# Free Rossby Wave Instability at Finite Amplitude

RICHARD C. DEININGER

*Department of Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge 02139*

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ABSTRACT

The finite-amplitude evolution of the instability of a nonparallel basic-state flow and the basic state are studied. The basic state consists of a free Rossby wave in an inviscid, barotropic beta-plane model. The method of multiple time scales is used to obtain the weakly nonlinear evolution of the system on the long time scale corresponding to the slow growth of the slightly unstable perturbation.

The results of the analysis show that, as the perturbation grows, both the amplitude and phase of the Rossby wave are modified, producing a nonlinear feedback which acts to stabilize the perturbation. Feedback due to nonlinearly produced harmonics can be either stabilizing or destabilizing to the perturbation. The total feedback is usually stabilizing and leads to an oscillatory exchange between the Rossby wave and perturbation. The mechanism responsible for the nonlinear feedback is the tilted trough mechanism.

## 1. Introduction

Examination of daily weather maps reveals the presence of wavelike features in a state of constant change. Classical explanations for the existence of waves in the atmosphere include Rossby's (1939) theory of free barotropic planetary waves, Charney's (1947) and Eady's (1949) theory of the baroclinic instability of a zonal flow, and Charney and Eliasen's (1949) theory of forced waves produced by the diversion of the zonal flow by the large-scale topography. In attempts to understand the variability of such waves, various authors have undertaken linear stability analyses of these three different wave types and, indeed, found them to be unstable. Lorenz (1972), Gill (1974), Coaker (1977), and Mied (1978) have studied the stability of a free barotropic Rossby wave. Pedlosky (1975), Merkine and Israeli (1978), and Lin (1980a,b) have studied the stability of a basic state consisting of a vertically sheared zonal flow and a neutral baroclinic wave. Charney and Flierl (1981) analyzed the linear stability of a topographically forced wave. Of course, as these instabilities grow to finite amplitude, the assumption of linearity is violated. Clearly, then, a fundamental question of interest is: How do finite amplitude effects alter the time evolution of both the perturbation and Rossby wave? This paper will address this problem in the context of the free Rossby wave stability problem as posed by Gill (1974). It would be of further interest to study the effects of nonlinearity beginning with a linear stability problem in which the basic state consisted of a vertically sheared zonal flow and a neutral baroclinic wave. In such a problem feedbacks between the perturbation, basic state wave, and vertical shear would be possible. The present problem may be regarded, in part, as a first step in this direction. The effects of nonlinearity on the topographically forced wave stability problem are different than those of the free wave problem and have been reported in a separate paper (Deininger, 1981).

Gill (1974) considered a basic state Rossby wave which varied both cross-stream and downstream and showed that, for small basic wave amplitude, the instability was confined to a small portion of wavenumber space, which corresponds to the locus of wavenumbers satisfying the kinematic resonance condition (or near resonance conditions) for three waves. For small amplitudes, instability is thus limited in wavenumber space, while this instability, for larger basic wave amplitudes, studied by Lorenz (1972), is much less limited in wavenumber space. Gill refers to the instability as resonant instability for small basic wave amplitude and Rayleigh instability for large basic wave amplitude. Here I shall consider only the case of nonresonant instability. How, one might ask, can the nonresonant instability be realized in a physical system where the resonant instability occurs for a small amplitude basic wave? Since the resonant instability is limited in the wavenumber domain, if we invoke a quantization condition such as cyclic continuity in the earth's atmosphere, the wavenumbers necessary for triad resonance may not be possible. In this case, the favored mode of response of the system or the response requiring least amplitude of the basic wave for in-

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stability, is presumably nonresonant in the triad sense.

In order to study the effects of nonlinearity, the nonlinearity is required to be weak. This means the product of the perturbation with itself is at least an order smaller than the perturbation and the product of the perturbation with the basic state. When this is true it is possible to close the problem using the method of multiple time scales, by choosing the perturbation amplitude parameter in order to achieve a balance between the weak nonlinearity and slow growth of the slightly unstable perturbation. Slow growth is attained by hinging the nonlinear stability analysis about the point of neutral stability. This technique is the same as that used by Pedlosky (1970) in the study of a baroclinically unstable zonal flow, and by Loesch (1978) in a problem similar to the current one. In fact, it will prove instructive to compare the results of this paper for a nonparallel flow to those of Pedlosky (1970) for a parallel flow. Loesch's (1978) results can be treated as a nontrivial special case of the present work. This case is the subject of a separate note (Deininger and Loesch, 1982).

## 2. The model

The nondimensional vorticity equation governing the barotropic motion of a quasigeostrophic, inviscid, homogeneous fluid on an infinite beta plane, bounded in the vertical direction by infinite horizontal plates, is

$$\frac{\partial}{\partial t}\nabla^2\Psi + \beta\frac{\partial}{\partial x}\Psi + J(\Psi, \nabla^2\Psi) = 0, \quad (2.1)$$

where

$$J(a, b) \equiv \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x},$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In (2.1) $\Psi$, $x$, $y$, $u = -\partial\Psi/\partial y$, and $v = \partial\Psi/\partial x$ are the nondimensional streamfunction, longitude coordinate, latitude coordinate, zonal velocity and meridional velocity, respectively. The dimensional variables (denoted by primes) are related to their nondimensional counterparts by $(x', y', t', u', v', \Psi') = (Lx, Ly, L/Ut, Uu, Uv, LU\Psi)$, and the planetary vorticity factor $\beta$ is given in terms of the latitudinal gradient of the Coriolis parameter $\beta' = df/dy$, by

$$\beta = \beta' L^2/U.$$

## 3. Formulation of the nonlinear analysis

An exact solution of (2.1) whose stability Gill (1974) studied, is the familiar Rossby wave solution

$$\Psi = R\sin\theta, \quad (3.1a)$$

which, upon substitution into (2.1), gives the dispersion relation

$$K^2 w = \beta k. \quad (3.1b)$$

In (3.1)

$$\theta = kx + ly + wt,$$
$$K^2 = k^2 + l^2.$$

Superimposing an infinitesimal perturbation $\psi$ on the Rossby wave solution (3.1a), and substituting into (2.1), results in the linear perturbation problem solved by Gill (1974). This is

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial}{\partial x}\psi + R\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(\nabla^2 + K^2)\psi = 0. \quad (3.2)$$

Following his analysis, a solution of the form

$$\psi = e^{i\lambda t}\sum_{n=-\infty}^{\infty} P_n e^{i\theta_n} + *, \quad (3.3)$$

is taken, where (*) denotes the complex conjugate and

$$\theta_n = k_n x + l_n y + w_n t,$$

with

$$(k_n, l_n, w_n) = (k_0, l_0, w_0) + n(k, l, w).$$

As Gill does, I refer to the $e^{i\theta_0}$ term of (3.3) as the secondary wave and choose the arbitrary $w_0$ such that

$$K_0^2 w_0 = \beta k_0.$$

This entails no loss of generality, because $\lambda$ in (3.3) is, as yet, undetermined. Substitution of (3.3) into (3.2) results in the following recursion relation between the successive amplitudes of (3.3), which is,

$$Rba_{n-1}Q_{n-1} + (\lambda + \delta_n)Q_n + Rba_{n+1}Q_{n+1} = 0, \quad (3.4a)$$

where

$$K_n^2 = k_n^2 + l_n^2, \quad (3.4b)$$

$$Q_n = K_n^2 P_n, \quad (3.4c)$$

$$a_n = \frac{K_n^2 - K^2}{K_n^2}, \quad (3.4d)$$

$$\delta_n = w_n - \frac{\beta k_n}{K_n^2}, \quad (3.4e)$$

$$b = \tfrac{1}{2}(kl_0 - lk_0), \quad (3.4f)$$

for all integers of (3.3). Solution of (3.4a) involves the solution of an infinite set of homogeneous algebraic equations. In order to make progress on this, a truncation is obviously necessary. For ease of presentation and to keep the algebra in the nonlinear analysis to a minimum, I shall retain only two terms in the perturbation field, i.e.,

$$\psi = e^{i\lambda t}(P_0 e^{i\theta_0} + P_1 e^{i\theta_1}) + *. \tag{3.5}$$

Indeed, this is a severe truncation but it does not affect the basic result sought. In fact, the analysis to follow can be carried out for *any* finite truncation. The result of the three-term truncation is given in the Appendix since it is of relevance with respect to the work of Loesch (1978).

If we truncate the perturbation field to that of (3.5), the infinite set of homogeneous equations generated by (3.4a) reduces to

$$(\lambda + \delta_1)Q_1 + Rba_0 Q_0 = 0,$$

$$Rba_1 Q_1 + \lambda Q_0 = 0.$$

For the existence of a nontrivial solution to this pair of homogeneous equations it is required that

$$\lambda = -\delta_1/2 \pm 1/2(\delta_1^2 + 4R^2 b^2 a_1 a_0)^{1/2}.$$

Thus, for instability to occur the condition

$$a_1 a_0 < 0 \tag{3.6}$$

must be true and the amplitude of the Rossby wave, $R$, must exceed a certain critical value defined by

$$R_c^2 = -\frac{\delta_1^2}{4b^2 a_1 a_0}. \tag{3.7}$$

Eq. (3.6) states that energy transferred from a wave must be transferred to waves of shorter and longer scale, as was originally discussed by Fjørtoft (1953).

As we shall see, the analysis pivots about the critical Rossby wave amplitude (the amplitude above which the wave shear is sufficiently large to overcome $\beta$), given in (3.7), which is based on a two-term truncation of the perturbation field. The critical amplitude calculated using this truncation is accurate at or near triad resonance where $\delta_1$ is zero or small, respectively. The instability at or near triad resonance occurs at infinitesimal Rossby wave amplitude and for a range of wavenumbers limited by the requirement that $\delta_1 = 0$, which is the condition that the sum of the frequencies of the basic state and two perturbation modes must vanish. For a given basic state the wavenumber quantizing condition of cyclic continuity around the globe may not allow the particular wavenumbers required for triad resonance, in which case the small-amplitude resonant triad instability is eliminated leaving only the larger-amplitude, nonresonant instability. This instability occurs over a broader band of wavenumbers including the quantized waves. For example, if in this doubly periodic geometry $k = l = 2$ (two wavelengths in each direction of the domain) instability is possible for $k_0 = -2, -1, 0, 2$ if $l_0 = 1$ and for $k_0 = 0, \pm 1$ if $l_0 = 2$ but triad resonance only occurs for the nonquantized wavenumber $k_0 = -0.1$ for $l_0 = 1$ or $k_0 = -0.9$ for $l_0 = 2$. Thus, only the nonresonant instability can fit in the domain. However, (3.7) is generally inaccurate for large Rossby wave amplitudes as is the critical amplitude calculated from the three-mode truncation (given in the Appendix) for some parameter regimes. In fact, many more modes generally need to be retained before convergence of the linear stability problem—and therefore critical amplitude—occurs (Tung, 1976). Furthermore, relaxing the severity of the truncation does not alter the form of the nonlinear evolution equations obtained in the next section but only greatly increases the computational effort. Thus, since the qualitative aspects of the nonlinearity are of primary interest here, I will employ the computationally simplest truncation possible, namely, the two-mode truncation (3.5).

On the neutral curve defined by (3.7), the real part of $\lambda$ is

$$\lambda_r = -\delta_1/2,$$

and (3.5) yields

$$P_1 = C_1 P_0, \tag{3.8a}$$

where

$$C_1 = \frac{\delta_1 K_0^2}{2R_c b a_1 K_1^2}. \tag{3.8b}$$

Now (3.5) on the neutral curve can be written

$$\psi = \sum_{n=0}^{1} C_n P_0 e^{i\hat{\theta}_n} + *, \tag{3.9}$$

where

$$\hat{\theta}_n = \theta_n - (\delta_1/2)t,$$

$$C_0 = 1.$$

If the Rossby wave amplitude $R$ is larger than $R_c$ by a small amount, $\Delta(\Delta \ll R_c)$, i.e.,

$$R = R_c + \Delta, \tag{3.10}$$

the rate of growth of the perturbation, $\lambda_i$, proportional to the square root of $\Delta$ is obtained. It is

$$\lambda_i^2 = -2b^2 R_c a_1 a_0 \Delta. \tag{3.11}$$

This suggests the long time scale over which the perturbation and Rossby wave must vary. Thus, the long time scale, $T$, is defined as

$$T = |\Delta|^{1/2} t.$$

Now, as is usual in the method of multiple time scales, the time operator in (2.1) is replaced by

$$\frac{\partial}{\partial t} + |\Delta|^{1/2} \frac{\partial}{\partial T}. \tag{3.12}$$

Since, however, the basic state Rossby wave is characterized by an amplitude and a phase, *both* the amplitude and phase of the Rossby wave structure must be allowed to undergo change on the long time scale, induced by weak non-linearity, as the perturbation evolves on the long time scale. To allow the phase or frequency of the Rossby wave structure to be altered, either a frequency expansion or a stretching of the time coordinate would be effective. Here, I shall stretch the time coordinate according to the formula

$$\frac{\partial t_*}{\partial t} = 1 + W(T), \tag{3.13}$$

where $t_*$ is the stretched fast time. The stretching function $W$ will be determined by the removal of secularities in the expansion scheme to follow. Using (3.12), the fast time derivative in expression (3.12) may be replaced by

$$[1 + W(T)]\frac{\partial}{\partial t_*}.$$

Now, (2.1) may be written

$$(1 + W)\frac{\partial}{\partial t_*}\nabla^2\Psi + |\Delta|^{1/2}\frac{\partial}{\partial T}\nabla^2\Psi + \beta\frac{\partial}{\partial x}\Psi + J(\Psi, \nabla^2\Psi) = 0. \tag{3.14}$$

In order to achieve the balance between weak linear instability and weak nonlinearity, I will choose the size of the perturbation such that the expansion parameter for $W$ and $\Psi$ is $|\Delta|^{1/2}$. Therefore, the expansions are:

$$\Psi = \psi^{(0)} + |\Delta|^{1/2}\psi^{(1)} + |\Delta|\psi^{(2)} + \cdots, \tag{3.15a}$$

$$W = |\Delta|^{1/2}\frac{W_1}{w} + |\Delta|\frac{W_2}{w} + |\Delta|^{3/2}\frac{W_3}{w} + \cdots. \tag{3.15b}$$

Eq. (3.15b) is invalid for a stationary Rossby wave. In this case the direct expansion of frequency would be of greater use.

The introduction of the stretched fast time formally necessitates the brief retracing of certain steps taken thus far. Substituting (3.15) into (3.14) and retaining terms of O(1) yields

$$\frac{\partial}{\partial t_*}\nabla^2\psi^{(0)} + \beta\frac{\partial}{\partial x}\psi^{(0)} + J(\psi^{(0)}, \nabla^2\psi^{(0)}) = 0.$$

This amounts to nothing more than respecifying the basic state in stretched time, i.e.,

$$\psi^{(0)} = R\sin\theta, \tag{3.16}$$

where $\theta$ now is

$$\theta = kx + ly + wt_*,$$

and the dispersion relation (3.1b) is satisfied. The amplitude of the Rossby wave is assumed to be slightly above its value on the neutral curve defined by (3.7) so that $R$ is given by (3.10).

The balance of terms at $O(|\Delta|^{1/2})$ yields

$$\frac{\partial}{\partial t_*}\nabla^2\psi^{(1)} + \beta\frac{\partial}{\partial x}\psi^{(1)} + R_c\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(\nabla^2 + K^2)\psi^{(1)} = K^2W_1R\cos\theta + K^2\frac{\partial R}{\partial T}\sin\theta. \tag{3.17}$$

One homogeneous solution to (3.17) is just the neutral perturbation (3.9), i.e.,

$$\psi^{(1)} = \sum_{n=0}^{1} C_nP_0e^{i\hat\theta_n} + *, \tag{3.18}$$

but in stretched time, so now

$$\hat\theta_n = \theta_n - (\delta_1/2)t_*.$$

There is a second homogeneous solution to (3.17) corresponding to the Rossby wave structure. However, such a solution, introduced at this order, would be found to be zero at the next order making its inclusion unnecessary at this point. Since the particular solution to (3.17) grows linearly with time, the inhomogeneous terms are secular and must be removed. Their removal requires that there be no $O(|\Delta|^{1/2})$ frequency changes of the Rossby wave ($W_1 = 0$) and that the O(1) Rossby wave amplitude is independent of long time ($\partial R/\partial T = 0$), as one would expect. Of course, these null results do not preclude the possibility of there being nonzero higher-order corrections to the phase and amplitude of the Rossby wave structure. In fact such corrections do occur and are of fundamental importance.

With the specification of the basic state and neutral perturbation in stretched time complete, the stage is set to study the nonlinear effects which will begin to appear at the next order.

## 4. The nonlinear analysis

Substituting the expansions (3.15) and (3.10) into (3.14) and only retaining terms of order $|\Delta|$, yields

$$\frac{\partial}{\partial t_*}\nabla^2\psi^{(2)}+\beta\frac{\partial}{\partial x}\psi^{(2)}+R_c\cos\theta\left(k\frac{\partial}{\partial y}-l\frac{\partial}{\partial x}\right)\times(\nabla^2+K^2)\psi^{(2)}=-\frac{\partial}{\partial T}\nabla^2\psi^{(1)}-\frac{W_2}{w}\frac{\partial}{\partial t_*}\nabla^2\psi^{(0)}-J(\psi^{(1)},\nabla^2\psi^{(1)}). \quad (4.1)$$

Using (3.18) and (3.9) to evaluate the inhomogeneous terms of (4.1), and retaining only those terms consistent with the original truncation, (4.1) can be rewritten as

$$\frac{\partial}{\partial t_*}\nabla^2\psi^{(2)}+\beta\frac{\partial}{\partial x}\psi^{(2)}+R_c\cos\theta\times\left(k\frac{\partial}{\partial y}-l\frac{\partial}{\partial x}\right)(\nabla^2+K^2)\psi^{(2)}$$
$$=[K^2A_cW_2+4b(K_0{}^2-K_1{}^2)C_1|P_0|^2]\cos\theta+\left[K_0{}^2\frac{\partial P_0}{\partial T}e^{i\hat\theta_0}+K_1{}^2C_1\frac{\partial P_0}{\partial T}e^{i\hat\theta_1}+2b(K_1{}^2-K_0{}^2)C_1P_0{}^2e^{i\bar\theta_1}+*\right], \quad (4.2)$$

where

$$\bar\theta_n=2\hat\theta_0+n\theta.$$

Because the inhomogeneous term proportional to $\cos\theta$ in (4.2) is resonant, its coefficient must be set equal to zero to avoid secular behavior in the expansion. Thus, the order $|\Delta|$ frequency change is determined by the self-interaction of the perturbation field, i.e.,

$$W_2=\gamma_1|P_0|^2, \quad (4.3a)$$

where, after using (3.8b),

$$\gamma_1=8b^2\frac{(K_1{}^2-K_0{}^2)(K^2-K_0{}^2)}{\delta_1K^2K_1{}^2}. \quad (4.3b)$$

The change in frequency of the wave is toward higher frequencies for $\delta_1>0$, toward lower frequencies for $\delta_1<0$, and singular for $\delta_1=0$. This singularity corresponds to the case of triad resonance between the basic state Rossby wave and the two perturbation modes and is due, in part, to the severe truncation. For this reason and because the nonresonant instability is of primary interest, I will assume $\delta_1\neq0$ so triad resonance is not possible.

Next the inhomogeneous terms in (4.2) proportional to $e^{i\hat\theta_1}$ and $e^{i\hat\theta_0}$ are dealt with. At first perusal they appear to be resonant, but this will prove to be false. To see this, assume a particular solution of the form

$$\psi_p{}^{(2)}=\sum_{n=0}^{1}P_n{}^{(2)}e^{i\hat\theta_n}+*. \quad (4.4)$$

Therefore, the $Q_n=K_n^2P_n$ must satisfy

$$½\delta_1Q_1{}^{(2)}+R_cba_0Q_0{}^{(2)}=iK_1{}^2C_1\frac{\partial P_0}{\partial T}, \quad (4.5a)$$

$$R_cba_1Q_1{}^{(2)}-½\delta_1Q_0{}^{(2)}=iK_0{}^2\frac{\partial P_0}{\partial T}. \quad (4.5b)$$

Combining (4.5a,b) results in the equation

$$\left(\frac{2A_cba_0}{\delta_1}+\frac{\delta_1}{2A_cba_1}\right)Q_0{}^{(2)}=\left(\frac{2K_1{}^2C_1}{\delta_1}-\frac{K_0{}^2}{A_cba_1}\right)\frac{\partial P_0}{\partial T}. \quad (4.6)$$

Using (3.7) and (3.9b), both bracketed quantities of (4.6) are found to vanish. This occurs because we are expanding about the point of neutral stability and as a result $P_0$ is allowed to vary on the time scale of the barotropic instability and $P_0{}^{(2)}$ is allowed to be arbitrary. Since all the structure of the neutral perturbation is assumed to be specified at $O(|\Delta|^{1/2})$, I can set

$$P_0{}^{(2)}=0. \quad (4.7a)$$

Mathematically, the vanishing of the coefficient of $\partial P_0/\partial T$ says the inhomogeneous terms of (4.5) are orthogonal to the adjoint solutions of the homogeneous side of (4.5) and, therefore, are nonresonant and produce a horizontally phase-shifted solution for which

$$P_1{}^{(2)}=2i\frac{K_1{}^2C_1}{\delta_1}\frac{\partial P_0}{\partial T}. \quad (4.7b)$$

The inhomogeneous term in (4.2) proportional to $e^{i\bar\theta_1}$ produces a forced solution. It is

$$\psi_f{}^{(2)}=\sum_{n=0}^{1}F_ne^{i\bar\theta_n}+*, \quad (4.8a)$$

where

$$F_n=if_nP_0{}^2, \quad (4.8b)$$

so

$$f_1=\frac{2b(\bar\delta_0-\delta_1)(K_1{}^2-K_0{}^2)C_1}{\bar K_1{}^2D}, \quad (4.8c)$$

$$f_0=\frac{-4b^2R_c\bar a_1(K_1{}^2-K_0{}^2)C_1}{\bar K_0{}^2D}, \quad (4.8d)$$

$$D=(\bar\delta_1-\delta_1)(\bar\delta_0-\delta_1)-4b^2R_c{}^2\bar a_0\bar a_1.$$

The barred variables are defined by

$$\left.\begin{aligned}(\bar{k}_n, \bar{l}_n, \bar{w}_n) &= 2(k_0, l_0, w_0) + (k, l, w)\\ \bar{K}_n^2 &= \bar{k}_n^2 + \bar{l}_n^2\\ \bar{a}_n &= \frac{\bar{K}_n^2 - K^2}{\bar{K}_n^2}\\ \bar{\delta}_n &= \bar{w}_n - \frac{\beta \bar{k}_n}{\bar{K}_n^2}\end{aligned}\right\}. \tag{4.9}$$

There is yet another solution to be included at this order which represents an $O(|\Delta|)$ amplitude correction to the basic state Rossby wave. It is

$$\psi_R^{(2)} = R^{(2)} \sin\theta. \tag{4.10}$$

A solution of this form might also have been added at $O(|\Delta|^{1/2})$. However, as previously mentioned, it would have been found to be zero at this order; thus, it was unnecessary to do so. The evolution of $R^{(2)}$ and $P_0$ will be determined by the removal of secularities at the next order. The complete $O(|\Delta|)$ solution is

$$\psi^{(2)} = \psi_R^{(2)} + \psi_p^{(2)} + \psi_f^{(2)}, \tag{4.11}$$

where $\psi_R^{(2)}$, $\psi_p^{(2)}$ and $\psi_f^{(2)}$ are given by (4.10), (4.4), and (4.8), respectively.

We now proceed to $O(|\Delta|^{3/2})$ where closure will be obtained. The $O(|\Delta|^{3/2})$ problem is

$$\begin{aligned}&\frac{\partial}{\partial t_*}\nabla^2\psi^{(3)} + \beta\frac{\partial}{\partial x}\psi^{(3)} + R_c\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(\nabla^2 + K^2)\psi^{(3)}\\ &= -\frac{\partial}{\partial T}\nabla^2\psi^{(2)} - \frac{W_2}{w}\frac{\partial}{\partial t_*}\nabla^2\psi^{(1)} - \frac{W_3}{w}\frac{\partial}{\partial t_*}\nabla^2\psi^{(0)}\\ &\quad - \frac{\Delta}{|\Delta|}\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(\nabla^2 + K^2)\psi^{(1)}\\ &\quad - J(\psi^{(1)}, \nabla^2\psi^{(2)}) - J(\psi^{(2)}, \nabla^2\psi^{(1)}).\end{aligned} \tag{4.12}$$

When evaluating the inhomogeneous terms of (4.12) it is only necessary to retain those terms which are resonant in either the Rossby wave or perturbation structure. Other inhomogeneities only produce forced solutions which would be of importance at the next order and therefore are not of any concern since the problem will be closed on the long time scale $T$ at this order. Then using (3.18), (4.11), (4.10), (4.4), and (4.18) to evaluate the resonant inhomogeneous terms of (4.12) yields

$$\begin{aligned}&\frac{\partial}{\partial t_*}\nabla^2\psi^{(3)} + \beta\frac{\partial}{\partial x}\psi^{(3)} + R_c\cos\theta\left(k\frac{\partial}{\partial y} - l\frac{\partial}{\partial x}\right)(\nabla^2 + K^2)\psi^{(3)}\\ &= \left[K^2R_cW_3 + \frac{4ib(K_0^2 - K_1^2)C_1}{\delta_1}\left(P_0^*\frac{\partial P_0}{\partial T} - P_0\frac{\partial P_0^*}{\partial T}\right)\right]\cos\theta + \left[K^2\frac{\partial R^{(2)}}{\partial T} - \frac{4b(K_0^2 - K_1^2)C_1}{\delta_1}\frac{\partial}{\partial T}|P_0|^2\right]\\ &\times\sin\theta + \left[K_1^2\frac{\partial P_1^{(2)}}{\partial T}e^{i\theta_1} + i\left(w_1 - \frac{\delta_1}{2}\right)K_1^2C_1\frac{W_2}{w}P_0e^{i\theta_1} + i\left(w_0 - \frac{\delta_1}{2}\right)K_0^2\frac{W_2}{w}P_0e^{i\theta_0}\right.\\ &+ i\frac{\Delta}{|\Delta|}b(K_0^2 - K^2)P_0e^{i\theta_1} + i\frac{\Delta}{|\Delta|}b(K_1^2 - K^2)C_1P_0e^{i\theta_0} + ib(K_0^2 - K^2)P_0R^{(2)}e^{i\theta_1}\\ &\left.+ ib(K_1^2 - K^2)C_1P_0R^{(2)}e^{i\theta_0} + 2ib(K_0^2 - \bar{K}_1^2)f_1P_0|P_0|^2e^{i\theta_1} - 2ib(K_1^2 - \bar{K}_1^2)C_1f_0P_0|P_0|^2e^{i\theta_0} + *\right].\end{aligned} \tag{4.13}$$

Removal of those terms resonant in the Rossby wave structure, $\sin\theta$ and $\cos\theta$, from (4.13) results in

$$\frac{\partial R^{(2)}}{\partial T} = -\gamma_2\frac{\partial}{\partial T}|P_0|^2, \tag{4.14a}$$

$$W_3 = i\gamma_3\left(P_0^*\frac{\partial P_0}{\partial T} - P_0\frac{\partial P_0^*}{\partial T}\right), \tag{4.14b}$$

where

$$\gamma_2 = \frac{2K_0^2(K_1^2 - K_0^2)}{R_cK^2(K_1^2 - K^2)} > 0, \tag{4.15a}$$

$$\gamma_3 = \frac{\gamma_2}{R_c}. \tag{4.15b}$$

Eqs. (4.14a,b) give the $O(|\Delta|)$ correction to the Rossby wave amplitude and the $O(|\Delta|^{3/2})$ correction to the Rossby wave frequency due to the self-interaction of the perturbation field. Since $\gamma_2 > 0$, (4.14a) says that as the perturbation grows ($d|P_0|^2/dT > 0$), the energy is drawn out of the basic wave ($dR^{(2)}/dT < 0$). Similarly, when the perturbation decays ($d|P_0|^2/dT < 0$), the energy is stored in the basic wave ($dR^{(2)}/dT > 0$).

For the purpose of removing resonant terms in (4.13) proportional to $e^{i\theta_1}$ and $e^{i\theta_0}$, a particular solution of the form

$$\psi_p^{(3)} = \sum_{n=0}^{1} P_n^{(3)}e^{i\theta_n} + *, \tag{4.16}$$

is taken. Substituting (4.16) into the left-hand side of (4.13) results in the following:

$$\left.\begin{aligned} &\tfrac{1}{2}\delta_1 Q_1^{(3)} + R_c b a_0 Q_0^{(3)} \\ &\quad = i\Big[K_1^2 \frac{\partial P_1^{(2)}}{\partial T} + i\frac{\Delta}{|\Delta|} b(K_0^2 - K^2)P_0 \\ &\qquad + i(w_1 - \tfrac{1}{2}\delta_1)K_1^2 C_1 \frac{W_2}{w} P_0 \\ &\qquad + ib(K_0^2 - K^2)P_0 R^{(2)} \\ &\qquad\qquad + 2ib(K_0^2 - K_1^2) f_1 P_0 |P_0|^2\Big] \\ &R_c b a_1 Q_1^{(3)} - \tfrac{1}{2}\delta_1 Q_0^{(3)} \\ &\quad = i\Big[i\frac{\Delta}{|\Delta|} b(K_1^2 - K^2)C_1 P_0 \\ &\qquad + i(w_0 - \tfrac{1}{2}\delta_1)K_0^2 \frac{W_2}{w} P_0 \\ &\qquad + ib(K_1^2 - K^2)C_1 P_0 R^{(2)} \\ &\qquad\qquad - 2ib(K_1^2 - \bar{K}_1^2)C_1 f_0 P_0 |P_0|^2\Big] \end{aligned}\right\} . \quad (4.17)$$

Combining (4.17), just as (4.5) was combined to get (4.6), and using (3.7), (3.8b), and (4.7b), results in the solvability condition

$$\frac{\partial^2 P_0}{\partial T^2} - \frac{\Delta}{|\Delta|}\sigma_i^2 P_0 - \sigma_i^2 P_0 R^{(2)} + \gamma_4 W_2 P_0 + \gamma_5 P_0 |P_0|^2 = 0. \quad (4.18)$$

In (4.18)

$$\lambda_4 = \delta_1/2, \quad (4.19a)$$

$$\gamma_5 = \frac{2b^2\delta_1(\bar{\delta}_0 - \delta_1)\bar{a}_1(K_1^2 - K_0^2)^2}{K_1^2(K_1^2 - K^2)D}, \quad (4.19b)$$

and

$$\sigma_i^2 = -2R_c b^2 a_1 a_0\,,$$

which agrees, as it should, with the linear result (3.11) obtained earlier. Eq. (4.18) closes the problem on the time scale of the instability. Since there is essentially no loss of generality in assuming $P_0 = P$ is real (e.g., Pedlosky, 1970), the closed set of nonlinear equations in $P$, $R^{(2)}$, and $W_2$ are

$$\frac{d^2P}{dT^2} - \frac{\Delta}{|\Delta|}\sigma_i^2 P - \sigma_i^2 P R^{(2)} + \tfrac{1}{2}\delta_1 P W_2 + \gamma_5 P^3 = 0, \quad (4.20a)$$

$$W_2 = \gamma_1 P^2, \quad (4.20b)$$

$$\frac{dR^{(2)}}{dT} = -\gamma_2 \frac{d(P^2)}{dT}. \quad (4.20c)$$

From (4.14b) $W_3 = 0$. This is not restrictive since $W_3$ is not coupled to the perturbation field, as $W_2$ is.

## 5. Discussion of results

By analogy with Pedlosky (1970) and inspection of (4.20), the behavior of (4.20) can be ascertained. With $\gamma_1 = 0$ and $\gamma_5 = 0$, (4.20) are similar in form to the amplitude equations obtained by Pedlosky (1970) in the context of inviscid equilibration for baroclinic instability. The feedback between the amplitude correction of the Rossby wave and the perturbation field is similar to the feedback between the vertical shear of the zonal flow and the baroclinic wave in Pedlosky's case. By this analogy and from (4.20a,c) it is clear that, as the perturbation to the Rossby wave grows, it extracts energy from the Rossby wave which acts to modify the Rossby wave towards stability and therefore slows the growth of the disturbance. Ultimately, the disturbance stops growing, but at this point the disturbance begins to feed energy to the Rossby wave, which later causes the perturbation to become unstable again. This feedback mechanism leads to an oscillatory behavior by itself. However, because the basic state is a wave field there is an additional feedback, not found when the basic state is a zonal flow, between the phase correction of the Rossby wave and the perturbation. This is the unique result of this paper. Suppose for the moment that $\gamma_2 = 0$ and $\gamma_5 = 0$. Then, the phase feedback is described by (4.20a,b), i.e.,

$$\frac{d^2P}{dT^2} - \sigma_{\text{eff}}^2 P = 0, \quad (5.1a)$$

$$W_2 = \gamma_1 P^2, \quad (5.1b)$$

where the effective growth rate, which accounts for the nonlinearity, is

$$\sigma_{\text{eff}}^2 = \sigma_i^2 - \tfrac{1}{2}\delta_1 W_2\,. \quad (5.1c)$$

The essence of this new feedback can be described as follows: The originally infinitesimal perturbation initially grows exponentially according to the linear theory. However, as the perturbation grows, the frequency of the Rossby wave is altered nonlinearly according to (5.1b). Inspection of (5.1c) shows that as the perturbation reaches finite amplitude, the frequency alteration becomes large enough to decrease the effective growth rate of the perturbation [note from (4.3b) that $(\delta_1/2)W_2 > 0$ when (3.6) is satisfied]. Continued growth of the disturbance further modifies the phase of the Rossby wave which changes the sign of $\sigma_{\text{eff}}^2$ and therefore $d^2P/dT^2$. As the Rossby wave frequency changes further, the disturbance stops growing ($dP/dT = 0$), and begins to decay (since $d^2P/dT^2 < 0$ when $dP/dT = 0$). Thus, the relative position of the Rossby wave and its disturbance is no longer conducive to instability. The amplitude at which this first occurs is

$$P_{\max}^2 = \frac{2\sigma_i^2}{\delta_1\gamma_1} + \left[\left(\frac{2\sigma_i^2}{\delta_1\gamma_1}\right)^2 + \frac{8E}{\delta_1\gamma_1}\right]^{1/2},$$

which was determined through the use of the first integral of (5.1), namely,

$$\tfrac{1}{2}(P_T)^2 + V(P) = E,$$

where

$$V(P) = -\tfrac{1}{2}\sigma_i^2 P^2 + \tfrac{1}{8}\delta_1\gamma_1 P^4,$$

$$E = \tfrac{1}{8}\delta_1\gamma_1 P^4(0),$$

and the initial condition

$$P_T(0) = \sigma_i P(0),$$

was used. As the perturbation decays the Rossby wave phase is further modified which, now, acts to modify the disturbance towards destabilization by increasing $\sigma_{\rm eff}$ in (5.1). The decay of the perturbation continues until the relative position of Rossby wave and perturbation is, again, one conducive to the instability of the perturbation. The amplitude at which this occurs is

$$P_{\rm min}{}^2 = -\frac{2\sigma_i^2}{\delta_1\gamma_1} + \left[\left(\frac{2\sigma_i^2}{\delta_1\gamma_1}\right)^2 + \frac{8E}{\delta_1\gamma_1}\right]^{1/2}.$$

At this value of $P$, $d^2P/dT^2 > 0$ so the disturbance begins to grow again. The cycle then is renewed after $P$ reaches its initial value. In summary, as the Rossby wave and perturbation travel through each other due to the nonlinear alteration of phase, the perturbation finds itself in an alternating unstable and stable environment which leads to an oscillatory behavior. Thus, Rossby wave instability is phase sensitive when nonlinear effects are allowed to act.

In general, both the amplitude and phase feedbacks between the Rossby wave and perturbation occur simultaneously with a third feedback between the perturbation and higher harmonics of the perturbation. The third feedback is represented by the last term of (4.20a) and can be either stabilizing ($\gamma_5 > 0$) or destabilizing ($\gamma_5 < 0$). All these feedbacks involve the tilting of ridges and troughs and, as was suggested by Lorenz (1972), need not involve a zonal flow.

A nonlinear equation for $P$ can be obtained when all the feedbacks are included by eliminating $W_2$ and $R^{(2)}$ from (4.20a), using (4.20b,c), i.e.,

$$\frac{d^2P}{dT^2} - \alpha P + \beta P^3 = 0, \tag{5.2}$$

where

$$\alpha = \sigma_i^2\left[\frac{\Delta}{|\Delta|} + \gamma_2 X^2(0)\right],$$

$$\beta = \frac{8b^2(K_1^2 - K_0^2)(K^2 - K_0^2)}{K^2K_1^2} \times \left[1 + \frac{\delta_1(\delta_1 - \bar{\delta}_0)K^2(K_1^2 - K_0^2)\tilde{a}_1}{4D(K^2 - K_0^2)(K_1^2 - K^2)}\right].$$

The nonlinear coefficient $\beta$ is stabilizing for most choices of basic state and perturbation. From (5.2) the solution for $P$ can be obtained in terms of elliptic functions. The further solution of (5.2) will not be given here because it is not particularly illustrative and the solution of an equation analogous to (5.2) is given in Pedlosky (1970).

As previously mentioned, the results of the three-term truncation are given in the Appendix. It is interesting to note that the form of the equations is not different from (4.20). In fact, this is true for any finite truncation with the exception of the symmetric case considered by Deininger and Loesch (1982). The present result differs from that of Loesch (1978) where a special symmetric case of the present three-term truncation was considered in which

$$k_0 = l = 0.$$

The discrepancy is rectified in Deininger and Loesch (1982). One contributing factor to the discrepancy can be most easily discussed in the context of this paper. This concerns the way in which phase changes of the Rossby wave are handled. Following Loesch (1978), I originally attempted to account for phase changes in the present problem by introducing a phase-shifted Rossby wave at $O(|\Delta|^{1/2})$, i.e.,

$$|\Delta|^{1/2}R^{(1)}\cos\phi,$$

where the basic state wave was $R_c \sin\phi$ and $\phi = kx + ly + wt$, and determining $R^{(1)}$ by the removal of secularities. Doing this resulted in the determination of $R^{(1)}$ as

$$R^{(1)} = |\Delta|^{1/2}\gamma_1 R_c \int |P_0|^2 dt,$$

which gives rise to a secularity and, therefore, invalidates the expansion procedure. Thus the time-stretching method used in this paper is the more desirable technique for calculating the phase change of the Rossby wave.

There are several extensions of this theory which broaden its applicability to the atmosphere. The first extension involves the truncation of the perturbation field. As previously mentioned, the form of the amplitude equations remains unaltered for any finite truncation. The coefficients, however, may change their numerical values depending on the truncation, which would be of concern in any specific application. To calculate these coefficients for a less severe truncation, numerical techniques would be of greater use to complete certain steps of the analysis than the analytic ones used here.

Second, this analysis for a barotropic Rossby wave is extendable, with little change, for a baroclinic perturbation, to a model of a continuously stratified, quasigeostrophic atmosphere with a lid. The same problem, but without an upper lid, is considerably more difficult and therefore has not been solved. Presumably, for waves which would be vertically trapped, the lack of an upper lid is not a serious obstacle; however, the inclusion of a vertically sheared zonal flow which could trap the waves results in further difficulties.

Yet a third extension was made to see if the same results are obtained for a basic state consisting of a neutral baroclinic mode in an atmosphere with no zonally uniform vertical shear in a two-layer model. The linear stability problem was studied by a number of authors, but most recently by Jones (1979). As Kim (1978) has pointed out, the linear eigenvalue problem separates into two separate branches when the depth of each layer is equal. The nonlinear analysis can be carried out on each of these branches just as was done in this paper. Although the mathematics is essentially the same, the physics is altered in this case. Now *both* momentum fluxes and temperature fluxes are responsible for the nonlinear feedbacks.

Perhaps the most serious obstacle to the application of this work to the atmosphere is the neglect of friction. A unit of long time could be as short as ~10 days, which is at least of the order of an Ekman spin-down time. Thus, it is important to include a forcing to maintain the wave against dissipation. If forcing is specified to maintain only the basic wave against dissipation, amplitude equations are obtained analogous to those obtained by Pedlosky (1970) for large friction and Pedlosky (1971) for small dissipation, but with some modification due to the additional phase feedback. The small dissipation case may lead to aperiodic vacillation. However, with a real forcing such as topography it becomes apparent that the higher-order effects of the forcing function on the perturbation are important. In the most interesting small dissipation case, where the long time scale and spin-down time scales are of the same order, the expansion procedure breaks down due to the effects of the forcing function. The questions posed by including friction and forcing deserve further study.

The assumption of weak nonlinearity may also limit the problem's applicability. The relaxation of the weakly nonlinear constraint would almost certainly lead to a behavior more complicated than the one presented here. The details of this would depend on the truncation. In fact, Baines (1976) in a truncated, fully nonlinear model shows that for less restrictive truncations, the system seems to forget the memory of the initial conditions, but his three-component system nevertheless exhibits periodic behavior.

## 6. Concluding remarks

In the previous sections the nonlinear amplitude equations have been obtained which govern the interaction between a slowly growing perturbation to a Rossby wave with itself and the Rossby wave. The resulting evolution of the Rossby wave and perturbation fields takes place on a time scale of the slow instability. In addition to the usual feedback between the amplitude of the basic flow and perturbation, as described by Pedlosky (1970), an additional feedback is present between the *phase* of the basic state Rossby wave and perturbation. The addition of the phase feedback is the main result of the present paper: its presence points out the difference between the weakly nonlinear stability of a parallel flow and that of a nonparallel flow in its simplest form (a wave). This phase feedback is a possible mechanism for the nonconstant speed of progression of midlatitude high and low pressure systems.

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## APPENDIX

### Three-Term Truncation Results

The coefficients of (4.3a), (4.14a), and (4.18) for the three-mode truncation, that is, when the $e^{i\theta_1}$, $e^{i\theta_0}$, and $e^{i\theta_{-1}}$ structures are retained, are

$$\gamma_1 = \frac{4b^2 a_0 K_0^2}{K^2}\left[\frac{K_0^2 - K_1^2}{K_1^2(\lambda_r + \delta_1)} - \frac{K_0^2 - K_{-1}^2}{K_{-1}^2(\lambda_r + \delta_{-1})}\right],$$

$$\gamma_2 = \frac{2b^2 R_c a_0 K_0^2}{K^2}\left[\frac{K_0^2 - K_1^2}{K_1^2(\lambda_r + \delta_1)^2} + \frac{K_0^2 - K_{-1}^2}{K_{-1}^2(\lambda_r + \delta_{-1})^2}\right] > 0,$$

$$\gamma_3 = \frac{2b^2 a_0 K_0^2}{K^2}\left[\frac{K_0^2 - K_1^2}{K_1^2(\lambda_r + \delta_1)^2} - \frac{K_0^2 - K_{-1}^2}{K_{-1}^2(\lambda_r + \delta_{-1})^2}\right],$$

$$\gamma_4 = \left[\frac{a_1}{(\lambda_r + \delta_1)^2} - \frac{a_{-1}}{(\lambda_r + \delta_{-1})^2}\right]\left[\frac{a_1}{(\lambda_r + \delta_1)^3} + \frac{a_{-1}}{(\lambda_r + \delta_{-1})^3}\right]^{-1},$$

$$\gamma_5 = \frac{1}{\left[\dfrac{a_1}{(\lambda_r + \delta_1)^3} + \dfrac{a_{-1}}{(\lambda_r + \delta_{-1})^3}\right]} \left\{ -\frac{2(\bar{K}_1^2 - K^2)(K_0^2 - K_1^2) f_1}{K_1^2 (K_0^2 - K^2)(\lambda_r + \delta_1) R_c} - \frac{4b(\bar{K}_0^2 - K^2)(K_1^2 - K_{-1}^2)}{K_1^2 K_{-1}^2 (\lambda_r + \delta_1)(\lambda_r + \delta_{-1})} f_0 + \frac{2(K_{-1}^2 - K_0^2)(K^2 - \bar{K}_{-1}^2) f_{-1}}{K_{-1}^2 (K_0^2 - K^2)(\lambda_r + \delta_{-1}) R_c} + \frac{8b^4 R_c^2 a_0^2 K_0^4 (K_{-1}^2 - K_1^2)[(K_1^2 - K^2)(4K^2 - K_{-1}^2) + (K_{-1}^2 - K^2)(4K^2 - K_1^2)]}{(4K^2 w - \beta k)(\lambda_r + \delta_1)^2 (\lambda_r + \delta_{-1})^2 K_1^4 K_{-1}^4} \right\}.$$

The last term of $\gamma_5$ is due to the production of the second harmonic of the basic wave which is not produced when only two modes are present.

$$\sigma_i^2 = \frac{2}{R_c}\left(\frac{a_1}{\lambda_r + \delta_1} + \frac{a_{-1}}{\lambda_r + \delta_{-1}}\right) \times \left[\frac{a_1}{(\lambda_r + \delta_1)^3} + \frac{a_{-1}}{(\lambda_r + \delta_{-1})^3}\right]^{-1}.$$

$R_c$ and $\lambda_r$ can be determined from the two equations

$$R_c^2 b^2 a_0 (a_1 + a_{-1}) = 3\lambda_r^2 + 2(\delta_1 + \delta_{-1})\lambda_r + \delta_1 \delta_{-1},$$

$$\lambda_r^3 + (\delta_1 + \delta_{-1})\lambda_r^2 + [\delta_1\delta_{-1} - R_c^2 b^2 a_0 (a_1 + a_{-1})]\lambda_r + R_c^2 b^2 a_0 (a_1 \delta_{-1} + a_{-1}\delta_1) = 0.$$

The $f_n$ are

$$f_0 = \frac{1}{D\bar{K}_0^2}[4b(K_1^2 - K_{-1}^2) C_1 C_{-1} (\bar{\delta}_1 + \lambda_r)(\bar{\delta}_{-1} + \lambda_r) - 4b^2 R_c \bar{a}_1 (K_1^2 - K_0^2)(\bar{\delta}_{-1} + 2\lambda_r) C_1 - 4b^2 R_c \bar{a}_{-1} (K_0^2 - K_{-1}^2)(\bar{\delta}_1 + 2\lambda_r) C_{-1}],$$

$$f_1 = \frac{2b}{\bar{K}_1^2 (\bar{\delta}_1 + 2\lambda_r)} \times [-R_c(\bar{K}_0^2 - K^2) f_0 + (K_1^2 - K_0^2) C_1],$$

$$f_{-1} = \frac{2b}{\bar{K}_{-1}^2 (\bar{\delta}_{-1} + 2\lambda_r)} \times [R_c(\bar{K}_0^2 - K^2) f_0 + (K_{-1}^2 - K_0^2) C_{-1}],$$

where

$$D = -4b^2 R_c^2 \bar{a}_1 \bar{a}_0 (\bar{\delta}_{-1} + 2\lambda_r) + (\bar{\delta}_1 + 2\lambda_r)(\bar{\delta}_0 + 2\lambda_r) \times (\bar{\delta}_{-1} + 2\lambda_r) - 4b^2 R_c^2 \bar{a}_{-1} \bar{a}_0 (\bar{\delta}_1 + 2\lambda_r),$$

$$C_1 = \frac{-R_c b a_0 K_0^2}{(\lambda_r + \delta_1) K_1^2}, \quad C_{-1} = \frac{-R_c b a_0 K_0^2}{(\lambda_r + \delta_1) K_{-1}^2}.$$

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