1}{\omega_0} \sum_{l=0}^{\infty} \frac{h_l^- \bar{\omega}_l}{2l+1} P_l(|\mu|), \\ -1 &\leq \mu \leq 1. \end{aligned} \quad (17)$$

A similar result was obtained in a different way by Wiscombe and Grams (1976), however, to obtain it they used the azimuthally averaged phase function $p^{(0)}(\mu' \rightarrow \mu)$. Such averaging is avoided here, since Schuster's approximation will be extended to more than two streams where azimuthal dependence is important. (Of course, if the net flux is the only parameter of interest, it is not necessary to consider azimuthal dependences).

From (8) and (17) we find that the average value of β is given by

$$\langle \beta \rangle = \frac{1}{2} - \frac{1}{2\bar{\omega}_0} \sum_{l=1}^{\infty} \bar{\omega}_{2l-1} \left[\frac{(2l-3)!!}{(2l)!!} \right]^2. \quad (18)$$

Values for the forward scatter fractions are obtained from $f(\mu) = 1 - \beta(\mu)$ and $\langle f \rangle = 1 - \langle \beta \rangle$.

4. *N*-stream approximation

In the spirit of Schuster's two-stream approximation, we divide the surface of the unit sphere into *n* nonoverlapping patches, and assume that the intensity $I(\tau, \mu, \phi)$ is constant on each patch, i.e.,

$$I(\tau, \mu, \phi) = \begin{cases} I^{(1)}(\tau), & \text{if } \mu \in \{\mu^{(1)}\} \text{ and } \phi \in \{\phi^{(1)}\} \\ I^{(2)}(\tau), & \text{if } \mu \in \{\mu^{(2)}\} \text{ and } \phi \in \{\phi^{(2)}\} \\ \vdots \\ I^{(n)}(\tau), & \text{if } \mu \in \{\mu^{(n)}\} \text{ and } \phi \in \{\phi^{(n)}\}. \end{cases} \quad (19)$$

Each patch in (19) is described by a range in $\mu \in \{\mu^{(i)}\}$, and a range in $\phi \in \{\phi^{(i)}\}$. It is especially convenient to consider patches separated by either latitudinal (constant μ) or longitudinal (constant ϕ) lines. Schuster's approximation is the special case $n = 2$ where the ranges are $\{\mu^{(1)}\} = (0, 1)$, $\{\mu^{(2)}\} = (-1, 0)$, $\{\phi^{(1)}\} = \{\phi^{(2)}\} = (0, 2\pi)$. In general, the patches will be described by the functions $H^i(\mu, \theta)$ where

$$H^i(\mu, \phi) = \begin{cases} 1, & \text{if } \mu \in \{\mu^{(i)}\} \text{ and } \phi \in \{\phi^{(i)}\} \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

(In our notation, superscripts denote stream parameters and subscripts denote Legendre polynomial indices.) For convenience, the surface area of the *i*th patch will be defined by

$$A^i = \int_{-1}^1 \int_0^{2\pi} H^i(\mu, \phi) d\mu d\phi. \quad (21)$$

The intensities given by (19), the patch functions given by (20), and the areas given by (21) can each be considered elements of the *n*-dimensional vectors **I**, **H** and **A**. Now, when $n > 2$ we no longer have only the forward and back scatter fractions: we must consider scattering from all the directions (μ', ϕ') lying in any one patch *j*, into any single direction (μ, ϕ) lying inside the *i*th patch. These scatter fractions may be expressed as elements of an $n \times n$ matrix **S** where

$$S^{ij}(\mu, \phi) = (\frac{1}{4}\pi\bar{\omega}_0) \int_{-1}^1 \int_0^{2\pi} H^j(\mu', \phi') \times p(\mu', \phi' \rightarrow \mu\phi) d\mu' d\phi' H^i(\mu, \phi). \quad (22)$$

In Schuster's two-stream approximation, $S^{11} = S^{22} = f$ and $S^{12} = S^{21} = \beta$. Using (21), the average value of each of these matrix elements is

$$\langle S^{ij} \rangle = \int_{-1}^1 \int_0^{2\pi} S^{ij}(\mu, \phi) d\mu d\phi / A^j. \quad (23)$$

Finally, the transfer equation (2) is approximated by a system of *n* coupled differential equations:

$$\frac{1}{A^i} \int_{-1}^1 \int_0^{2\pi} \mu H^i(\mu, \phi) d\mu d\phi \frac{dI^i(\tau)}{d\tau} = I^i(\tau) - \bar{\omega}_0 \sum_j \langle S^{ij} \rangle I^j(\tau) - \frac{\pi F \bar{\omega}_0}{A^i} e^{-\tau/\mu_0} \sum_j S^{ji}(\mu_0, \phi_0). \quad (24)$$

The reversal of the indices *i* and *j* in the last term of (24) comes from an application of the reciprocity principle $p(\mu_0, \phi_0 \rightarrow \mu, \phi) = p(\mu, \phi \rightarrow \mu_0, \phi_0) = p(\cos\Theta)$. As in the two-stream approximation, the success of this *n*-stream approximation hinges on our ability to perform the integrations in (22) and (23) for the scatter fractions. As before, the procedure involves expanding the patch functions in orthogonal polynomials, substituting this expansion with (12) into (22), and finally evaluating the integral analytically. Since the patch functions depend on both θ and ϕ , an expansion in spherical harmonics rather than Legendre polynomials is required:

$$H^i(\mu, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l h_{lm}^i Y_{lm}(\mu, \phi). \quad (25)$$

Following the same procedure as in the two-stream derivation, (25) and (12) are substituted into (22). After integrating over μ' and ϕ' with the aid of the orthogonality of the spherical harmonics, the resulting scatter fractions are

$$S^{ij}(\mu, \phi) = \frac{1}{\omega_0} \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\bar{\omega}_l}{2l+1} h_{lm}^i Y_{lm}(\mu, \phi) H^j(\mu, \phi). \quad (26)$$

This result, which is the generalization of (17) for an arbitrary number of streams, shows that the scatter fractions can be obtained easily for any phase function.

Before applying (24) and (26) to the particular case of a four-stream model, it is important to emphasize that realistic phase functions (such as those generated from Mie theory) can be handled easily with this *n*-stream method. This advantage comes from the procedure of integrating the actual phase function over an individual patch, rather than picking a set of values μ_i and/or ϕ_i a priori, and then evaluating the phase function at these values only.

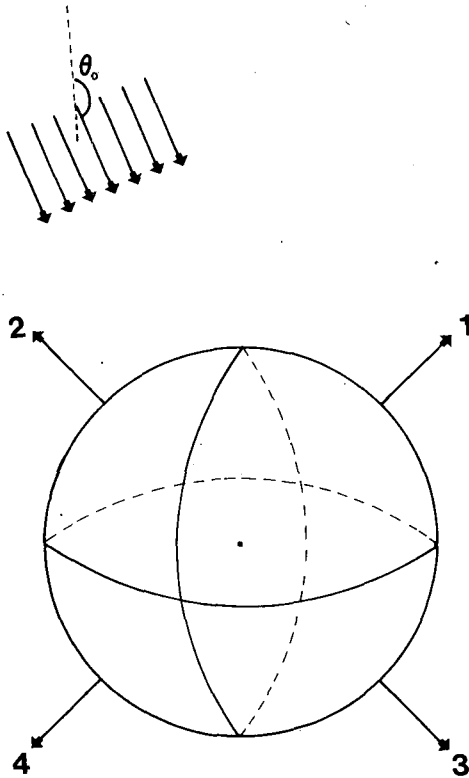


FIG. 2. A four-stream model. The unit sphere is quartered into four regions of equal surface area. The incident radiation consists of a parallel beam with $\pi/2 < \theta_0 \leq \pi$ and $\phi_0 = \pi/2$.

5. Four-stream approximation

As an example of the role of the azimuth angle ($\phi' - \phi$) in the overall scattering problem, consider the four-stream approximation where the unit sphere is divided into four quadrants as shown in Fig. 2. Following the method outlined in the previous section, the patch functions given in (20) are

$$\begin{aligned}
 H^{(1)}(\mu, \phi) &= \begin{cases} 1, & \text{if } 0 < \mu \leq 1 \text{ and } 0 \leq \phi < \pi \\ 0, & \text{otherwise} \end{cases} \\
 H^{(2)}(\mu, \phi) &= \begin{cases} 1, & \text{if } 0 < \mu \leq 1 \text{ and } \pi \leq \phi < 2\pi \\ 0, & \text{otherwise} \end{cases} \\
 H^{(3)}(\mu, \phi) &= \begin{cases} 1, & \text{if } -1 \leq \mu < 0 \text{ and } 0 \leq \phi < \pi \\ 0, & \text{otherwise} \end{cases} \\
 H^{(4)}(\mu, \phi) &= \begin{cases} 1, & \text{if } -1 \leq \mu < 0 \text{ and } \pi \leq \phi < 2\pi \\ 0, & \text{otherwise.} \end{cases} \quad (27)
 \end{aligned}$$

Since the area of each patch is π , $A^{(1)} = A^{(2)} = A^{(3)} = A^{(4)} = \pi$.

To illustrate the azimuthal dependence, we will compare the scatter fractions S^{ij} of this four-stream model with β and f , the scatter fractions of the two-stream model for a Henyey Greenstein phase func-

tion with $g = 0.5$. The Henyey Greenstein phase function

$$p(\cos\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g \cos\Theta)^{3/2}} \quad (28)$$

is chosen for the simplicity of its Legendre coefficients

$$\bar{\omega}_l = (2l + 1)g^l. \quad (29)$$

This comparison assumes that the incident radiation lies in the third quadrant, $\mu_0 < 0$, with $\phi_0 = \pi/2$ (see Fig. 2). The four fractions S^{13} , S^{23} , S^{33} and S^{43} are evaluated using (26), and the results are plotted in Fig. 3 as functions of μ_0 . For this choice of g , the sums in (26) were seen to converge when the

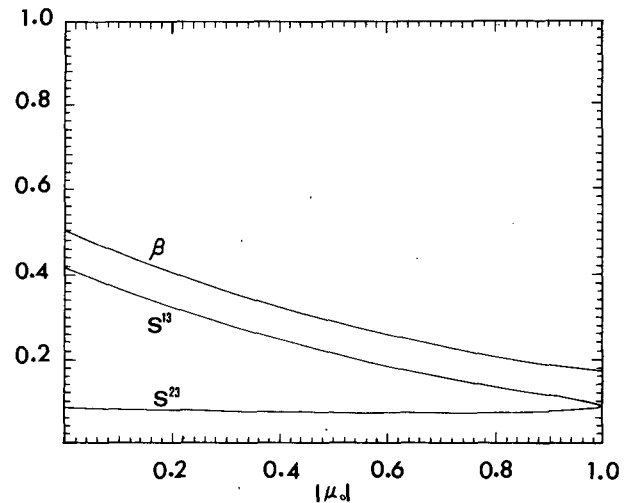
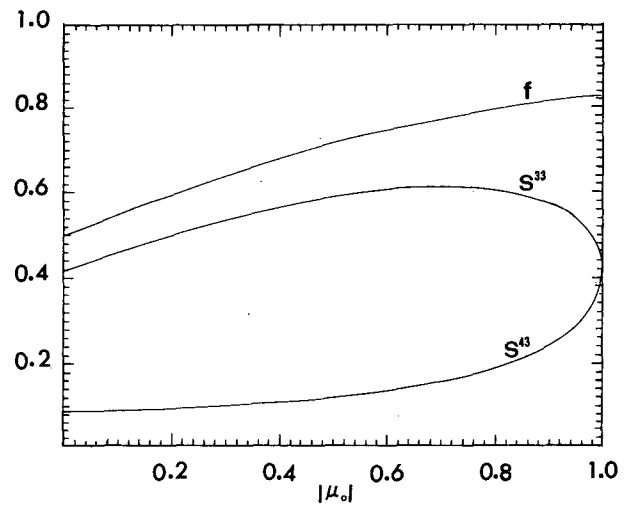


FIG. 3. A comparison of the four-stream scatter fractions S^{13} , S^{23} , S^{33} and S^{43} with the two-stream scatter fractions β and f for a Henyey-Greenstein phase function with $g = 0.5$. The azimuthal angle of the source, ϕ_0 , is chosen to be $\pi/2$. The forward scatter fractions f , S^{33} and S^{43} are plotted in Fig. 3a as functions of the cosine of the solar zenith angle μ_0 , while the backscatter fractions β , S^{13} and S^{23} are plotted in Fig. 3b.

spherical harmonic degree reached $l = 6$. These are to be compared with the fractions β and f of the two-stream model also shown in Fig. 3. (Our two-stream calculations are in agreement with those tabulated by van de Hulst, 1980, p. 307.) As expected, $\beta(\mu_0) = S^{13}(\mu_0) + S^{23}(\mu_0)$ and $f(\mu_0) = S^{33}(\mu_0) + S^{43}(\mu_0)$. Note that the differences $S^{13} - S^{23}$ and $S^{33} - S^{43}$ that measure the azimuthal dependence of the phase function are strongly dependent on the value of ϕ_0 . From symmetry, it follows that $S^{13} = S^{23} = \beta/2$ and $S^{33} = S^{43} = f/2$ when $\phi_0 = 0$. Differences between S^{13} and S^{23} and between S^{33} and S^{43} are most pronounced for $\phi_0 = \pi/2$. Finally, in view of the symmetric way in which the patch functions were chosen, Fig. 3 shows that $S^{13}(0, \phi_0) = S^{33}(0, \phi_0) = S^{33}(1, \phi_0) = S^{43}(1, \phi_0) = f(1)/2$, and $S^{43}(0, \phi_0) = S^{23}(0, \phi_0) = S^{23}(1, \phi_0) = S^{13}(1, \phi_0) = \beta(1)/2$. These relationships persist for all values of ϕ_0 . For larger values of g , differences between the scatter fractions grow more pronounced, and the sums given by (26) converge more slowly. For example, when $g = 0.85$ our calculations show that these sums must be evaluated to $l = 16$.

6. Discussion

In this paper, Schuster's two-stream approximation has been extended to an arbitrary number of streams. A cursory comparison of (24) and (26) with the analogous equations of similar approximations [discrete ordinate method (Chandrasekhar, 1950), spherical harmonic method (Kourganoff, 1952), six-stream approximation (Chu and Churchhill, 1955), and the 12-stream approximation (Whitney, 1972)] suggests several advantages of the method under consideration. First, the resulting set of n coupled linear differential equations of this approximation are considerably simpler than those found in the other approximations, leading to the expectation that this method will consume substantially less computer time for the same required accuracy. Second, our n -stream approximation is flexible with respect to the choice of the patch functions. Thus, unequal patches may be chosen to maximize the computational efficiency in a particular application. Third, no approximations are made on the form of the phase function, other than its expansion in

Legendre polynomials. Unlike the quadrature method or the spherical harmonic method, all integrals involving the phase function are evaluated analytically. Since the details of the phase function are important at small optical depths, our method is especially applicable for realistic phase functions and moderate optical depths. Explicit comparisons of our n -stream approximation with the standard techniques of solving the transfer equation are currently in progress.

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