Entrainment and Detrainment in a Simple Cumulus Cloud Model

DAVID A. RANDALL
Laboratory for Atmospheric Sciences, NASA/Goddard Space Flight Center, Greenbelt, MD 20771

GEORGE J. HUFFMAN
Department of Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge 02139

(Manuscript received 22 December 1981, in final form 26 July 1982)

ABSTRACT

A cumulus cloud's size, shape and internal properties can be predicted, provided that the rate of entrainment is determined by a suitable entrainment parameterization theory. A cumulus cloud model based on such a theory is analogous to the mixed-layer models of the planetary boundary layer (PBL) and the upper ocean.

The entrainment rate is closely related to turbulent transport near the cloud boundary. The mixing-length theory suggested by Asai and Kasahara (1967) is examined in this light. An alternative theory is suggested, which completely removes the strong scale-dependence of the Asai-Kasahara model. Scale-dependence is re-introduced by including the perturbation pressure term of the equation of vertical motion.

For a given sounding, the new model predicts deeper clouds than the Asai-Kasahara model. This results both from the entrainment assumption used, and from the effects of the perturbation pressure.

The expected cloud-top entrainment rate is zero for the simple model considered, although finite-difference errors lead to a positive cloud-top entrainment rate in actual simulations. Lateral entrainment nevertheless dominates cloud-top entrainment. The need for a realistic parameterization of cloud-top entrainment is noted.

The fractional entrainment rate for updrafts is shown to vary only slightly with height, and to decrease only slowly as the cloud radius increases. The fractional detrainment rate for updrafts increases with height. Downdrafts are found to entrain heavily near the PBL top, and to detract primarily into the PBL, in agreement with the observations of Betts (1976).

1. Introduction

The importance of entrainment for the dynamics of growing cumulus clouds was pointed out by Stommel (1947, 1951), who inferred the existence of entrainment on the basis of the large departure of observed in-cloud lapse rates from the moist adiabat. Malkus (1954) calculated the amount of entrainment needed to account for aircraft-observed properties of individual drafts within trade cumulus clouds. These authors assumed that entrainment occurs mainly on the sides of clouds. Squires (1958a,b), Paluch (1979), Randall (1980a) and Emanuel (1981) have argued that cloud-top entrainment is also significant.

Two entrainment mechanisms were distinguished by Byers and Braham (1949) and Houghton and Cramer (1951). The first is "dynamic entrainment," which is envisioned to occur as an organized flow of environmental air into a cumulus cloud; the second is a more random "turbulent entrainment." Although it is recognized that these two entrainment mecha-

\textsuperscript{1} Current affiliation: Department of Meteorology, University of Maryland, College Park 20742.

0022-4928/82/12779-14$07.50
© 1982 American Meteorological Society
the fractional entrainment rate is independent of height.

Johnson (1976) and Nitta (1977) incorporated cloud-scale downdrafts into simple cumulus ensemble models of the type proposed by Arakawa and Schubert. They showed that, when the models are applied to deduce cumulus statistics from observed heat and moisture budgets for the GATE area, models with downdrafts predict less shallow cloud activity than models without downdrafts. However, the downdraft parameterization used by Johnson involves several arbitrary assumptions, and that used by Nitta does not resolve a spectrum of downdrafts for clouds of different sizes. A suitable foundation is needed so that more realistic cloud-scale downdraft models can be developed.

Of course, entrainment also occurs at the top of the planetary boundary layer. Over the past twenty years, many authors have built on the pioneering work of Ball (1960) to study the planetary boundary layer and the upper ocean through the use of mixed-layer models. These models represent the structures of turbulent layers in terms of the depths of the layers and their bulk (vertically-integrated) properties. A key element of any mixed-layer model is the entrainment theory used to predict the time-evolution of the layer depth. The mixed-layer models have proven very useful for improving our understanding of boundary layer processes, despite (or perhaps because of) their many simplifying assumptions.

It is, therefore, surprising that the methods and insights developed through the study of entraining mixed layers have not been applied to entraining cumuli. Lilly (1968), Deardorf (1976), Schubert (1976), Randall (1980a,b), and others have discussed stratocumulus-topped mixed layers, showing that radiative and evaporative cooling help to drive entrainment. Similar processes probably occur at the top and lateral boundaries of growing cumulus clouds.

In this paper, the equations governing a growing cumulus cloud are derived in such a way that the entrainment mass flux enters explicitly, showing how entrainment tends to modify the properties of the cloudy air, and the size and shape of the cloud.

Numerical experiments with simple cloud models show that the primary mechanism determining the differences between clouds of different radii is associated with the perturbation vertical pressure-gradient force. Holton (1973) pointed out that the perturbation pressure force tends to cause clouds of different radii to grow differently.

The lateral and cloud-top entrainment rates simulated by the models are critically examined, as is the lateral detrainment rate. The time-averaged profiles of lateral entrainment and detrainment are determined, for updrafts and downdrafts separately, and are discussed from the point of view of spectral cumulus ensemble models.

A goal of this study is to examine some of the assumptions of the simple cloud models used in cumulus parameterizations.

2. Entrainment and the conservation laws

We begin by considering the ways in which entrainment and turbulent exchange in and near a cumulus cloud tend to modify the distribution of a scalar variable $\Psi$ which satisfies a conservation equation of the form

$$\frac{\partial}{\partial t}(\rho \Psi) + \nabla \cdot (\rho \Psi \mathbf{V} + \mathbf{F}_\Psi) = S_\Psi. \quad (2.1)$$

Here $\rho$ is density, $\mathbf{V}$ is the three-dimensional operator, $\mathbf{F}_\Psi$ is the vector turbulent flux of $\Psi$, and $S_\Psi$ represents any sources or sinks.

A fundamental question is whether the subcloud-scale turbulence which accounts for $\mathbf{F}_\Psi$ is sufficiently scale-separated from the cloud-scale motions that it can be accurately parameterized in terms of the bulk structure of the cloud. A definitive answer can probably best be obtained from high-resolution three-dimensional modeling studies. As in many previous studies, we simply assume that such parameterization is possible.

We adopt a cylindrical coordinate system, in which the boundary of the cloud is given by $r = R(\phi, z, t)$, where $r$, $\phi$ and $z$ are the radial, azimuthal and vertical coordinates, respectively. First consider the effects of entrainment on a "pie-slice" of cloud, as schematically illustrated in Fig. 1. Multiplying (2.1) by $r$, and integrating the result with respect to $r$, from $r = 0$ to $r = R - \epsilon$ (just inside the boundary), we find that

$$\frac{\partial}{\partial t}(\rho \overline{r \Psi^2} \cdot R^2/2)$$

$$+ \frac{\partial}{\partial \phi} \left[ \rho \overline{\phi \Psi^2} + \left( \mathbf{F}_\Psi \right)_\phi \cdot \epsilon \right] R^2/2$$

$$+ \frac{\partial}{\partial z} \left[ \rho \overline{w \Psi^2} + \left( \mathbf{F}_\Psi \right)_z \cdot \epsilon \right] R^2/2$$

$$+ R \left[ \Psi E + \left( \mathbf{F}_\Psi \right)_r \right] \epsilon = S_\Psi \epsilon. \quad (2.2)$$
where subscripts $r$, $\phi$ and $z$ denote vector components in the cylindrical coordinate system;

$$E = \left[ \frac{\rho \partial R/\partial t + \rho \partial R/\partial \phi + w \partial R/\partial z}{1 + R/R_{\text{in}}} \right]_{R_{\text{in}}}$$  \hspace{1cm} (2.3)

is the entrainment mass flux;

$$(F_{\phi})_{n,R_{\text{in}}} = \left[ (F_{\phi})_{r} - (F_{\phi})_{\phi} \right] + \frac{1}{R} \partial R/\partial \phi$$

$$- (F_{\phi})_{z} \partial R/\partial z |_{R_{\text{in}}}$$  \hspace{1cm} (2.4)

is the part of $F_{\phi}$ which, as shown below, is directly related to the entrainment rate; and we define an averaging operator

$$\langle \cdots \rangle_{R_{\text{in}}} = \frac{2}{R_{\text{in}}} \int_{0}^{R_{\text{in}}} (\cdots) \, dr.$$  \hspace{1cm} (2.5)

The overdot denotes the substantial derivative. Further integration of (2.2) with respect to $\phi$, from $\phi = 0$ to $\phi = 2\pi$, would give the integral of (2.1) over the cross sectional area of the cloud.

Putting $\Psi = 1$ in (2.2), we obtain

$$\frac{\partial}{\partial t}(R^{2}/2) + \frac{\partial \rho}{\partial \phi} (\frac{R^{2}}{2}) - RE = 0,$$  \hspace{1cm} (2.6)

which expresses the conservation of cloud mass.

The entrainment mass flux $E(\phi, z, t)$ is the rate at which air enters the cloud. For a cylindrical cloud of time-independent radius (i.e., $R$ constant in $\phi$, $z$ and $t$), $E$ is equal to the radial mass flux $(-\rho \dot{R})_{R_{\text{in}}}$; more generally, several of the terms on the right-hand-side of (2.3) contribute to $E$. For $E > 0$, the cloud is entraining. For $E = 0$, the cloud boundary moves as a material surface. For $E < 0$, the cloud is detraining. The actual value of $E$ must be parameterized in terms of the properties of the cloud.

Mixed-layer entrainment theories have been developed to explain the advance and retreat of turbulent layers. In attempts to apply these theories to cumulus clouds, it is important to keep in mind that the boundary of a cumulus cloud need not necessarily coincide with the boundary of the turbulence associated with the cloud. To say that an air parcel is, at one moment, outside the visible cloud and, at the next moment, inside the visible cloud, is not necessary to say that the parcel is entrained. Entrainment is by its nature a turbulent process; it is the active growth of turbulence by annexation of quiet air. It can occur in the absence of clouds, as it often does at the top of the planetary boundary layer. Detrainment, on the other hand, is the passive retreat of turbulence; air which has been turbulent becomes quiet as dissipation destroys turbulence energy. Just as it is possible for entrainment to occur entirely within clear air, it is also possible for detrainment to occur entirely within cloudy air, i.e., for the final quenching of turbulence energy to precede the evaporation of all liquid water. During a cloud’s period of growth, the cloud and turbulence boundaries are observed to nearly coincide (e.g., Bunker et al., 1949), even in the anvil outflow region. As long as the anvil air remains turbulent, it cannot be considered to have been detrained. But as a cloud decays, the turbulence in its interior can be dissipated long before the cloud disappears (Fig. 2).

In this paper, we speak, for convenience, of the “cloud boundary.” But more precisely we refer to the boundary of the mass of turbulent air associated with the cloud.

Now considering this boundary region, multiply (2.1) by $r$, integrate with respect to $r$, from $r = R - \epsilon$ to $r = R + \epsilon$, and take the limit as $\epsilon \to 0$. In accordance with the preceding discussion, assume that the turbulence is confined to the cloud ($r < R$), so that $F_{\phi}$ vanishes at $r = R + \epsilon$. The result is

$$-\Delta(\rho \Psi) \partial R/\partial t + \Delta(\rho \dot{R} \Psi) - \Delta(\rho \dot{\phi} \Psi) \partial R/\partial \phi$$

$$- \Delta(\rho w \Psi) \partial R/\partial z - (F_{\phi})_{n,R_{\text{in}}} = \int_{R_{\text{in}}}^{R + \epsilon} \mathcal{S}_{\phi} \, dr, \hspace{1cm} (2.7)$$

---

**Fig. 2.** Schematic diagram depicting the relative distributions of turbulence and liquid water associated with growing and decaying cumulus clouds.
where
\[ \Delta( ) = ( )_{R+i} - ( )_{R-i}, \] (2.8)
denotes the outward radial increase of a quantity across the annulus \( R - \varepsilon < r < R + \varepsilon \). Observations (e.g., Warner, 1977) show that temperature, total mixing ratio, vertical velocity, and turbulence intensity can all undergo very sharp "jumps" at cumulus cloud boundaries, so that it seems necessary to allow \( \Delta \Psi \neq 0 \). Where entrainment is occurring, the active advance of the turbulent cloud region into the quiet clear region creates and maintains such jumps. But where detrainment is occurring, the passive retreat of turbulence leaves behind any jumps created by earlier entrainment; the boundary of the retreating turbulence is not sharply defined, and is not marked by jumps of \( \Psi \) (or of \( F_\phi \)).

The single term on the right-hand-side of (2.7) represents a possible concentrated source or sink of \( \Psi \) within the annulus. For example, if \( \Psi \) is temperature, this term can represent intense radiative and/or evaporative cooling concentrated at the cloud edge. On the other hand, if \( \Psi \) is a highly conservative variable, the term vanishes.

Putting \( \Psi = 1 \) in (2.7) gives
\[ \Delta \rho \partial R/\partial t + \Delta(\rho \dot{r}) - \Delta(\rho \dot{\phi}) \partial R/\partial \phi - \Delta(\rho w) \partial \phi/\partial z = 0, \] (2.9)
which expresses the conservation of mass for the annulus. We can use (2.3) and (2.9) to rewrite (2.7) as
\[ -E \Delta \Psi - (F_\phi)_{a,R-i} = \int_{R-i}^{R+i} S_\phi dr. \] (2.10)

Although the components of the vector \( (F_\phi) \) must be such that \( (F_\phi)_{a,R-i} \) satisfies (2.10), the vector itself is not completely determined by (2.10).

Suppose that \( E, \Delta \Psi, \) and \( \int_{R-i}^{R+i} S_\phi dr \) are known. Then (2.10) fixes the value of \( (F_\phi)_{a,R-i} \); we are not free to parameterize it independently. This means that turbulent transport near the cloud edge is closely coupled to the entrainment rate. When a jump in \( \Psi \) exists at the boundary of a cloud, a given rate of organized mass flow across the jump must be balanced by a combination of turbulent flux convergence and any concentrated source or sink of \( \Psi \) at the jump. Further discussion is given in the next section.

For completeness, the Appendix gives the equations for the time rates of change of the discontinuities at cloud edge, and shows how these discontinuities are influenced by entrainment.

3. Entrainment and detrainment in the Asai–Kasahara model

The preceding analysis can be used to interpret and extend the Asai–Kasahara (hereafter AK) model (Asai, 1967, 1968; Asai and Kasahara, 1967; Kasahara and Asai, 1967). Because of its clean design and relative simplicity, the AK model has had lasting appeal; it has been studied by Ogura and Takahashi (1971, 1973), Holton (1973), Silverman and Glass (1973), Cotton (1975), Stewart (1977), Stailey (1978), Chiu (1978) and Yau (1980). As shown below, the AK model implicitly incorporates certain assumptions about the rate of entrainment and the effects of entrainment on a cloud.

Asai and Kasahara (1967) simulated dynamic entrainment into a cloud of constant radius; they parameterized turbulent entrainment with a mixing length theory. Cotton (1975) concluded on the basis of case studies that the resulting model consistently underpredicts ultimate cloud top heights; a similar finding was reported by Stailey (1978). Cotton obtained more realistic cloud top heights through the use of a revised entrainment parameterization, which is discussed later in this section.

AK employed the anelastic approximation, and considered an axisymmetric cloud of constant radius \( a \). Since they allowed turbulent fluxes at \( r = a \), we let "\( a \)" correspond to "\( R - \varepsilon \)" in our notation. Then (2.2), (2.3), (2.6) and (2.10) become
\[ \rho_0 \partial \rho/\partial t (\Psi^a) + \partial/\partial z [\rho_0 \dot{w}^a \Psi^a + (F_\phi)^a] \]
\[ + 2/\alpha [-\Psi E + (F_\phi)^a] = \Psi^a, \] (3.1)
\[ E = (\rho_0 \dot{r})^a, \] (3.2)
\[ \partial/\partial z (\rho_0 \dot{w}^a) - 2E/a = 0, \] (3.3)
\[ -E \Delta \Psi - (F_\phi)^a = \int_a^{r+2a} S_\phi dr, \] (3.4)
respectively. Here and throughout this section, \( \bar{\Psi}^a \) denotes an areal average over \( 0 < r < a \), and \( \bar{\Psi}^b \) denotes an areal average over \( a < r < b \), where \( b \) is the radius of the "outer cylinder." In the AK notation,
\[ \Delta \Psi = \Psi^b - \Psi^a. \] (3.5)

According to (3.2), the entrainment mass flux is just equal to the inward radial mass flux at the cloud edge. This is a direct result of the assumption \( R = \) constant; therefore, this assumption is the lateral entrainment closure assumption of the AK model. If \( w \) is predicted, (3.3) allows \( E \) [and so \( (\rho_0 \dot{r})^a \)] to be diagnostically computed. Then, according to (3.4), \( (F_\phi)^a \) is determined for given \( \Delta \Psi \) and \( S_\phi \).

Suppose that \( E > 0 \), i.e., there is inflow. As argued in Section 2, we expect a discontinuity in \( \Psi \) at the cloud edge. Use of (3.5) allows us to write (3.4) as
\[ (F_\phi)^a = -E (\Psi^b - \Psi^a) - \int_a^{r+2a} S_\phi dr. \] (3.6)
This differs from AK's assumption that
\[ (F_\Psi)_{r,a} = \rho_0(K/a)(\Psi^b - \Psi^a), \tag{3.7} \]
where \( K \) is a mixing coefficient.

Now suppose that \( E < 0 \), i.e., that there is outflow across the cloud boundary. As argued in Section 2,
\[ (F_\Psi)_{r,a} = \begin{cases} -E(\Psi^b - \Psi^a) - \int_s \Psi_{\|} dr, & E > 0 \\ 0, & E \leq 0 \end{cases} \tag{3.9} \]

Use of (3.9) to eliminate \( (F_\Psi)_{r,a} \) in (3.1) gives
\[ \rho_0 \frac{\partial}{\partial t}(\bar{\Psi}^a) + \frac{\partial}{\partial z}[\rho_0 \bar{\omega} \bar{\Psi}^a + (F_\Psi)^{a-}] - \frac{2E}{a} \Psi_x = \overline{S_{\Psi + 2}} \tag{3.10} \]
where, for convenience, we define
\[ \Psi_x = \begin{cases} \Psi^b, & E > 0 \\ \Psi^a, & E \leq 0. \end{cases} \tag{3.11} \]

By use of (3.3), we can eliminate \( E \) in (3.10), giving
\[ \rho_0 \frac{\partial}{\partial t}(\bar{\Psi}^a) + \frac{\partial}{\partial z}[\rho_0 \bar{\omega} \bar{\Psi}^a + (F_\Psi)^{a-}] - \Psi_x \frac{\partial}{\partial z}[\rho_0 \bar{\omega} \bar{\Psi}^a] = \overline{S_{\Psi + 2}}. \tag{3.12} \]

It is worth noting that (3.10–3.12) are equivalent to equations used by AK, except that the mixing length terms do not appear. Taken together, (3.10) and (3.11) represent what could be called an upstream differencing scheme for the radial advection of \( \Psi \). In (3.10), the interior and boundary source terms have been combined into \( \overline{S_{\Psi + 2}} \). Further discussion of the source term is given at the beginning of the next section.

Of course, we could have arrived at (3.10) and (3.12) more directly by integrating the conservation law for \( \Psi \) from \( r = 0 \) to \( r = a + 2 \). Such a derivation would not have required consideration of \( (F_\Psi)_{r,a} \).

An interesting property of (3.12) is that the radius \( a \) does not appear explicitly. Some \( a \)-dependence may be hidden in \( S_{\Psi} \) or \( (F_\Psi)_{a} \) (whose forms we have not specified), but if this is not the case then the solution obtained from (3.12) is independent of \( a \).

At this point, we can comment on Cotton's (1975) model. Cotton retained AK's assumption that the cloud radius is constant, which means that he adopted AK's entrainment assumption. He acknowledged that the scalar properties of cumuli are observed to exhibit top-hat profiles. However, he chose the "edge" of his cloud to lie inside the thin transition zone at the cloud boundary. To determine \( \Psi_x \) (see 3.11), he adopted a "vector flux weighted mean," designed so that \( \Psi_x = \bar{\Psi}^a \), when the vertical mass flux in the cloud dominates, and \( \Psi_x = \bar{\Psi}^b \), when the radial mass flux dominates. This "vector flux weighted mean" value there is no discontinuity of \( \Psi \) or \( F_\Psi \) in this case, so that
\[ (F_\Psi)_{r,0} = 0. \tag{3.8} \]

AK used (3.7) regardless of the sign of \( E \), so that
\[ (F_\Psi)_{r,a} \neq 0 \] even for \( E < 0 \).

It is convenient to summarize (3.6) and (3.8) as

of \( \Psi_x \) must be interpreted as occurring somewhere inside the thin transition region at the cloud edge, say, where \( r = r_x \). Cotton allowed radial turbulent transports at \( r_{x0} \), so evidently this radius lies within the turbulence.

Presumably, the weighting factors used in evaluating \( \Psi_x \) depend on the particular value of \( r_x \), since the values of \( \Psi_x \) itself depends on \( r_x \). Similarly, the mixing coefficients used to determine the radial turbulent transports at \( r_x \) should also depend on the particular value of \( r_x \). For consistency, the same value of \( r_x \) must be used for evaluating both \( \Psi_x \) and the radial turbulent transports at \( r_x \). Cotton does not discuss this point, and apparently his assumptions do not guarantee this consistency.

Moreover, Cotton's approach does not take advantage of the relationship (3.6), which is unfortunate since (3.6) follows simply from an integral of the conservation law for \( \Psi \), and does not involve any assumptions about the physics of turbulent mixing.

Turning back to the current model, the assumption \( R = \text{constant} \) does not fix the cloud-top entrainment rate. In the AK model, the rate of rise of cloud-top is determined implicitly as the rate of rise of the highest level where cloud water occurs. One might expect vertical turbulent transport to be vigorous below this level, and negligible above. However, AK did not include vertical turbulent transport in their model. Further discussion is given in Section 5.

4. A comparison of four models

To demonstrate the implications of the ideas presented in the preceding sections, we now compare results obtained with four "two-cylinder" cloud models. Each of the models follows the design described by Yau (1980), except as noted. The model domain consists of a vertical cylinder and a concentric annulus, the boundaries of which are fixed in time and space at constant radii. Each level in the vertical has two values for every variable, a horizontal average over the inner region and a horizontal average over the outer region. Thus, the model may be referred to as 'axisymmetric and 1½ dimensional.' The model
is initialized by setting the model reference state to some large-scale sounding, then imposing a vertical velocity and humidity pulse at the bottom of the inner cylinder. On each succeeding time step, the anelastic system of equations is solved using simple upstream differencing and a staggered grid. First, the inner region vertical velocity is predicted. This new velocity, continuity, and the fixed geometry of the model are combined to diagnose the radial velocity and the vertical velocity in the outer region. Finally, the temperature and the moisture variables are predicted.

An important feature of Yau’s model is that it includes the perturbation pressure force. Most of the simple cloud models developed over the last 30 years entail the assumption that the pressure inside a cloud adjusts instantly to the pressure in the environment. Recent observations (e.g., Marwitz, 1973; Davies-Jones, 1974) have shown that negatively buoyant air can be accelerated upward through cloud base by the perturbation pressure force. Holton (1973) introduced this force into a simple cloud model, but he retained only the first term in a Bessel series expansion. In Yau’s model, the anelastic pressure equation is solved by assuming top-hat radial profiles for vertical velocity, potential temperature, and all water variables, and discretizing in the vertical. Radial integration of the anelastic continuity equation yields a piece-wise smooth radial velocity profile at each level. The horizontal structure of the perturbation pressure is then given by a Fourier–Bessel series. The Bessel functions are calculated only once, and carried as constants. At each time-step, the pressure-mode coefficients are calculated and used to construct horizontal averages inside and outside the cloud. The current model uses 20 pressure modes.

The source terms [(3.10) and (3.12)] are expressed as functions of the prognostic variables of the model. The vertical momentum sources considered are the perturbation pressure force, buoyancy and drag. Heat and moisture sources arise only from the phase changes of water substance; radiative cooling is neglected. The ice phase is ignored, and all warm rain processes are parameterized according to the simple “Kessler” scheme described by Yau. Vertical turbulent exchanges are not included; the consequences of this important assumption are assessed later.

In two of the models discussed below, we use the “jump” entrainment hypothesis presented in Section 3. The only concentrated source–sink terms considered to occur at the cloud boundary are those associated with phase changes. In view of (3.9), we cannot separate the concentrated source/sink at the boundary from the concentrated turbulent flux divergence at the boundary. All we can say (or need to say) is that the two act together to bring the entrained air to saturation, while maintaining the top-hat profile through time. This is exactly analogous to the assumption used in cloud-topped mixed-layer models.

Model 1 is the AK model, which includes the mixing-length assumption, and does not include the perturbation pressure. In Model 2, the mixing-length assumption is dropped, in accordance with the analysis of entrainment presented in Sections 2 and 3. Models 3 and 4 are identical to Models 1 and 2, respectively, except that the perturbation pressure force is included following the method of Yau (1980). In particular, Model 3 is essentially identical to “Model K,” as described by Yau. The properties of the four models are summarized in Table 1.

The assumption $R = constant$, which is common to all four models, is approximately consistent with the observed shapes of cumulus clouds, except for those clouds which produce anvils. The assumption implies that in regions of radial inflow ($f > 0$), entrainment is sufficient to prevent the cloud boundary from being swept inward, but is not vigorous enough to enlarge the cloud. For regions of radial outflow, the assumption implies that the turbulence of air particles falls to zero just as the particles cross $r = a$. Kondo (1974), and Ryan and Lalouis (1979) allowed $R$ to increase with time, where $\dot{r}_a > 0$, so that anvil-like formations could be simulated. It is clear that a physically-based entrainment–detainment assumption should be derived from consideration of the generation, transport and dissipation of cloud turbulence energy, but this is left for the future. In the present paper, we simply adopt the assumption $R = constant$, and explore the consequences.

The models have been run using the ratio of inner to outer cylinder radii set to 0.32. We used an idealized sounding (Fig. 3) which features a “mixed layer” of uniform potential temperature, 1 km deep. The mixing ratio decreases slightly through the depth of this layer, in agreement with observations. The cloud-triggering initial perturbation is also shown in Fig. 3. The vertical resolution for all runs is 250 m, and the domain top is at 15 km.

Fig. 4 shows the time–height evolution of the cloud water, and the time-history of the rainfall rate, for each model. During the period of active cloud growth, the cloud base level is at the top of the mixed layer. The tallest, most vigorous clouds, for each radius, are those without the mixing-length assumption (Models 2 and 4) and with the perturbation pressure (Models 3 and 4). For each model, the ultimate cloud-top height and rainfall rate increase with cloud radius, except for Model 2, which shows no scale-dependence.
at all. This result for Model 2 follows from the analysis of Section 3.

Other experiments (not shown) reveal interesting behavior as $\sigma$ is changed, but the conclusions of this paper do not substantively depend on a particular choice of $\sigma$. Further discussion of $\sigma$-dependence will be given elsewhere.

5. Simulated lateral and top entrainment rates

Using the results obtained with Model 4 of the preceding section, we have determined the lateral and top entrainment mass fluxes by the methods described below. The variables defined and discussed in this section are purely diagnostic, and play no role in the prognostic calculations entailed in running the model.

For convenience, define the positive and negative contributions to the lateral entrainment mass flux as

$$E_{L+}(z, t) = \text{Max}\{-\rho_0 \dot{h}_a, 0\},$$

$$E_{L-}(z, t) = \text{Max}\{\rho_0 \dot{h}_a, 0\},$$

so that the total lateral entrainment mass flux is

$$E_L(z, t) = -\rho_0 \dot{h}_a = E_{L+} - E_{L-}.$$  (5.3)

In order to compare the importance of lateral and cloud-top entrainment for the cloud’s mass budget, we calculated $\langle E_{L+} \rangle$, $\langle E_{L-} \rangle$ and $\langle E_L \rangle$, where

$$\langle \cdot \rangle(t) = 2\pi a/\pi a^2 \int_0^{\tau(t)} (\cdots) dz$$  (5.4)

denotes an integral over the lateral surface of the cloud, normalized by the cloud-top cross-sectional area. Of course, the $\langle \cdot \rangle$ operator integrates out height-dependence. The time-dependence of the cloud-top height $\tau_t$ is taken into account in (5.4).

The cloud-top level was determined, at each time step, by inspection of the simulated distributions of temperature and moisture. First, we constructed the time–height distribution of the “excess water”

$$q_e = q_v + q_c - q^*,$$  (5.5)

where $q_v$, $q_c$, and $q^*$ are the vapor, cloud water and saturation mixing ratios, respectively, for the inner cylinder. The excess water is positive inside the cloud, zero at the cloud-top, and negative above cloud-top. By interpolation from the model grid levels, we determined the cloud-top level $\tau_t$ as the highest level where $q_e = 0$, for each time step. The cloud-top entrainment mass flux was then obtained from

$$E_T = [\partial z_t/\partial t - w(\tau_t)]\rho_0(\tau_t).$$  (5.6)

(In a model with higher lateral resolution, a similar approach could be used to determine the lateral entrainment mass flux.)

Since we have not included parameterizations of vertical turbulent transport and cloud-top entrainment in our model, the highest level where liquid water occurs, i.e., the cloud-top level, rises in time purely as a result of vertical advection (apart from possible adiabatic cooling to saturation in the air just above the cloud-top). We therefore expect

$$\partial z_t/\partial t = w(z_t),$$  (5.7)

or, by comparison with (5.6),

$$E_T = 0.$$  (5.8)

In other words, our simple model, like the AK model, incorporates the implicit assumption that the cloud-top entrainment rate is zero. This could go far to explain why the models underpredict observed cloud-top heights (Cotton, 1975).

The simulated lateral and cloud-top entrainment rates (Fig. 5) are found to be qualitatively independent of cloud radius. Lateral entrainment occurs through most of the cloud depth, during most of the cloud’s life cycle. However, lateral detrainment occurs near the cloud-top level throughout the development of the cloud (not only after the cloud has “topped-out”), in agreement with the results of Simpson et al. (1982). Strong lateral outflow also occurs.
FIG. 4. Time–height distributions of cloud water and time evolution of rainfall rate for Models 1–4, and for cloud radii of 0.5, 1.0 and 2.0 km. The cloud water contour interval is 0.2 g kg⁻¹, starting at 0.2 g kg⁻¹. In the rainfall rate plots, the solid line is for $a = 2.0$ km, the dashed line for $a = 1.0$ km, and the dotted line for $a = 0.5$ km.
in a thin layer near the ground following the onset of precipitation. The net lateral entrainment mass flux \( \langle E_L \rangle \) is nearly equal to the top entrainment mass flux \( E_T \) throughout the growth phase, but \( \langle E_{L+} \rangle \) greatly exceeds \( E_T \). In this sense, lateral entrainment dominates top entrainment.

As discussed above, we expected to find \( E_T = 0 \), so it is not surprising that \( \langle E_{L+} \rangle \) dominates \( E_T \). At the same time, \( E_T \) is definitely positive rather than zero. This is apparently due to vertical truncation errors in the model. As soon as a grid volume contains some liquid water, the model advects that liquid up into the next grid volume, where it evaporates, and so moistens and cools towards saturation. This computational process clearly need not lead to a rate of rise of cloud-top exactly equal to \( w(z_T) \) and, therefore, \( E_T \) need not be zero.

From the point of view of cumulus parameterization theories, it is of interest to determine the lifecycle-averaged vertical profiles of the upward and downward mass fluxes, as well as the entrainment and detrainment mass fluxes. Arakawa and Schubert (1974) and others have employed simple cumulus ensemble models in which a spectrum of cumulus subensembles is characterized by fractional entrainment rate, defined by

\[
\lambda_i = \frac{1}{M_i(z)} \frac{dM_i(z)}{dz}. \quad (5.9)
\]

Here \( M_i \) is the life-cycle-averaged cumulus mass flux,
defined by

\[ M_i = \bar{N_i \sigma_i^2 \rho_0(z) \bar{w}^2(z)}, \]  

(5.10)

where subscript \( i \) denotes the \( i \)th subensemble, \( N_i \) is the number of clouds in the subensemble, \( \sigma_i \) is the ratio of the radii of the inner and outer cylinders, and the \( (\cdot) \) operator denotes a time-average over the life cycle of the cloud. For simplicity, \( \lambda_i \) is typically assumed to be independent of height. Clearly, \( \lambda_i \) is a measure of the lateral entrainment rate.

Arakawa and Schubert considered distributed lateral entrainment, with lateral detrainment confined to an infinitesimal layer at cloud-top. Johnson (1977) and Lord (1982) allowed the possibility of distributed lateral detrainment as well. Lord assumed that the fractional detrainment rate is proportional to the fractional entrainment rate, at all levels. Although Lord found that lateral detrainment is unimportant for the thermodynamic interactions between cumulus clouds and their environment, Yanai et al. (1982) have argued that it is very important for the interaction of the cumuli with the environmental vorticity field.

To date, cloud-top entrainment has not been considered in the simple cumulus ensemble models.

For Model 4, we have investigated the simulated life-cycle-averaged vertical mass fluxes, as well as the life-cycle-averaged lateral entrainment and detrainment mass fluxes. For simplicity, we approximate the "life-cycle average" by an average over the first hour of the simulation (see again Figs. 4 and 5). In the following discussion we consider a single subensemble of clouds, set \( N_i \) to 1, and drop the subscript \( i \). Because both updrafts and downdrafts occur during the lifecycles of our simulated clouds, we consider entrainment and detrainment for updrafts separately from entrainment and detrainment for downdrafts, and adopt the following notations and definitions:

\[ S = \begin{cases} 1, & \bar{w}^u > 0 \\ 0, & \bar{w}^d \leq 0, \end{cases} \]  

(5.11)

---

**Fig. 6.** Vertical profiles of various time-averaged quantities: upward, downward, and net cumulus mass fluxes; lateral entrainment and detrainment mass fluxes for updrafts; lateral entrainment and detrainment mass fluxes for downdrafts. In the lower six panels, solid lines are for entrainment and dashed lines are for detrainment. All units are mb h\(^{-1}\). Results are for Model 4 only, and for cloud radii of 0.5, 1.0 and 2.0 km.
\[ M = \sigma^2 \rho_0 \bar{\omega}_d^i, \]  
\[ M_{up} = \sigma^2 \rho_0 \bar{\omega}_d^i \bar{S}^i, \]  
\[ M_{dn} = \sigma^2 \rho_0 \bar{\omega}_d^i (1 - \bar{S})^i, \]  
\[ (E_{L+})_{up} = \bar{E}_{L+} \bar{S}^i, \]  
\[ (E_{L-})_{up} = \bar{E}_{L-} \bar{S}^i, \]  
\[ (E_{L+})_{dn} = \bar{E}_{L+} (1 - \bar{S})^i, \]  
\[ (E_{L-})_{dn} = \bar{E}_{L-} (1 - \bar{S})^i. \]

We have the identities
\[ M = M_{up} - M_{dn}, \]  
\[ \bar{E}_{L+}^i = (E_{L+})_{up} + (E_{L+})_{dn}, \]  
\[ \bar{E}_{L-}^i = (E_{L-})_{up} + (E_{L-})_{dn}. \]

Now define fractional rates of entrainment and detrainment as follows:
\[ (\lambda_{+})_{up} = [(E_{L+})_{up}/M_{up}] (2\pi a/\pi a^2) \sigma^2, \]  
\[ (\lambda_{-})_{up} = [(E_{L-})_{up}/M_{up}] (2\pi a/\pi a^2) \sigma^2, \]  
\[ (\lambda_{+})_{dn} = [(E_{L+})_{dn}/M_{dn}] (2\pi a/\pi a^2) \sigma^2, \]  
\[ (\lambda_{-})_{dn} = [(E_{L-})_{dn}/M_{dn}] (2\pi a/\pi a^2) \sigma^2. \]

For \( a = 0.5, 1.0, \) and \( 2.0 \) km, Fig. 6 shows the simulated vertical distribution of \( M, M_{up} \) and \( M_{dn}. \) Also shown are the simulated vertical distributions of \( (E_{L+})_{up}, (E_{L-})_{up}, (E_{L+})_{dn}, \) and \( (E_{L-})_{dn}. \) For all three radii, the net vertical mass flux \( M \) is downward near and below cloud base and upward through the bulk of the cloud. For the largest cloud, there is also a small net downward mass flux near the cloud top. The net mass flux attains its largest value slightly below the midlevel of the cloud. The upward mass flux peaks at about the same level. The downward mass flux is largest near the lower boundary.

For updrafts, the time-averaged entrainment mass flux is large at and below cloud base, decreases slowly with height through the bulk of the cloud, and falls off sharply near the cloud top. The detrainment mass flux for updrafts is naturally very small at low levels, but is nearly uniform with height above the 2 km level. This outflow apparently occurs near cloud-top as the cloud deepens (cf. Fig. 5).

Downdraft entrainment and detrainment are noticeable at all levels, but the most spectacular features are a sharp entrainment maximum near cloud base, and a sharp detrainment maximum just above the ground. Studying the Venezuelan International Meteorological and Hydrological Experiment (VIMHEX) data, Betts (1976) found evidence that "precipitating convection appears to strip off the subcloud

Fig. 7. Fractional entrainment and detrainment rates for updrafts and downdrafts (percent per km). Solid lines are for entrainment, and dashed lines are for detrainment. Results are for Model 4 only, and for cloud radii of 0.5, 1.0 and 2.0 km.
layer which ascends in updrafts, and to replace it with . . . air from just above cloud base, which descends in downdrafts associated with the evaporation of falling rain." It is possible that our model is simulating this process, although more definite conclusions should await studies with higher vertical resolution and a parameterization of the effects of boundary-layer turbulence. Experiments with more detailed microphysical parameterization (not shown) reduce the magnitude of the near-surface downdraft entrainment–detainment couplet, but it continues to be a spectacular feature. Using a detailed three-dimensional model, Simpson et al. (1982) also find that downdrafts detrain below cloud base.

The physical interpretation of the outflow from the downdrafts is clear; the presence of the lower boundary forces a divergence of the downward current. As shown in Fig. 5, the inflow near cloud base occurs after the outflow at the boundary. The outflow pulse forces ascent in the environment. Because the environment is stably stratified, it is lifted only as far as necessary to flow back inward over the outflow. This is analogous to what happens when a rock is dropped into shallow water. The water is pushed to the side, rises and then rushes back in to cover the rock. Having an artificial lateral wall at $r = b$ in our model presumably exaggerates this effect, but a qualitatively similar "splash" should occur even with more realistic open boundaries.

There is a tendency for more of the downdraft air to originate at high levels in large-radius clouds than in small-radius clouds. This is also in agreement with the results of Simpson et al. (1982).

The fractional entrainment and detrainment rates are shown in Fig. 7. For updrafts, the fractional entrainment rate is large at low levels (where the updraft mass flux is small), but is fairly uniform with height throughout most of the cloud depth. The fractional detrainment rate steadily increases with height from cloud base to cloud top. So, although the assumption that the fractional rate of entrainment is uniform with height in the cloud layer is supported by our results, the fractional detrainment rate is not simply a constant times the fractional entrainment rate, as suggested by Lord (1982). Note also that the fractional entrainment rate decreases only very slowly as the cloud radius increases. This weak radial dependence of the fractional entrainment rate helps to explain why ultimate cloud-top height depends only weakly on cloud radius.

For downdrafts, the fractional entrainment and detrainment mass fluxes are large near cloud top, but this is not very meaningful because the downdraft mass flux is very small there (cf. Fig. 6). The fractional entrainment rate becomes very large just above cloud base, and the fractional detrainment rate becomes very large just above the ground.

6. Summary and conclusions

We have shown that the ideas developed for the study of entraining mixed layers can be fruitfully applied to entraining cumulus clouds. The classical distinction between turbulent and dynamic entrainment is shown to be misleading and unnecessary; there is a single, unitary entrainment process in which the motion of the cloud boundary relative to the mean flow is permitted, produced and controlled by turbulent processes.

A simple cloud model incorporating a jump entrainment theory produces deeper clouds than earlier models using mixing-length theories. Following Asai and Kasahara (1967), we assume that the cloud radius is independent of height and time. The horizontal scale-dependence of the resulting model arises purely from the perturbation pressure gradient force.

The simulated lateral and cloud-top entrainment and detrainment mass fluxes were determined. Cloud-top entrainment occurs, but is purely computational in the present model. Lateral entrainment dominates cloud-top entrainment. Lateral detrainment occurs throughout the depth of the cloud, and dominates near the cloud-top. The fractional entrainment rate varies only slowly with height, while the fractional detrainment rate increases strongly with height.

The observational results of Squires (1958a,b), Warner (1977), and Paluch (1979) suggest that much of the air inside growing cumulus clouds enters through the rising cloud top. Evaporatively unstable cloud-top entrainment (Squires, 1958b; Deardorff, 1980; Randall, 1980a; Moeng and Arakawa, 1980) and destabilization due to cloud-top radiative cooling (Lilly, 1968; Randall, 1980b) are not included in the present model, but they may control the rate of cloud-top entrainment. Parameterizations of these processes, and the closely related vertical turbulent transfer mechanisms, should allow a realistic rate of cloud-top entrainment. Deeper simulated clouds may result.

Simpson and van Helvoirt (1980) and Simpson et al. (1982) have discussed the entrainment and detrainment simulated with the three-dimensional numerical model, and they have emphasized the importance of shear for determining the time, place and intensity of mass exchanges between a cloud and its environment. Although these authors did not discuss the time-averaged entrainment–detainment profiles, their work suggests that a three-dimensional model is needed to fully explore this problem.

Despite the many idealizations of the cloud model discussed in this paper, its study has suggested questions and methods of analysis which should be pursued with more detailed models.

Acknowledgments. We are indebted to Dr. M.-K. Yau for giving us an early version of the cloud model.
code. Dr. Chin-hoh Moeng helpedfully reviewed the manuscript. Professors Fred Sanders and Kerry Emanuel, and the other members of the MIT Convection Club, offered useful comments. J. Abeles programmed the plotting routines.

A portion of this work was performed while D. Randall was affiliated with the Department of Meteorology at MIT. Both authors have been supported by NSF Grant No. ATM-79-10844. G. Huffman has also been supported by NSF Grant No. ATM-80-19301.

APPENDIX

Discontinuities at Cumulus Cloud Boundaries

As shown by Rambaldi and Randall (1981), the shear at the boundary of an axisymmetric or slab-symmetric cloud satisfies

\[ \frac{\partial \Delta v}{\partial t} + \hat{\omega} \frac{\partial \Delta v}{\partial \hat{\omega}} = -\Delta[-(E/p \partial v/\partial \hat{\omega})] \]

\[ + \Delta v \frac{\partial \hat{\omega}}{\partial \hat{\omega}} = \frac{\Delta B}{\partial t} + \left[ \nabla \cdot \mathbf{F}_v \right] \hat{\rho}_{R=t}, \]  
(A1)

where \( \hat{v} \) is the tangential velocity component, the simple overbar denotes an arithmetic mean across the cloud boundary, \( s \) and \( n \) are tangential and normal coordinates, \( B \) is the buoyancy force per unit mass, and \( \mathbf{F}_v \) is the Reynolds stress tensor. Similar equations for the shear at the top of the atmospheric mixed layer were derived by Deardorff (1973) and Randall (1976).

Considering again the scalar variable satisfying (2.1), let

\[ r' = r - R(\phi, z, t) \]  
(A2)

be a transformed radial coordinate, defined so that the surface \( r' = 0 \) moves with the cloud boundary. Rewriting (2.1) in advective form, and using (A2), we obtain

\[ \rho \left[ \frac{\partial \rho}{\partial r} \right] + \frac{\partial}{\partial r} \left( \rho \frac{\partial \rho}{\partial r} \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial \rho}{\partial z} \right) \]

\[ - \left( \frac{\partial}{\partial r} \right) \rho \frac{\partial R}{\partial r} + \frac{\partial}{\partial z} \left( \rho \frac{\partial R}{\partial z} \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial R}{\partial z} \right) - \rho \]

\[ + \nabla \cdot \mathbf{F}_\rho = S_\rho. \]  
(A3)

Evaluating (A3) at \( r = R + \epsilon \), and again at \( r = R - \epsilon \), and subtracting, we find that

\[ \frac{\partial \Delta \rho}{\partial t} + \Delta \left[ \frac{\partial \rho}{\partial \hat{\nu}} \right] + \frac{\partial}{\partial \hat{\nu}} \left( \Delta \frac{\partial \rho}{\partial \hat{\nu}} \right) \]

\[ - \left( \frac{\partial \Delta \rho}{\partial r} \right) \rho \frac{\partial R}{\partial r} + \frac{\partial}{\partial z} \left( \rho \frac{\partial R}{\partial z} \right) + \frac{\partial}{\partial z} \left( \rho \frac{\partial R}{\partial z} \right) - \rho \]

\[ + \nabla \cdot \mathbf{F}_\rho = S_\rho. \]  
(A4)

Similar equations for the atmospheric mixed layer were derived by Tennekes (1973) and Randall (1976).

REFERENCES


—, 1951: Entrainment of air into a cumulus cloud II. J. Meteor., 8, 127–129.