Stable and Unstable Near-Resonant States in Multilevel, Severely Truncated, Quasi-Geostrophic Models

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(Manuscript received 17 August 1981, in final form 27 October 1981)

ABSTRACT

Stationary planetary waves are investigated with severely-truncated quasi-geostrophic models extending from the surface to 100 km. For a typical winter zonal-wind profile, it is shown that large amplitude or resonant planetary waves of intermediate zonal wavenumbers (~4 or 5) occur with an equivalent barotropic structure. In the presence of Ekman friction and Newtonian damping these stationary waves have associated with them a mountain torque and temperature transport which can influence the zonal flow. In time-dependent calculations it is shown that this wave-zonal flow interaction is stable to small perturbations on the low side of resonance and unstable on the high side of resonance. Here high and low refer to large and small values of the zonal wind.

Resonant zonal wavenumbers of lower wavenumber (~2 or 3) also occur for the same zonal profile and have a node in the vertical with a small amplitude maximum near the surface and a larger amplitude maximum in the stratosphere; still lower quasi-resonant wavenumbers also occur with two nodes in the vertical. These waves destabilize the wave-zonal flow interaction on both the high and low sides of the resonance peak. This instability depends upon the presence of the orography and the basic asymmetric state as Newtonian damping and surface friction are sufficient to damp the baroclinic instability associated with a linear inviscid model.

1. Introduction

One of the most extensively used theoretical models for understanding climatic variations in the asymmetric perturbations about a zonal-mean state has been the stationary linear model. This model assumes that the fundamental hydrodynamic and thermodynamic equations can be linearized around a mean zonal state and that asymmetric perturbations from this zonal state can be found as a function of the asymmetric forcings, which include the surface orography, diabatic heating sources, time-averaged transient eddy fluxes and all other terms not included explicitly in the linear model. Since the diabatic heating term is a fairly large contributor to the forcing term and since this diabatic heating term undergoes variations from year to year, it has been suggested (e.g., Roads, 1980a, 1981a) that the variations in climate are due to variations in the diabatic heating associated with anomalous boundary conditions and that these climatic anomalies can be described by linear models. It has also been suggested (e.g., Namias, 1964) that blocking, which is on the short-time scale of climatic variations, may also be due in part to changes in various diabatic heating sources.

Recently, however, it was pointed out by Charney and Devore (1979) and by many others (e.g., Charney et al., 1981, Roads, 1980c and references therein) that the basic state, around which the asymmetric perturbations are linearized, has slight variations from time to time and may have a large influence upon the resulting solutions. This influence is largest near the parameter regions of resonance where the amplitudes of the asymmetric perturbations can become very large for small forcings. In inviscid models, infinite responses could be obtained. Because of these very large responses one might expect that if the atmosphere moved by chance close to resonance, this would be an unstable state and the atmosphere would then immediately change to a more stable configuration. However, in the simple barotropic models and baroclinic models investigated so far, stable near-resonant states exist and dominate the climatic solutions of the model. For barotropic models (e.g., Roads, 1981b), the low side of the resonance peak (low referring to low values of the zonal wind) is stable whereas the high side of the resonance peak (high referring to large values of the zonal wind) is unstable. In two-level baroclinic models similar results occur for intermediate wavenumbers but Roads (1980b) found that slight instability also occurred on the low side of the resonance peak for low wavenumbers and for a sufficiently weak asymmetric forcing.

Resonance for low wavenumbers in barotropic and two-level baroclinic models demands unrealistically large zonal winds. It is known however that resonant
planetary waves of low wavenumber occur in multilevel baroclinic models since vertical structure can now be present in the waves and since the larger stratospheric winds and associated shears during winter can trap the stationary waves of low wavenumber (see Tung and Lindzen, 1978).

Since the low wavenumbers are observed to have most of the eddy kinetic and potential energy (see Saltzman, 1970) it would seem to be useful to extend previous studies of wave–zonal flow interactions in barotropic and two-level models near a resonance point to multi-level models which have the capability of describing the interactions near a resonance point for low wavenumbers and for realistic zonal winds. It would also seem to be useful to extend the strictly linear calculations of Tung and Lindzen (1978) for multilevel models to the wave–zonal flow interaction problem in order to determine how these resonant planetary waves modify their basic state. To this purpose stationary planetary waves and their interaction with the zonal mean flow are modeled in severely truncated quasi-geostrophic multilevel models extending from the surface to 100 km. The basic model is discussed in Section 2 and analytical solutions are discussed in Section 3. Section 4 gives the stationary solutions in the numerical model and the eigenvalues for the wave–zonal flow problem and the eigenvalues for the standard inviscid and viscous baroclinic instability problem. Some time-dependent solutions are also given. The conclusions are presented in Section 5.

It is concluded for this model and perhaps the atmosphere that resonant planetary waves of low and intermediate wavenumbers exist for reasonable values of the zonal wind. As in the two-level and barotropic models studied previously, the stationary wave–zonal flow interaction for intermediate wavenumbers is stable to small perturbations on the low side of the resonance peak and unstable on the high side of the resonance peak. These intermediate waves have an equivalent barotropic structure and can be modeled with barotropic and two-level models. However, large instability in the wave–zonal flow interaction is present on both sides of the resonance peak for the low wavenumbers, due to the presence of external forcings and an asymmetric basic state. It is suggested that stationary states of the atmosphere occur for intermediate wavenumbers near resonance points.

2. Model

The quasi-geostrophic equations in logp coordinates are (Holton, 1975):

Vorticity equation:

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi + \beta y) = \frac{f}{\rho} \frac{\partial}{\partial z} \rho \omega$$  \hspace{1cm} (1)

Thermodynamic equation:

$$\frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} + J(\psi, \frac{\partial \psi}{\partial z}) + \omega N^2 f = \epsilon \left( \frac{\partial \psi^E}{\partial z} - \frac{\partial \psi}{\partial z} \right)$$  \hspace{1cm} (2)

Surface boundary condition:

$$\frac{\partial}{\partial t} \left[ \psi_s \frac{f^2 \rho}{g} - \frac{\rho}{N^2} \frac{\partial \psi_s}{\partial z} \right] = f D \rho \nabla^2 (\psi_s) + \frac{\rho}{N^2} J(\psi, \frac{\partial \psi_s}{\partial z})$$

$$+ f^2 \rho \frac{J(\psi_s, P)}{g} - \epsilon \frac{\rho}{N^2} \left( \frac{\partial \psi^E}{\partial z} - \frac{\partial \psi_s}{\partial z} \right)$$  \hspace{1cm} (3)

Upper boundary condition:

$$\frac{\partial \psi}{\partial z} = 0$$

In the above:

$$\psi = \text{streamfunction}$$

$$\psi^E = \text{equilibrium streamfunction}$$

$$\psi_s = \text{surface streamfunction}$$

$$P = \text{orography} \ [-gh/f]$$

$$f = \text{Coriolis parameter} \ [-10^{-4} \text{ s}^{-1}]$$

$$g = \text{gravitational acceleration} \ [-9.81 \text{ m s}^{-2}]$$

$$h = \text{orographical height} \ [-10^3 \text{ m}]$$

$$\beta = \text{rotation rate} \ [-1.57 \times 10^{-11} \text{ m s}^{-1}]$$

$$\rho = \text{density of air} \ [=\rho_0 e^{-u/t_0}, \ (\rho_0 = \text{surface density})]$$

$$N^2 = \text{Brunt-Vaisala frequency} \ divided \ by \ f^2 \sim 3 \times 10^4$$

$$\epsilon = \text{Newtonian damping coefficient} \ [-10^{-6} \text{ s}^{-1}]$$

$$D = \text{depth of Ekman layer} \ [= (K/2f)^{1/2} \sim 210 \text{ m}]$$

$$K = \text{eddy diffusion coefficient at surface}$$

$$x = \text{longitude}$$

$$y = \text{latitude}$$

$$z = [-H \text{ln} p, p_s]$$

$$\omega = \frac{dz}{dt}$$

$$H = \text{constant scale height} \ [=7.52 \times 10^3 \text{ m}]$$

$$p = \text{pressure}$$

$$p_s = \text{surface pressure} \ [=1000 \text{ mb}]$$

$$t = \text{time}.$$

A rigid upper boundary condition can be troublesome because of the artificial reflectivity induced by this lid (e.g., Lindzen et al., 1968; Kirkwood and Derome, 1977). However, if this lid is placed high enough so that most of the upward-propagating energy from the surface is reflected or absorbed before the upper boundary is reached, the upper boundary condition itself may be relatively unimportant. For simplicity, a free stress and zero vertical velocity boundary condition at the top is assumed. As will be shown later, it is highly likely that all waves are trapped by typical zonal wind profiles so qualitatively it does not matter what the upper boundary condition is. Also, waves that are not trapped below the lid tend to be far away from resonance points and thus have very small amplitudes compared to the trapped waves.

Although some analytic solutions to this model are discussed later, most of the results come from a vertical finite-differenced model in which the equations are written in potential vorticity form [see Eq. (4)].
as follows (by eliminating the vertical velocity from the thermodynamic equation and vorticity equation):

For $n = 2$, $N - 1$ (where $n$ denotes the vertical gridpoint and $N$ the surface gridpoint):

\[
\frac{\partial}{\partial t} \left\{ \nabla^2 \psi_n - \lambda_{n-2} \psi_n - \psi_{n+1} \right\} + \lambda_{n-2} \psi_n = \psi_{n-1} - \psi_n \\
+ \lambda_{n-2} \epsilon_n = \psi_{n-1} + \psi_n \right\}
\]

For $n = 1$:

\[\psi_1 = \psi_2,\]

For $n = N$:

\[
\frac{\partial}{\partial t} \left\{ \psi_n - \lambda_n \psi_{n-1} - \psi_{N-1} \right\} - \kappa_n \nabla^2 \psi_N \]

\[- \lambda_n J(\psi_N, \psi_{N-1}) - J(\psi_N, P) + \epsilon_{n-1} \lambda_n \psi_{N-1} = 0, \]

where

\[\lambda_{n-2} = \frac{2\rho_n}{N_n^2 \Delta z^2 (\rho_n + \rho_{n-1})},\]

\[\lambda_{n-1} = \frac{4\rho_n}{N_{n-1}^2 \Delta z^2 (\rho_{n-1} + \rho_{n-2})},\]

\[\lambda_n = \frac{2\rho_n}{N_n^{-2} \Delta z^2 (\rho_n + \rho_{n-1})},\]

\[\lambda_n = \frac{2g}{\Delta z N_n^{-2} f^2},\]

\[\kappa_n = \frac{gD}{f}.\]

Note that at the surface, a one-sided derivative has been used to define the vertical derivative of the streamfunction and that elsewhere a centered difference has been used. Note also that $\rho_1$, $\epsilon$, and $N_1^2$ are below $\psi_1$.

For time-dependent calculations and linear calculations a complex tridiagonal system of equations is solved. For eigenvalue computations, a complete matrix is used.

3. Analytic results

Before some of the numerical results are presented, characteristics of the model that can be determined by analytic methods are discussed. This allows insight into why the stationary waves have their characteristic amplitudes and phases in the numerical model and how these waves interact with the zonal flow to give particular stationary solutions. For simplicity it is assumed that $N^2 = \text{constant}$, $\epsilon = \text{constant}$.

Then, combining Eqs. (1) and (2) into the quasi-geostrophic potential vorticity equation gives

\[
\frac{\partial}{\partial t} \left\{ \nabla^2 \psi + \frac{1}{\rho N^2} \frac{\partial}{\partial z} \psi \right\} = -J(\psi_n, \frac{1}{\rho N^2} \frac{\partial}{\partial z} \psi_n)
- J(\psi, \nabla^2 \psi + \beta y) + \frac{\epsilon}{\rho N^2} \frac{\partial}{\partial z} (\psi^E - \psi), \quad (4)
\]

\[
\frac{\partial}{\partial t} \left\{ \frac{\rho \psi_x f^2}{g} - \frac{\rho \psi_x f}{g} \right\} = fD \rho_0 \nabla^2 (\psi_x) + \frac{\rho_s}{N^2} \frac{\partial}{\partial z} \psi_x
+ \frac{f^2 \rho_s}{g} J(\psi_x, P) - \frac{\epsilon \rho_s}{N^2} \frac{\partial}{\partial z} \psi_x \psi_x \beta_y \psi_x, \quad (5)
\]

Integrating (4), subtracting (5), and using the upper boundary condition (at $z = Z$) gives

\[
\frac{\partial}{\partial t} \left\{ \int_0^Z \rho \nabla^2 \psi dz - \int_0^Z \rho \psi_x dz \right\} = - \int_0^Z \rho J(\psi, \nabla^2 \psi + \beta y)
\times dz - fD \rho_0 \nabla^2 (\psi_x) - \frac{f^2 \rho_s}{g} J(\psi_x, P), \quad (6)
\]

where the vertical structure of the streamfunction is given by

\[
\frac{\partial}{\partial t} \left\{ \int_z^Z \rho \nabla^2 \psi - \frac{\rho}{N^2} \frac{\partial \psi}{\partial z} \right\} = J(\psi, \frac{\rho}{N^2} \frac{\partial \psi}{\partial z}) \]

\[- \int_z^Z \rho J(\psi, \nabla^2 \psi + \beta y) - \frac{\epsilon \rho}{N^2} \frac{\partial}{\partial z} (\psi^E - \psi), \quad (7)
\]

a. Zonal equations

For one zonal wavenumber and one meridional mode the stationary zonal equations reduce to (for $\tilde{\psi} = \psi$ cos $\lambda y$)

\[
0 = fD \rho_0 \left( \frac{\partial \tilde{\psi}}{\partial z} - \frac{\partial \tilde{\psi}}{\partial z} \right) - \frac{f^2 \rho_s}{g} \nabla I_1 (\psi \psi^* - \tilde{\psi} \tilde{\psi}^*) \quad (8)
\]

and

\[
0 = \nabla I_1 \left( \frac{\partial \tilde{\psi}}{\partial z} - \frac{\partial \tilde{\psi}}{\partial z} \right)
- \frac{\epsilon \rho}{N^2} \frac{\partial}{\partial z} (\psi^E - \psi), \quad (9)
\]

where

\[
(\psi - \tilde{\psi}) = (\tilde{\psi} e^{-i\lambda y} + \psi^* e^{-i\lambda y}) \sin \lambda y,
\]

\[
I_1 = \frac{2 \int_0^L \sin \lambda y \cos \lambda y \cos \lambda y \sin \lambda y}{\int_0^L \cos^2 \lambda y}.
\]

The horizontal truncation to one meridional mode and one zonal wave is the simplest nonlinear study that can be performed. This truncation has been ex-
tensively used in linear studies in order to gain the knowledge necessary to understand more complex interactions. In order to simulate the atmospheric response it may eventually prove necessary to incorporate more nonlinear features. Our philosophy, however, is that understanding of the complete system can best be gained by the inductive process of first understanding the simplest nonlinear system.

The second term on the right-hand side of (8) is the mountain torque, which arises from the friction in the system. That is, in the presence of friction the asymmetric streamfunction forced by the orography is not exactly in phase or out of phase with the orography. The nonlinear response in the zonal equations can then influence the zonal flow. As will be shown later, for typical zonal profiles the mountain torque tends to be negative which means that a balance cannot be obtained except where the surface wind is close to zero. In fact, in many numerical integrations (discussed later) the only equilibrium solution occurred where \( \psi \) was zero, that is the balance at the surface was zero friction equal to zero mountain torque. In the atmosphere the momentum transports play a crucial role in maintaining a non-zero surface wind in various regions of the globe. Thus to model this momentum transport effect on the zonal state, i.e.,

\[
-\int_0^Z \rho \sum J(\psi, \nabla^2 \psi) dz,
\]

where \( \sum \) refers to the contributions by all waves, the term \( (\ddot{\psi}_z - \ddot{\psi}_s) \) in (8) and the equilibrium term in (9) will refer to a radiative-dynamical equilibrium term. The equilibrium or forcing term \( \psi^f \) is quite arbitrary and designed such that only realistic zonal states are analyzed (i.e., non-zero surface wind and reasonable vertical shears) for each individual wavenumber.

Integrating (9) using (8) results in

\[ \psi = \psi^f + T + \int_0^Z TT dz, \]

where

\[ T = \frac{\text{in} I f}{\text{D} l g} \left( \ddot{\psi}_s \hat{\rho} - \ddot{\psi}_s \hat{\rho} \right) = -\frac{2 \text{in} I f}{\text{D} l g} \text{Im} \{ \ddot{\psi}_s \hat{\rho} \}, \]

\[ TT = -\frac{1}{\epsilon} \text{in} I f \left( \ddot{\psi}_s \frac{\partial \psi^*}{\partial z} - \ddot{\psi}_s \frac{\partial \psi^*}{\partial z} \right) = \frac{2 \text{in} I f}{\epsilon} \text{Im} \{ \ddot{\psi}_s \frac{\partial \psi^*}{\partial z} \}. \]

That is, at the surface \( \ddot{\psi} \) is affected by the momentum forcing and mountain torque and is affected at higher altitudes by the Newtonian forcing and transport of temperature.

b. Asymmetric equations

The stationary torque and temperature transport can be investigated through the stationary asymmetric equations that govern them. From (4) and using

\[ \ddot{\psi} = \ddot{\psi} e^{-i2 H}, \]

\[ 0 = -\text{in} I f \int_0^Z \frac{\partial^2 \ddot{\psi}}{\partial z^2} - \frac{\ddot{\psi}}{H^2} + \text{in} I f \int_0^Z \frac{1}{\rho} \frac{\partial \ddot{\psi}}{\partial z} \]

\[ + \text{in} I f (\alpha^2 - \hat{\rho}) \ddot{\psi} - \text{in} \ddot{\psi} - \frac{\epsilon}{N^2} \left( \frac{\partial^2 \ddot{\psi}}{\partial z^2} - \frac{\ddot{\psi}}{4 H^2} \right), \]

(11)

and from (5)

\[ 0 = -f Da \ddot{\psi} + \frac{\text{in} I f}{\int_0^Z} \frac{\partial \ddot{\psi}}{\partial z} + \frac{\ddot{\psi}}{2 H} - \frac{\text{in} I f}{\int_0^Z} \frac{\partial \ddot{\psi}}{\partial z} \]

\[ + \text{in} I f \int_0^Z \frac{\ddot{\psi}}{\rho} \hat{\rho} + \frac{\epsilon}{N^2} \left( \frac{\partial \ddot{\psi}}{\partial z} + \frac{\ddot{\psi}}{2 H} \right), \]

(12)

where

\[ I_f = \frac{2}{L} \int_0^L \sin^2 \theta \theta \theta \theta d\theta, \]

\[ \alpha^2 = n^2 + \hat{\rho}. \]

Now, let

\[ \text{in} I f \ddot{\psi} = \ddot{\psi}, \]

then (11) and (12) can be reduced to

\[ 0 = \frac{\partial^2 \ddot{\psi}}{\partial z^2} + \frac{N^2 \beta + 1}{\frac{\partial \hat{\rho}}{\partial z}} - \frac{\partial^2 \ddot{\psi}}{\partial z^2} - \frac{\ddot{\psi}}{4 H^2} - \frac{\ddot{\psi}}{4 H^2 \text{in}}, \]

(13)

\[ \left[ \ddot{\psi} + \frac{\epsilon}{\text{in}} \right] \]

\[ \left[ \frac{i Da \ddot{\psi} \frac{D^2}{\text{in}} + \frac{u_s}{\text{in}} \frac{\partial \ddot{\psi}}{\partial z} - \frac{\epsilon}{\text{in}} \frac{\partial \ddot{\psi}}{\partial z} \right. \]

\[ \frac{\text{in} 2 H}{\text{in} 2 H} \]

(14)

Following Tung and Lindzen (1978) and Bender and Orszag (1978), an approximate solution to the above set can be found by the WKB method. That is, (13) can be written as
\[
0 = \frac{\partial^2 \psi}{\partial z^2} - Q \psi, \quad (15)
\]
where
\[
Q = -\left[ N^2 \beta + \frac{1}{H} \frac{\partial \bar{u}}{\partial z} - \frac{\partial^2 \bar{u}}{\partial z^2} - \bar{u} N^2 (\alpha^2 - \beta^2) \right] \left[ \frac{\bar{u} + \frac{\epsilon}{in}}{n} \right],
\]
or
\[
Q = -Q_0 \bar{u} + \frac{1}{4H^2} - \frac{i \epsilon}{n} \{Q_0 \},
\]
where
\[
Q_0 = \frac{\left[ N^2 \beta + \frac{1}{H} \frac{\partial \bar{u}}{\partial z} - \frac{\partial^2 \bar{u}}{\partial z^2} - N^2 \bar{u} (\alpha^2 - \beta^2) \right]}{(\bar{u}^2 + \frac{\epsilon^2}{n^2})}.
\]

In the following discussion it is assumed that \( \frac{\epsilon Q_0}{n} \) is small with respect to \(-uQ_0 + \frac{1}{4H^2} \) except at the designated turning point \( z \), where \( 0 = -uQ_0 + \frac{1}{4H^2} \). Now for \( \text{Re}Q \gg 0 \) and \( \left| \frac{\text{Re}Q}{\text{Im}Q} \right| \gg 1 \) the solutions behave as a damped exponential and for \( \text{Re}Q < 0 \) and \( \left| \frac{\text{Re}Q}{\text{Im}Q} \right| \gg 1 \) the solutions have wavelike behavior. For most reasonable zonal-wind profiles \( Q \) < 0 near the surface and \( Q > 0 \) some distance above the surface. Thus wavelike solutions (in the vertical) exist near the surface and then decay exponentially in the upper atmosphere. For \( \text{Re}Q \approx 0 \) mixed behavior occurs, the WKB method is not valid, and another solution must be found. This occurs at the turning point \( z \). The solution here is found by approximating \( Q \) in a two-term Taylor expansion with respect to \( z \) which results in the approximate equation
\[
\frac{\partial^2 \psi}{\partial z^2} - \left[ Q \right]_z + \frac{\partial Q}{\partial z} \left( z - z \right) \psi = 0,
\]
which has tabulated analytical solutions known as Airy functions (these are Bessel functions of fractional order). The solutions in this region are then asymptotically matched to the WKB solutions. The limits of validity in the complex plane for the asymptotic forms of the Airy functions are known as Stokes lines. Crossing these lines results in different asymptotic behavior. It may be shown that if the Newtonian damping is weak, the asymptotic expansions for the Airy functions are valid far enough away from the turning point, \( z \), and can be matched to the WKB solutions.

The WKB solutions to 15 are
\[
\psi = A_2 (-Q)^{-1/4} \times \sin \left[ \int_0^z (-Q)^{1/2} dz + \pi/4 \right], \quad z \ll z
\]
or
\[
\psi = \frac{A_2 Ai(x)}{\pi^{-1/2} (a + ib)^{1/6}}, \quad z \approx z
\]
where \( Ai \) denotes the Airy function and
\[
x = (a + ib)^{-1/3} [-ic + (a + ib)(z - z)],
\]
\[
a = -\frac{\partial}{\partial z} (Q_0 \bar{u}),
\]
\[
b = -\frac{\partial}{\partial z} \left[ \frac{\epsilon}{n} \{Q_0 \}, \quad z \approx z,
\]
\[
c = \frac{\epsilon}{n} Q_0 ,
\]
\[
Q \approx [-ic + (a + ib)(z - z)].
\]
\( A_2 \) is a constant determined from the lower boundary condition (14), that is, substitution of (16) into (14) results in
\[
-\frac{u \bar{p} N^2 f^2}{gd} = A_2 ,
\]
where
\[
d = -\left[ \left( \bar{u} + \frac{\epsilon}{in} \right) [-Q(0)]^{1/4} \times \cos \left( \int_0^z \sqrt{-Q dz} + \pi/4 \right) \right.
\]
\[
+ \left. \left[ [-Q(0)]^{-1/4} \sin \left( \int_0^z \sqrt{-Q dz} + \pi/4 \right) \right] \times \left[ \left( \frac{\bar{u}}{2H} - \frac{\partial \bar{u}}{\partial z} \right) + i \left( \frac{D \alpha^2 N^2 f}{n} - \frac{\epsilon}{n2H} \right) \right] \right] .
\]
\[\text{[In the WKB approximate solution the derivative of the term } (-Q)^{-1/4} \text{ multiplying the exponential is assumed small and thus ignored. See also Tung and Lindzen (1978).]}\]

The asymmetric solution depends solely on the region below the turning point because of the assumption that the upper-level lid is placed high enough above the turning point so that the region immediately above the turning point is described mainly by the damped exponential. More to the point, a radiation boundary condition is assumed above the turning point for these analytical solutions. Thus, the internal waves are trapped below the turning point and arbitrary boundary conditions can be used at the upper level. (The uppermost model level must still be far enough away so that the solution near the turning point is described mainly by the damped exponential.)
c. Stationary solutions

From (10) the stationary solutions for the system occur where $\bar{V} = \bar{V}^E + T + \int_0^z TTdz$, with

$$
T = \frac{n\bar{f}^2u_0N^2\bar{f}^2|(-Q)^{-1/4}|^2}{D|g^2|d^2} \left( -\sin\left[ \int_0^z 2\text{Re}(\frac{-Q}{2}) + \pi/2 \right] \right) \left( \text{Re}(\frac{-Q}{2}) - u_0 \text{Im}(\frac{-Q}{2}) \right) \\
- \sinh\left[ \int_0^z 2\text{Im}(\frac{-Q}{2}) \right] \left( u_0 \text{Re}(\frac{-Q}{2}) + \frac{\epsilon}{n} \text{Im}(\frac{-Q}{2}) \right) \\
- \left( -\cos\left[ 2\int_0^z \text{Re}(\frac{-Q}{2})dz + \pi/2 \right] + \cosh\left[ 2\int_0^z \text{Im}(\frac{-Q}{2})dz \right] \left( \frac{D\alpha^2N^2}{n} - \frac{\epsilon}{2nH} \right) \right). \tag{19}
$$

$$
TT = -\frac{n\bar{f}^2u_0^2N^4\bar{f}^2|(-Q)^{-1/4}|^2}{\epsilon g^2|d|^2} \left[ -\text{Im}\sqrt{-Q} \sin\left( 2\int_z^z \sqrt{-Q}dz + \pi/2 \right) + \text{Re}\sqrt{-Q} \right] \\
\times \sinh\left( 2\text{Im}\int_z^z \sqrt{-Q}dz \right) \exp(z/H), \quad z \ll z_t \tag{20}
$$

$$
TT = -\frac{n\bar{f}^2u_0^2N^4\bar{f}^2|(-Q)^{-1/4}|^2}{2\epsilon g^2|d|^2} \left[ -\text{Im}\sqrt{-Q} \exp\left( -2\text{Re}\int_z^z \sqrt{-Q}dz \right) \right] \exp(z/H), \quad z \gg z_t
$$

The temperature transport, $TT$, and mountain torque, $T$, are quite complicated functions of the mean parameters but a major factor contributing to their amplitudes is the condition of resonance in which $d$ is small. For reasonable values of the Newtonian damping and surface friction, this occurs near to where

$$
0 = -|(-Q(0))^{1/4}| \cos\left( \int_0^z \sqrt{-Q}dz + \pi/4 \right) + |(-Q(0))^{-1/4}| \sin\left( \int_0^z \sqrt{-Q}dz + \pi/4 \right) \\
\times \left( \frac{1}{2H} - \frac{\partial \ln u_0}{\partial z} \right), \tag{21}
$$

which is the inviscid form of (18). As discussed by Tung and Lindzen (1978), since typical zonal wind profiles can have

$$
|(-Q(0))^{1/4}| > \left( \frac{1}{2H} - \frac{\partial \ln u_0}{\partial z} \right),
$$

the approximate resonance condition becomes

$$
\cos\left( \int_0^z \sqrt{-Q}dz + \pi/4 \right) \equiv 0,
$$

or

$$
\int_0^z \sqrt{-Q}dz \sim \frac{\pi}{4} + m\pi, \quad m = 0, 1, 2, \ldots
$$

This means that between the turning point and the surface, the phase must increase by $\pi/4$, $\pi/4$, $3\pi/4$, $5\pi/4$ ··· or solutions with $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ ··· vertical wavelengths between the turning point and ground are near resonance.

For cases in which

$$
|(-Q(0))^{1/4}| < \left( \frac{1}{2H} - \frac{\partial \ln u_0}{\partial z} \right),
$$

the approximate resonance condition has $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ ··· vertical wavelengths between the ground and turning point. This latter case occurs as $u_0 \to 0$.

Some values of the zonal wind

$$
\bar{u} = \bar{u}_r + \bar{u}_r z,
$$

which are resonant (according to the WKB inviscid solution, Eq. 21) are plotted in Fig. 1. These zonal winds are the average zonal wind that the asymmetric streamfunctions feel in the channel. That is, for asymmetric perturbations with the same meridional scale as the zonal wind, the maximum zonal wind in the center of the channel is $3\pi/8$ times as large as these zonal winds. For perturbations with half the meridional scale as the zonal wind, the maximum zonal wind is $15\pi/32$ as large as these zonal winds. Only some of the resonant values are plotted. For example, in the lower left-hand corner for small surface winds and vertical shear some resonant values of $\frac{\pi}{4}$ or more vertical wavelengths are present for which the turning point is only slightly less than 50 km. (There is some ambiguity here as the resonance lines contain only approximate vertical wavelengths of $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, etc., depending upon the relative magnitude of $\sqrt{-Q(0)}$ and $\bar{u}_r/\bar{u}_r$. In the upper left-hand corner these wavelengths increase toward $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, etc.) This means that the higher the lid and the smaller the zonal wind, the more likely it is that resonant waves of very small vertical wavelength (i.e., many vertical wavelengths) existing to great
heights are present. The drastic change in the resonant behavior of the waves \((4, 1, 1; 1, 2, 1; 2, 2, 1; 3, 2, 1; \text{and } 4, 2, 1)\) as \(\bar{u}_s \to \infty\) a value where \(zt \approx 0\) is presumably spurious and due to the invalidity of the WKB solutions near this point.

With these caveats in mind, consider the qualitative behavior of the resonant values. Note that the shear and surface wind speed needed to resonate a particular wave decrease with increasing zonal wavenumber, meridional mode, and vertical wavenumber. Moreover, the resonant values of the zonal wind are not unique to a particular 3-dimensional wavenumber, because at least two waves can have the same value for the resonant wind, suggesting that in certain parameter regions, nonlinear interactions may be extremely important. These resonant curves and specific values of the turning points shift toward lower values of the shear and surface wind speed for a decrease in the static stability or for a decrease in \(\beta\).

Fig. 1 also indicates the vertical structure of waves away from a resonance point. For example, for zonal wavenumber 4 and meridional mode 1, the vertical structure between the ground and the turning point has less than \(\frac{1}{6}\) wavelength for large values of the shear and surface wind speed. For a decreasing shear and surface wind speed, wave 4, 1 has an increase in the number of vertical wavelengths. The height of the turning point also increases as the shear and surface wind speed decrease and ultimately a point is reached at which the height of the resonant behavior becomes greater than or equal to the height of an arbitrary lid. For values of the turning point beyond this lid, the solutions become increasingly inaccurate and do not show resonant behavior.

The resonant behavior is controlled by the damping in the system. At resonance, with Ekman friction,

\[
d \sim [-Q(0)]^{-1/4} \sin \left( \int_0^{z_t} \sqrt{-Qdz} + \frac{\pi}{4} \right) \frac{D\alpha^2 N f}{n},
\]

which is nonzero. This and Newtonian damping prevent infinite responses to finite forcings. However, with reasonable values of the damping terms, the response can still be quite large in the vicinity of the resonance points.

Finally, let us return to (19) and (20) in order to understand how the mountain torque and temperature transport behave. For \(\epsilon \approx 0\) the mountain torque is

\[
T \propto u_s \cos \left[ 2 \int_0^{z_t} \Re(-Q)^{1/2} dz + \frac{\pi}{2} \right].
\]

\(^1\) Where resonance is defined as

\[
\int_0^{z_t} \sqrt{-Qdz} \approx \frac{\pi}{4} + m\pi.
\]

![Fig. 1: Resonant values of the zonal wind for \(\bar{u} = \bar{u}_s + \bar{u}_z\). The values of the zonal wind are given in nondimensional units \(u_s = 0.03 - \bar{u}_s = 19.1 \text{ m s}^{-1}\) and \(u_z = 20 - \bar{u}_z = 20 \times 10^{-4} \text{ s}^{-1}\). For a top at 50 km and zero surface wind \(\bar{u} = 100 \text{ m s}^{-1}\). The solid lines give the resonant values of the zonal wind and are labeled by the zonal wavenumber, the meridional mode, and the approximate vertical wavelength, where 1 denotes \(\frac{1}{6}\) and 5 denotes \(\frac{4}{3}\). No resonant solutions are calculated for vertical wavelengths greater than \(\frac{4}{3}\). The dashed straight lines refer to the boundaries placed upon the computation with the lines placed where the turning point of the solution is at 50 km. The values given next to these lines refer to the zonal wavenumber and meridional mode. Above this line the solutions have a turning point below 50 km and below this line a turning point exists above 50 km. For the dotted lines, the lines are placed where the turning point occurs at the surface. To the right of this line the turning point occurs below the surface.

The dominant resonant responses occur for

\[
2 \int_0^{z_t} \Re(-Q)^{1/2} dz + \frac{\pi}{2} \approx \pi + 2m\pi,
\]

\[m = 0, 1, 2, \ldots\]

So for \(u_s > 0, T < 0\), and to force a westerly solution at the surface requires \(\psi, \varepsilon > 0\). The situation is more complicated when \(\varepsilon \neq 0\) but, as will be shown later for typical zonal wind profiles, \(T < 0\).

No temperature transport exists for \(\varepsilon = 0\) (in fact the problem is undefined here if \(\varepsilon\) is also zero in the zonal equations). However, if \(\varepsilon \neq 0\) then a temperature transport occurs which has a large effect on the stationary solutions. Below the turning point

\[
TT \propto \left[ + \Im(-Q)^{1/2} \sin \left[ 2 \Re \int_z^{z_t} (-Q)^{1/2} + \pi/2 \right] \right. \\
- \Re(-Q)^{1/2} \sinh \left[ 2 \Im \int_z^{z_t} \sqrt{-Qdz} \right] e^{zH}. \tag{22}
\]

Note first the density effect, \(e^{zH}\), which becomes very large at high elevations. Thus the temperature transport effect can become very influential for de-
terminating the solutions at high elevations. Near resonance the first term on the right-hand side is \(\approx 0\) at the surface, indicating that \(TT < 0\) at the surface. As the elevation increases then the first term in (22) becomes more important. So, if \(m = 1\) (i.e., a vertical wavelength of \(\frac{2\pi}{3}\) or one node in the vertical structure) then the sine term is initially positive which acts to reduce the temperature transport by the sinh term, which is decreasing with elevation. Further up in the atmosphere, however, the sine term becomes negative and thus here the westerlies are being reduced. At and above the turning point

\[
TT \propto \text{Im}\sqrt{Q} \sim \frac{1}{2} \text{Re}(Q)^{1/2} \frac{\epsilon}{nu},
\]

and thus the westerlies are increased here. Additional structure occurs if \(m > 1\) and less structure occurs for \(m = 0\).

4. Numerical solutions

In this section numerical solutions for the stationary waves and their interaction with the zonal flow through the mountain torque and temperature transport are discussed for various values of the parameters \((u, \epsilon, N^2, k_N)\) for \(I = 4\). All values are nondimensionalized by \(a\), the radius of the Earth and \(f\), the Coriolis parameter. Various models are discussed in order to determine the effects of resolution and the location of the upper boundary condition: 1) a model extending to 100 km with 5 km vertical resolution, 2) a model extending to 100 km with 2.5 km vertical resolution, 3) a model extending to 50 km with 2.5 km resolution, and 4) a model extending to 25 km with 1.25 km resolution. In inviscid models the resolution can play an important role but with various damping mechanisms such as surface friction and Newtonian cooling the vertical resolution is less critical. The resolution seems to play an important role in this model only for the lowest planetary wavenumbers (\(<1\)) or for the largest planetary wavenumbers (\(\geq 8\)). The elevation of the upper lid is more important, since this determines whether or not various planetary waves of sufficient amplitude are present. As will be seen, if the lid is too low it is impossible to resonate the lowest wavenumbers with reasonable values of the zonal wind.

Fig. 2 shows the basic zonal wind profiles used in these investigations. The profiles are somewhat arbitrary, but are supposed to represent typical winter and summer profiles. Note that during the winter there is a tropospheric jet and a much larger stratospheric jet while during the summer only the tropospheric jet is present.

Fig. 3 shows the inviscid index of refraction (multiplied by \(-u\)) associated with the winter zonal profile, (i.e., \(-Qu\) for the WKB solutions). The index of refraction indicates whether or not an internal wave is present and trapped below a turning point where the wave changes to an exponentially decaying solution. Note that for this zonal wind profile the highest wavenumbers are trapped in the lower troposphere and that as the wavenumber decreases the waves are trapped at higher and higher altitudes. [The small region of \((-Qu)\) at \(\sim 22\) km does not seem to be dynamically significant.] Above the stra-
Atmospheric jet all wavenumbers are trapped because of the negative shear in the zonal wind profile.

To find resonant responses for specific wavenumbers, one needs to vary these profiles slightly. Since resonance is a function of the vertical integral of the zonal wind, there are a number of profiles that could resonate a particular wavenumber. For example, one could change the tropospheric wind or the stratospheric wind to resonate a low wavenumber. An equivalent procedure, however, is to keep the same zonal profiles and study all wavenumbers, even those that are not necessarily an integral number. This is equivalent to studying different zonal wind profiles and the interpretation about the resonance peak remains the same. That is, a point on the low side of a resonance peak can be associated with a wavenumber slightly smaller than the resonant wavenumber or with a vertically averaged zonal wind slightly smaller than the resonance value of the zonal wind. Once the wavenumber has been identified with respect to the resonance point, time-dependent integrations can be done in order to locate the zonal wind profiles that have stable near-resonant states.

a. Mountain torque

The mountain torque induced by these zonal wind profiles is given in Fig. 4 as a function of wavenumber. The curves are labeled 3 for $N^2 = 3 \times 10^4$, 6 for $N^2 = 6 \times 10^4$, S for the summer profile and LT for a tropospheric model of 1.25 km vertical resolution extending to 25 km. Note first that the effect of increasing the static stability is to shift the resonant wavenumbers toward higher wavenumbers. Experiments with $N^2 = 1.5$ and $9 \times 10^4$ also show this same shift. The increase in static stability also increased the mountain torque with the lowest wave-numbers having the largest changes. Also note that with an increase in static stability there are more well-defined resonant peaks indicating that there may be as many as four wavenumbers that are close to resonance.

For the summer zonal wind profile only the highest wavenumbers are resonant, indicating that the stratospheric jet must be present in order for resonant planetary waves of low wavenumber to exist. This is the reason why the tropospheric model does not have resonant waves of low wavenumber. Increasing the resolution has little effect on the shorter wavelengths which are trapped below the stratospheric jet. The tropospheric model which has a low-level lid does not produce any spurious resonant responses, suggesting that the effect of having a lid is simply to retain only those modes that are trapped below the lid. That is, a low-level lid produces no spurious waves; rather it eliminates resonant responses in low wavenumbers.

b. Vertical structure of the stationary wave amplitudes

The amplitudes of the streamfunctions associated with this mountain torque term are shown in Fig. 5a for the 100 km model with 5 km resolution. The resonant responses in the mountain torque curve are also noticeable in the amplitudes of the asymmetric perturbations with large amplitudes occurring for resonant ultralong waves and resonant long waves. Also note the difference in the structures. The largest waves tend to resonate with one vertical node whereas the shorter wavelengths resonate with no vertical nodes. For very small wavenumbers there is a slight sign of a resonance mode with two vertical nodes in the vertical. As pointed out by Tung and
Lindzen (1978), this wave is more sensitive to the Newtonian damping than are waves with less vertical structure.

Fig. 5b shows the amplitude response for $N^2 = 6 \times 10^{-4}$ and as in the mountain torque figure (Fig. 4) one can see a shift toward higher wavenumbers of the resonant peaks with the resonant wavenumber with two vertical nodes being more noticeable. Figs. 5a and 5b show that the largest amplitudes for the lower troposphere occur for the intermediate wavenumbers whereas the largest amplitudes for the stratosphere occur for the lower wavenumbers.
Fig. 6. Phases of the streamfunction for the 100 km model for various zonal wavenumbers. (a) \( N^2 = 3 \times 10^4 \). (b) As in (a) except for different zonal wavenumbers. (c) As in (b) except for the 50 km model. (d) As in (b) except for the 25 km model.

The same value of the orographic height has been used for all wavenumbers. Since in reality the orographic heights are larger at low wavenumbers, it must be expected that the tropospheric waves in nature must contain contributions from both the long waves and intermediate waves.

Fig. 5c shows the response of the amplitudes in a 50 km model \( (N^2 = 3 \times 10^4 \Delta z = 2.5) \). Here the resonant responses near wavenumber 3 and wavenumber 5 are similar to those in the 100 km model although the weak resonant response for the lowest wavenumbers is not present. This indicates that the lid at 50 km is preventing weak resonant responses from occurring at the lowest wavenumbers which are present in the 100 km model due to the negative shear in the zonal wind above the stratospheric jet trapping the lowest wavenumbers.

Fig. 5d shows the response with the tropospheric model. Here the dominant response occurs only in the shorter waves because of the inability of the
model to see the strong upper-level winds and turning point. The responses are qualitatively similar with the maximum response occurring at about the 3 km level for wavenumber 4.75. Thus the absence of the stratosphere means that the resonant ultralong waves are absent.

c. Vertical structure of the stationary wave phases

The phases of the waves in the various models are given in Fig. 6. The phase refers to the phase of the high with respect to the mountain peak. Minus numbers indicate that the high lags the mountain peak (i.e., occurs before). Fig. 6a shows phases in the 100 km model ($N^2 = 3 \times 10^4$, $\Delta z = 5$ km for the lowest wavenumbers. For wavenumber 0.5 the phase shift is $\frac{\pi}{4}$ but with an increase of wavenumber (i.e., wavenumber 1.5) the phase shift becomes $<\frac{\pi}{4}$ indicating that resonance occurs near wavenumbers 1 to 1.5 with two nodes in the vertical. Fig. 6b shows the phases for wavenumbers 2.25 to 5 and a phase shift of $\sim \frac{\pi}{4}$ occurs for wavenumbers 2.75 to 3 indicating resonant waves of one node in the vertical near these wavenumbers. Further increases result in resonant modes occurring near wavenumber 4 with no nodes in the vertical and an equivalent barotropic structure with constant phases in the vertical. Fig. 6c shows the phases for the 50 km model (twice the resolution of the 100 km model) and it is seen that very similar phases occur for wavenumbers 2.25 to 5 indicating that for these wavenumbers the phases in the lower atmosphere are not affected by the resolution or the lower location of the lid. Fig. 6d shows the responses in the 2.5 km model (1.25 km vertical resolution) and again similar responses occur for low and high wavenumbers.

The interesting aspect of these solutions for the models with the low-level lid is that the phases for the relatively low wavenumbers are similar to those in models with a much higher lid. This indicates that it may be possible, at least, to predict the correct phases for the ultralong waves in models with a low-level lid even though the amplitudes are not being correctly predicted.

d. Temperature transport by the stationary waves

Figs. 7a–d show the temperature transport as a function of height for the various models (positive values of the transport decrease the zonal wind). For the intermediate wavenumbers the temperature transport is strongest near the surface, while for the low wavenumbers it is a maximum in the lower stratosphere and for the lowest wavenumbers it has a maximum in the upper stratosphere and has a very strong minimum near the upper boundary. This re-
result is consistent with the previous analysis which shows that the temperature transport should be positive near the surface and below the turning point and should then change to a negative value above the turning point. Note also that the transport is strongest near the resonant wavenumbers.

e. Instability properties of numerical model

The eigenvalues for the topographic instability problem are shown in Fig. 8. Topographic instability refers to the stability of the wave-zonal flow interaction problem to small perturbations. That is, the stationary linear asymmetric response of the model to the orographical forcing is first calculated and then the zonal and asymmetric equations are linearized around this basic state. If only the asymmetric equations are linearized around a zonal state then this would be the standard baroclinic instability problem in which the linear asymmetric state and the orography are not present. This will be discussed later. By linearizing the wave-zonal flow interaction problem, however, the asymmetric state and the orographic forcing enter the problem in part due to the feedback by the temperature transport and mountain torque on the zonal flow. These terms can also influence the asymmetric perturbations through the orographic forcing term and through the dynamical terms in the asymmetric equations. These stability calculations are the simplest finite-amplitude stability calculations that can be made.

These instability calculations give an indication as to whether or not the stationary solutions are stable. For example, having found the linear stationary solution it is possible to construct a zonal forcing, as will be done later, to give a solution that also satisfies the zonal equations. The question remains as to whether or not this stationary solution for the “complete” set of equations is stable to small perturbations. If it is stable then a possible climatic solution is present. Of course, the zonal forcing may change over time giving rise to an additional time dependence, but that is not considered here.

The eigenvalue calculations can be written sche-
matically as follows

\[
\begin{bmatrix}
\hat{A}(\psi') & \hat{A}(\psi'_c) & \hat{A}(\psi'_s) \\
\hat{A}_c(\psi') & \hat{A}_c(\psi'_c) & \hat{A}_c(\psi'_s) \\
\hat{A}_s(\psi') & \hat{A}_s(\psi'_c) & \hat{A}_s(\psi'_s)
\end{bmatrix}
\begin{bmatrix}
\psi' \\
\psi'_c \\
\psi'_s
\end{bmatrix}
= \sigma e^{\sigma t}
\]

where \(\hat{A}\) refers to the zonal, \(\hat{A}_c\) to the cosine and \(\hat{A}_s\) to the sine equations. \(\psi', \psi'_c, \psi'_s\) refer to the zonal, cosine and sine eigenfunction and \(\sigma\) refers to the eigenvalues. Note that the equations for the model have to be first multiplied by the inverse of the matrix of spatial operators for the time-dependent part in order to get the right-hand side. A similar procedure is used for the baroclinic instability calculations except here the matrix consists of only \(\hat{A}_c, \hat{A}_s(\psi'_c, \psi'_s)\). A standard numerical eigenvalue routine is used and only the most unstable eigenvalue is shown.

The topographic instability for the 100 km models (5 and 2.5 km resolution) are given in Fig. 8a and the topographic instability for the 50 and 25 km models are given in Fig. 8b. Comparing these figures with Fig. 4 (which shows the mountain torque and therefore location of the resonance points) it can be seen that instability occurs on the high side of the resonance peak for the intermediate wavenumbers of \(\sim 4\) or 5 and on the low and high side of the resonance for the low wavenumbers. Note that in the 100 km models there are three distinct instability peaks. In the 50 km model there are only two distinct instability peaks, and in the 25 km model there is only one distinct instability peak, indicating that if the lid is too low the wave is not reflected by a turning point, is not resonant, and therefore will not be unstable.

The instability present in these wave–zonal flow interaction problems has little to do with the standard baroclinic instability problem which is shown in Fig. 9. If the friction and Newtonian damping are the same as in the topographic instability problem then the baroclinic instability is virtually damped.
out. In the absence of these viscous effects the familiar picture of instability emerges. Instability occurs in the ultralong waves, the so called Green modes, with an instability of \( \sim 10 \) days; a neutral point occurs near wavenumber 6; and large instability occurs in the long waves, the so called Charney modes. (See Geisler and Garcia, 1977; Kuo, 1979; Hartmann, 1979 and Strauss, 1981 for a thorough discussion of baroclinic instability in inviscid atmospheres.)

With Newtonian damping alone, only the ultralong waves are damped considerably as the long waves tend to be influenced mainly by the surface friction. It might be argued that the time constant for the Newtonian cooling (\( \sim 10 \) days) is fairly large in this model. However, Dickinson (1973) suggests that radiative cooling should be even larger in the stratosphere and near the surface the influence of sensible heat and radiation should give smaller time constants than are present in this model. It might also be argued that the Ekman friction is fairly large but so far as can be discerned, the values seem to be typical of those used in general circulation models.

A more influential effect on the instability is the choice of basic state. If the inviscid state is more unstable due to a different choice of parameters then the damping has less influence on the solution. For example, Fig. 10a shows the topographic and baroclinic instability in the 100 km model (5 km resolution) when \( N^2 \) is decreased to \( 1.5 \times 10^4 \) (it was \( 3 \times 10^4 \) previously). Note that the baroclinic instability for the standard instability problem with damping has increased in both the ultralong waves and long waves and has decreased somewhat in the topographic instability problem for the ultralong waves. Fig. 10b shows the responses in the 50 km model when a realistic static stability profile is taken (from Kuo, 1979 and Hartmann, 1979). Finally, the responses in the 50 km model when the surface wind is decreased to \( 2.5 \) m s\(^{-1} \) (but kept the same everywhere else however) and the realistic static stability profile is used is given in Fig. 10c. The increased shear at the lower level is responsible for the increase in baroclinic instability for the long waves and the reduced surface wind is responsible for the decrease in the topographic instability. Note that the topographic instability is still less than the instability in the standard baroclinic instability problem for the long waves and larger near the various resonance points of the intermediate and ultralong waves.

Summarizing, the instability present in the topographic instability problem with various damping effects gives a possible new picture of the instability present in the Earth’s atmosphere. Topographic instability is greater for the ultralong waves and intermediate waves than is the instability associated with the standard baroclinic instability problem. The instability in these problems seems to be strongly influenced by the presence of various frictional terms.

\( f. \) Time-dependent calculations

To understand better this topographic instability, several time dependent calculations were done. Using the zonal profile given in Fig. 2 and \( N^2 = 3 \times 10^4 \),

---

**Fig. 9.** (a) Baroclinic instability for the 100 km models. Unless shown otherwise \( k_u = 5 \times 10^{-3} \) and \( \epsilon = 10^{-2} \) nondimensional units. \( \Delta = 5 \) refers to the model with 5 km resolution and \( \Delta = 2.5 \) refers to the model with 2.5 km vertical resolution. (b) As in (a) except for the 50 and 25 km models.
Fig. 10. (a) Topographic and baroclinic instability for $N^2 = 1.5 \times 10^4$ in the 100 km model with 5 km resolution. (b) Topographic and baroclinic instability in the 50 km model with $N^2$ taken from Kuo (1979) and Hartmann (1979). (c) As in (b) except $\bar{u}_s = 2.5$ m s$^{-1}$ at the surface (in all other models $\bar{u}_s = 5$ m s$^{-1}$).

A particular wavenumber was picked and then the stationary linear response calculated. The mountain torque and heat transport induced by this linear response were then used to calculate the $\bar{V}^E$ which would give a stationary solution in the wave-mean flow interaction problem (that is, (10) is solved for...
$\psi$. Then a small perturbation of $10^{-5}$ (order 1% of the basic asymmetric state) was added to $\psi$, at the surface. The numerical solution for the stationary state was sufficiently accurate that unless this perturbation was added, the solution remained stationary for finite times.

The time-dependent profiles for $\psi$ (the zonal mean streamfunction), $\psi_r$ (the cosine function of the asymmetric streamfunction), $\psi_s$ (the sine function of the asymmetric streamfunction) for wavenumber 2.5 (low side of resonance peak) are given in Fig. 11. Note that at around day 28 the model zonal wind started decreasing at high elevations and explosive changes occurred in the interval from day 28 to day 32 with the upper-level westerlies changing to upper-level easterlies. The atmosphere then began to slowly return to upper-level westerlies with the jet lowered to $\sim 50$ km and gradually the zonal wind approached the equilibrium forcing indicating that the system was far away from resonance. Note also that from days 28 to 32, $\psi_r/\partial z$ became much larger negative than it was positive previously, indicating a reversal of the equator–pole temperature gradient with much stronger positive temperatures.

The time evolution of the asymmetric streamfunctions is shown in Fig. 11b and c. These figures show the asymmetric cosine function is fairly large and then decreases in amplitude from day 28 to day 40 whereas the sine function increases initially in amplitude, although with a reversal of phase, and then slowly decreases toward zero. The fundamental reason for the stratospheric warming in this model is that the stationary state is unstable to small perturbations. Thus, in addition to the large pulse of energy from the troposphere (see Matsuno, 1971), the basic state and scale of the perturbations is also important (see also Schoeberl and Strobel, 1980; and Holton and Wehrbein, 1981.)

The surface response is given in Fig. 11d, $A$ refers to the amplitude of the surface streamfunction, $\theta$ the phase of the surface high with respect to the orography, and $\psi$ the zonal mean streamfunction. Note that initially the zonal streamfunction at the surface increases, $A$ decreases, and $\theta$ increases (moving the higher closer to the mountain peak). After a short time, $A$ increases, $\psi$ decreases and $\theta$ remains fairly steady with small-scale oscillations. Eventually $\psi$ begins increasing again as $A$ diminishes in amplitude.

A similar calculation was done for wavenumber 3 which in this case was on the high side of the resonance peak and is shown in Fig. 12. Here the zonal flow evolves similarly to wavenumber 2.5 except that the instability is first noticed from day 32 to day 36, and because of the weak zonal forcing takes somewhat longer to return to the forcing equilibrium solution, if in fact it will ever return. Note also that here the amplitude of the wave in the upper stratosphere initially increases, and then gradually diminishes. The vertical wavelength also becomes smaller. Thus the same process that occurred on the low side of the resonance peak is also present on the high side, with the model essentially moving away from the resonance region. There was no sign of a stationary state and one may speculate that resonant responses in the low wavenumbers (gravest meridional mode) are not associated with stationary states.

In fact, Fig. 12d shows that at the surface the amplitude initially increases slightly then decreases and returns eventually to its former amplitude with several small time-dependent oscillations.

These integrations are to be contrasted with the intermediate-wavenumber responses (around wavenumbers 4 and 5). Here the low side of the resonance peak is stable and integrations with wavenumbers 4–4.5 remained at the initial state. On the other hand, wavenumber 5, which is on the high side of the resonance peak, was unstable and moved toward the low side of the resonance peak by decreasing the zonal wind. These responses are shown in Fig. 13 for the 25 km model (similar responses occurred in the 50 and 100 km models). These intermediate wavenumbers seem to search for stationary states near the low side of the resonance points whereas the low wavenumbers seem to move as far away as possible from the resonance points.

Several other integrations of low- and intermediate-scale wavenumbers for the various models were also done and can basically be described by the low and intermediate wavenumber responses shown in Figs. 11, 12 and 13. Low wavenumbers (gravest meridional mode) are unstable on the low and high side of the resonance peak and intermediate wavenumbers (or low zonal wavenumbers and smaller meridional modes) are stable on the low side of the resonance peak and unstable on the high side of the resonance peak.

This model was unable to describe the situation discussed by Tung and Lindzen (1978) of blocking being preceded by a lowering of the stratospheric wind maximum and subsequent amplification of a particular low wavenumber by resonant growth. In all cases investigated, the slight amplification of the low-wavenumber waves occurred in the troposphere and stratosphere simultaneously (although it was most noticeable in the stratosphere because of the low densities there), resulting in decreased westerlies everywhere below the turning point, changing to stratospheric easterlies. As the amplitudes of the waves decreased, the zonal wind increased again and finally at the end of the instability episode the amplitudes of the waves were noticeably decreased and the zonal wind was increased with a lower stratospheric jet maximum. However, as noted by Tung and Lindzen, this state might then be able to resonate another wavenumber.

It may well be that to describe the blocking sit-
Fig. 11. Vertical structure of the zonal streamfunction for various days in an initial value experiment for zonal wave 2.5 and the 100 km model (Δ = 5 km) with $N^2 = 3 \times 10^{-4}$. (b) As in (a) except for the cosine streamfunction. (c) As in (a) except for the sine streamfunction. (d) Values for the surface $\tilde{\psi}$, $\tilde{A}$, $\tilde{\theta}$ as a function of time.

uation and coincidental stratospheric warming adequately one needs to include several waves simultaneously. Such an attempt was carried out by Egger (1980) who showed that during a blocking event a stratospheric warming could occur. Also, a seasonal cycle in the diabatic forcing mechanisms may eventually have to be included (see Trenberth, 1973) in order for various wavenumbers to be excited. As was shown in this model, it is highly likely that the lowest wavenumbers are unstable and thus the atmosphere
is likely to change its configuration so that it approaches more stable states. Such a configuration could be, in fact, one with a large stratospheric zonal wind. It is tempting to speculate that the response seen in Fig. 11 occurs for a winter condition where a major stratospheric warming occurs, resulting in easterlies initially and then returning to stronger zonal westerlies with a lowered stratospheric jet. The
response seen in Fig. 12, however, might be the minor seasonal stratospheric warming during the spring with the flow becoming easterly and then remaining easterly during the summer (see Holton, 1975, for a description of these various warming events).

5. Conclusions

Multilevel, severely truncated, quasi-geostrophic models with damping were used to study the stationary-wave response to orography and the concom-
itant feedbacks on the zonal flow. Dominant responses occurred near resonance points of the linear model.

As in the two-level models studied previously (e.g., Roads, 1980c; Charney and Strauss, 1980) resonance points occurred for intermediate zonal wavenumbers (∼4–5 or lower depending upon the meridional structure of the waves) and were stable on the low side of the resonance point and unstable on the high side of the resonance point. Here high and low refer to large and small values of the zonal wind. The equivalent barotropic structure found in two-level models was also present in these multilevel models. A minor difference between these multilevel models and two-level models for these intermediate wavenumbers was that in the multilevel models the heat transport was largest near the surface and decreased upwards, although, presumably, this is due more to the surface boundary condition than to the use of two levels.

Major differences between models occurred in the resonant responses for the low wavenumbers (1 and 2 of gravest meridional mode). Two-level models and multilevel models with a low-level lid do not have resonant responses in the lowest wavenumbers for reasonable zonal winds. Low wavenumbers can only be excited near resonance points in a model with sufficient vertical extent because the vertical structure associated with these low wavenumbers (one or more nodes in the vertical) can now be modeled, because the turning point where the waves are trapped is below the lid and because the presence of the stratospheric westerlies and associated shears during the wintertime are sufficient to resonate these low wavenumbers.

However, in the particular wave–zonal flow interaction problem studied here the model was unstable on the low and high sides of the resonance points for low wavenumbers. In initial value experiments started near these points, the model immediately changed its configuration to move far away from these resonance points. During this change, the model went through a stratospheric warming with the development of zonal easterlies initially which later changed again toward strong zonal westerlies if the zonal forcing was sufficiently strong. The instability present in these wave–zonal flow experiments was associated with the orography and asymmetric basic state. The Newtonian cooling and Ekman friction were sufficient to damp the baroclinic instability associated with the standard inviscid baroclinic instability problem.

If blocking is associated with these near-resonant responses then it is likely to be a relatively short-lived phenomenon for the low wavenumbers because of the instability in the wave–zonal flow interaction, and a relatively long lived phenomenon for the intermediate wavenumbers because of the stable near-resonant state. (It should be emphasized here that low zonal wavenumbers with large meridional structure have equivalent responses to those for intermediate wavenumbers of gravest meridional mode. The distinguishing characteristic seems to be a vertical structure with no nodes—the so-called external mode.) However, it is known that these blocking waves are likely to be unstable to smaller-scale waves (e.g., Roads, 1980c) and model calculations with more nonlinear features are needed to determine the contribution by various waves to the development and decay of these intermediate-scale stationary states. In particular, it would be of interest to determine if more realistic models also show a tendency to maintain a state on the low side of the resonance points for the intermediate wavenumbers and on the high side, but far away, from the resonance points for the low wavenumbers.

Acknowledgments. This research was supported by NASA under Grant NAG5-105. I am grateful to E. Sarachik of Harvard and to various members of the Climate Research Group for their comments; especially T. Barnett, R. Somerville, and G. Vallis. The manuscript was text-edited by G. Johnston and the figures were drafted by F. Crowe. I am also grateful for the comments of the anonymous reviewers.

REFERENCES


