NOTES AND CORRESPONDENCE

Note on Mixed-Layer Entrainment Closure

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ABSTRACT

The entrainment closure for the turbulence energy budget commonly used in unsaturated mixed layer models is re-derived from the perspective of potential energy changes, and it is concluded that the correct physical interpretation of the energetics based on experimental evidence is that the turbulence forces entrainment at about the 20% efficiency level (*not* the 4% level). In the case of a cloud-topped mixed layer, this physical interpretation leads to a re-examination of the condition for stratocumulus cloud-top entrainment instability and the conclusion that recent corrections for liquid water loading may yet be too restrictive.

1. Introduction

This note comments upon and provides an independent argument for Stage and Businger's (1981a,b) entrainment closure for use in mixed-layer models. The discussion here is restricted to the very simple case of entrainment in the atmosphere forced by buoyancy fluxes and can easily be extended to the oceanic mixed layer and to cases of free entrainment and encroachment. Two aspects of the problem are considered: the energetics of the entrainment closure and the case of "cloud-top entrainment instability" (Deardorff, 1980).

2. Potential energy

As a starting point, the simplest case is considered, i.e., the dry well-mixed atmospheric-boundary layer heated from below neglecting radiation, advection and large-scale subsidence. The fundamental hypothesis used takes potential energy (PE) production associated with entrainment to be a fraction of the turbulent kinetic energy (TKE) production by positive buoyancy fluxes.

The PE of an atmospheric column is

$$\mathrm{PE} = \int_0^\infty \rho g z dz = \int_0^{z_B} \rho_m g z dz + \int_{z_B}^\infty \rho_a g z dz, \quad (1)$$

where the mixed-layer top is z_B and the mixed-layer density is ρ_m; ρ_a is the density of the air above the mixed-layer, and g is the acceleration of gravity. For shallow mixed layers, approximately incompressible, density changes can be entirely related to potential temperature (ϑ) changes: $\delta\rho/\rho_0 \simeq -\delta\vartheta/\vartheta_0$, where ρ_0 and ϑ_0 are reference values. Using this in (1), and differentiating via Leibnitz' rule, yields

$$\dot{\mathrm{PE}} \equiv d/dt \int_0^\infty \rho g z dz$$

$$= -\frac{g\rho_0}{\vartheta_0} z_B \left\{ \frac{z_B \dot{\vartheta}_m}{2} - \Delta\vartheta \dot{z}_B \right\}, \quad (2)$$

where $\Delta\vartheta = \vartheta_u - \vartheta_m$, ϑ_u being the potential temperature just above z_B. Neglecting radiative and advective effects, the vertically integrated first law of thermodynamics gives

$$z_B \dot{\vartheta}_m = -(F_{\vartheta B} - F_{\vartheta S}), \quad (3a)$$

where $F_{\vartheta S}$ is the potential temperature flux at the surface; and

$$F_{\vartheta B} = -\dot{z}_B \Delta\vartheta \quad (3b)$$

is the flux just below z_B (Lilly, 1968). Inserting (3a,b) into (2) gives

$$\dot{\mathrm{PE}} = -\frac{g\rho_0}{\vartheta_0} z_B \left(\frac{F_{\vartheta S} + F_{\vartheta B}}{2} \right). \quad (4)$$

Now, for interfacial stability, $\Delta\vartheta > 0$, and entrainment implies $\dot{z}_B > 0$; hence, $F_{\vartheta B} < 0$ for entrainment. Therefore, the PE production associated with mixed layer entrainment is proportional to $F_{\vartheta B} z_B/2$. The surface-flux term in (4) describes the amount of PE released to TKE ($F_{\vartheta S} > 0$, by assumption), and the entrainment closure hypothesis reduces simply to

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$$F_{\vartheta B} = -\kappa F_{\vartheta S}\,, \tag{5}$$

i.e., the closure originally proposed by Betts (1973) and others. It is seen that the closure hypothesis takes κ to be the fraction of TKE converted to PE, with $1 - \kappa$ of the TKE assumed to be dissipated.

Stull (1976), in an analysis of entrainment energetics, asserts (p. 1261) that "it can easily be shown that [the PE change] . . . per unit area due to turbulent entrainment" is

$$(\dot{\mathrm{PE}})_{\mathrm{entr}} = \frac{g\rho_0}{\vartheta_0}\int_{z_1}^{z_B} F_\vartheta(z)dz,$$

where z_1 is defined as the level for which $F_\vartheta(z_1) = 0$. Since $F_\vartheta(z) = F_{\vartheta S}(1 - z/z_B) + F_{\vartheta B}(z/z_B)$ in this simple case, this reduces to

$$(\dot{\mathrm{PE}})_{\mathrm{entr}} = \frac{g\rho_0}{\vartheta_0}\frac{F_{\vartheta B}{}^2}{F_{\vartheta S} - F_{\vartheta B}}\frac{z_B}{2}\,, \tag{6}$$

which differs from the entrainment PE production in (4) by a factor of $-F_{\vartheta B}/(F_{\vartheta S} - F_{\vartheta B})$. Stull's (1976) assertion is therefore not supported by the present analysis. Eq. (6) represents the "negative area" under the $F_\vartheta(z)$ curve, and Stull's closure equates this to a fraction of the "positive area" lower down, resulting in

$$F_{\vartheta B}{}^2 = \kappa' F_{\vartheta S}{}^2. \tag{7}$$

Eqs. (5) and (7) are consistent if $\kappa' = \kappa^2$.

From a physical standpoint, the differences in interpretation are clear: the approach presented here asserts the PE production associated with entrainment occurs everywhere within the layer, while Stull's approach assumes that it occurs only above the lowest zero-flux level. Equation (4) contradicts the latter assumption. Experimental results suggest that $\kappa \simeq 0.2$; clearly, the present analysis shows that buoyantly-produced turbulence is efficient at the 20% level in forcing entrainment, in contrast to Stull's 4%. While these differences are largely semantic in the dry-layer case, they are quite important when water-phase changes are considered, i.e., within a stratocumulus mixed-layer model.

Recently, Stage and Businger (1981a) have developed an entrainment closure by, firstly, equating the layer-integrated buoyancy term to the layer-integrated dissipation in the TKE equation and, secondly, partitioning the buoyancy term between PE production of TKE and TKE "consumption" by PE generation associated with entrainment. Eq. (5), which was derived purely from PE considerations, is identical to Stage and Businger's result for the dry layer discussed here. Stage and Businger (1981a) extended the analysis to include cases of fog layers and stratocumulus-topped mixed layers, and have also considered cases of free entrainment and encroachment. The essential difference between Stage and Businger's (1981a) and the current approach on the one hand, and that of Stull (1976) and Kraus and Schaller (1978) on the other, is the realization that the energy conversions PE → TKE → (PE + dissipation) are not limited to specific regions in the layer but occur throughout the entire layer (Manins and Turner, 1978). Stage and Businger (1981b) have compared closure results for the stratocumulus case, and no further analysis will be presented here.

3. Cloud-top entrainment instability

Three aspects of the stratocumulus-topped mixed layer model introduced by Lilly (1968) have received considerable attention recently: the details of the cloud-top infrared-radiative flux divergence (Deardorff, 1976; Schubert *et al.*, 1979; Randall, 1980b; Deardorff, 1981); the stability of the cloud-top entrainment process (Deardorff, 1980; Randall, 1980a; Hanson, 1981); and the entrainment closure (Deardorff, 1976; Schubert, 1976; Kraus and Schaller, 1978; Stage and Businger, 1981a,b). The interactions of the latter two are discussed here; cloud-top radiation remains an important unresolved question.

The analysis below is carried out using moist static energy h and total water mixing ratio r as conservative variables, and virtual static energy (s_v) as the buoyancy variable. Following Randall (1980a):

$$c_p\Delta_1 \equiv \Delta h - (1 - \epsilon\delta)L\Delta r \quad \text{(unsaturated)}, \tag{8a}$$

$$c_p\Delta_2 \equiv \beta\Delta h - \epsilon L\Delta r \quad \text{(saturated)}, \tag{8b}$$

where ϵ and β depend on temperature and pressure. Based on these relationships, Hanson (1981) and Deardorff (1980) (who used the equivalent of liquid static energy, $s_l = h - Lr$, instead of moist static energy) constructed mixing diagrams to suggest that cloud-top entrainment is unstable when the cloud-top jump $\Delta_2 < 0$. This criterion and its implications for the entrainment closure are now discussed in some detail.

Fig. 1 shows contours of Δ_1 and Δ_2 as functions of Δh and Δr for p = 900 mb from Eq. (8). The lightly shaded area is stable to perturbations involving no water phase changes, and the unshaded area is the stable region for the stratocumulus cloud-top entrainment, according to the criterion discussed above. The heavily shaded area is absolutely unstable. The important point here is that the "saturated" stability is a more restrictive measure of the stability; larger values of $(-\Delta h)$ are stable in the "unsaturated" case. Two questions arise: What happens in the lightly shaded area; and how does the entrainment closure handle this possibility?

Fig. 2 shows a part of the mixing diagram presented by Hanson (1981). Above the saturation line (dashed), the upper air is sub-saturated; within the

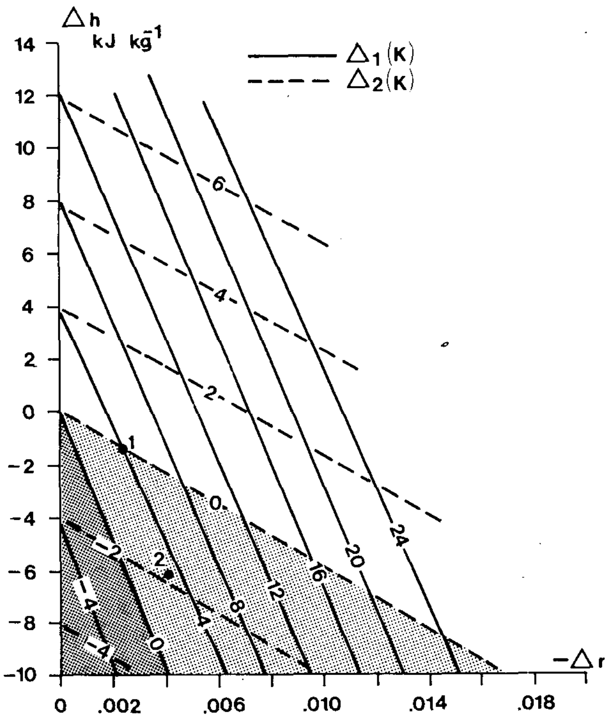

FIG. 1. Contours of saturated (Δ_2) and unsaturated (Δ_1) cloud-top stability parameters as functions of the cloud top jumps in moist static energy (Δh) and total water (Δr). The lightly shaded region is stable to dry convection and the unshaded area shows stability for stratocumulus convection, according to Hanson (1981) and Deardorff (1980).

cloud (below the dashed line) the liquid water content, for any h, is given by $r - q^*$. Mixing between the upper air (e.g., point U) and the cloud (points C) occurs along the line between the two points. Point C_1 illustrates the "neutral case" of Deardorff (1980, Fig. 2), and is given by point 1 on Fig. 1. Point C_2 represents a cloud parcel at the same buoyancy (virtual static energy) as the upper air (U) and so, in the absence of mixing, is marginally stable. During mixing, however, evaporative cooling decreases the $(U - C_2)$ parcel's buoyancy by about 2 kJ kg^{-1}, and according to the criterion above, this represents cloud-top instability. This is shown in Fig. 1 as point 2, which lies in the intermediate zone, between wet and dry instability.

The hatched area in Fig. 2 represents the cloud conditions for which $s_{vU} - s_{vC} \geqslant 0$ and $\Delta_2 < 0$. The criterion above asserts that a stratocumulus cloud deck will be unstable for any of these. Randall (1980a) showed that this corresponds to TKE production by the "entrainment" flux at the cloud top. In a closure formulation of the mixed-layer stratocumulus model, following Kraus and Schaller (1978), Randall (1980b) asserted that this implies unstable entrainment, there being no sink for the TKE aside from dissipation. However, Stage and Businger (1981b), in their entrainment closure, have shown that, since $\Delta_1 > 0$, there is TKE consumption by negative buoyancy fluxes in the subcloud layer.

If, as implied above and by Stage and Businger, the stratocumulus cloud is not necessarily limited to conditions of $\Delta_2 > 0$, it must be asked: under what conditions does the cloud break up?

One important implicit aspect of mixed-layer stratocumulus models is that, barring further arbitrary parameters, the lapse rate within the cloud layer is moist adiabatic. The virtual static energy stratification is, to first approximation,

$$\Upsilon_v = g(1 - \beta)[1 - (1 + \delta)\epsilon]$$

$$\cong 3.7 \text{ J kg}^{-1} \text{ m}^{-1} (= \text{m s}^{-2}).$$

Considering the $(U - C_2)$ mixing in Fig. 2, the minimum s_v reached during mixing, just before saturation, is ~2 kJ kg^{-1} less than s_{vC} (=294 kJ kg^{-1}). As the parcel is mixed into the cloud, therefore, it will sink, and further mixing will tend to raise its buoyancy toward s_{vC}. If further mixing proceeds slowly relative to the parcel's sinking, the possibility exists that it may sink right through the cloud, which would obviously tend to break up the solid stratocumulus deck. If however, the cloud is ~0.55 km thick, the cloud base s_v will be less than 292 kJ kg^{-1}; i.e., the parcel will reach zero relative buoyancy before it sinks all the way through the cloud regardless of the mixing rate. In this circumstance, even though the cloud-top entrainment is associated with TKE production, the fact that $\Delta_1 > 0$ (Fig. 1) implies creation of PE below cloud base (see Stage and Businger, 1981b, Fig. 7) and a sufficient supply of moisture from below could conceivably ensure the continued existence of the solid cloud deck.

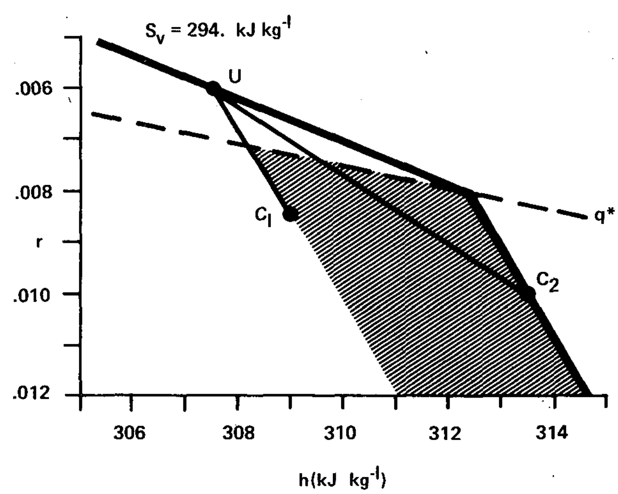

FIG. 2. Mixing diagram showing the s_v = 294 kJ kg^{-1} contour for unsaturated (above dashed line) and saturated (below dashed line) conditions. Since h and r are conservative for water phase changes, mixing between points $(U - C_1)$ and $(U - C_2)$ occurs along the line between them.

4. Conclusions

1. The mixed-layer entrainment closure developed and discussed by Stage and Businger (1981a,b), based on energetic arguments concerning the TKE budget, is consistent with the changes of PE in the layer. The arguments for closure following Stull's (1976) clear mixed layer analysis (i.e., Kraus and Schaller, 1978; Lilly and Schubert, 1980; Randall, 1980b) are not consistent with the PE budget.

2. The liquid-water loading correction to Lilly's (1968) stratocumulus instability criterion discussed by Randall (1980a), Deardorff (1980) and Hanson (1981) may still be too restrictive, in the sense that TKE production by cloud-top entrainment is not *prima facie* evidence of stratocumulus instability. The cloud thickness and the effects of non-isobaric mixing may also be important factors in determining the cloud deck's stability.

No attempt is made here to determine a more exact criterion for stratocumulus instability. Before this can be accomplished, it will be necessary to determine more details of the cloud-top infrared radiative flux divergence and the role of shear-induced turbulence in the entrainment process. Resolution of these questions requires experimental data and detailed numerical calculations.

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