Properties of the Eliassen-Palm Flux for Planetary Scale Motions[1]

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(Manuscript received 3 November 1981, in final form 3 February 1981)

ABSTRACT

Properties of the quasi-geostrophic Eliassen-Palm (EP) flux for planetary scale motions are discussed, in order to clarify how these properties generalize from their beta-plane counterparts when no restriction on the variation of the Coriolis parameter is imposed. These properties include the relationships between the divergence of the EP flux and the meridional flux of potential vorticity, and between the EP flux, group velocity and refractive index.

1. Introduction

Use of the Eliassen-Palm (EP) flux as a diagnostic both of wave propagation and wave, mean-flow interaction has been made by a number of authors (see, e.g., Dunkerton *et al.*, 1981; Edmon *et al.*, 1980; Palmer, 1981a). While many of the properties of the quasi-geostrophic EP flux are straightforwardly and unambiguously defined on a beta plane (see Edmon *et al.*, 1980), a number of possible ambiguities have appeared in the literature when it is necessary to consider the variation $\Delta(\ln f)$ of the Coriolis parameter over some path segment of an EP flux trajectory. For planetary-scale motions these trajectories are observed to extend over a considerable region of the meridional plane, from high to low latitudes (see, e.g., Palmer, 1981a); hence such variation may be somewhat larger than a typical Rossby number.

In this paper we consider some of the properties of the EP flux for these scales of motion, in particular the relation between the EP flux divergence and the meridional flux of eddy potential vorticity, and the relations between the EP flux, group velocity and the zonal mean refractive index in the WKBJ limit. This latter diagnostic has appeared in a number of different forms (e.g., Butchart *et al.*, 1982; Karoly and Hoskins, 1982; and O'Neill and Youngblut, 1982) as that quantity whose gradient determines the refraction of group velocity paths or EP flux trajectories, and it is clearly of interest to ascertain which, if any, of these forms holds for planetary scale motions. In this paper a planetary-scale motion is formally defined to be one for which Burger's (1958) quasigeostrophic theory is appropriate. In this theory $\Delta(\ln f)$ is taken to be O(1).

It is found that all the beta-plane results can be carried over though a number of subtleties arise. Firstly, with the usual form for eddy potential vorticity on the sphere (e.g., as defined in Edmon *et al.*, 1980), the meridional flux of eddy potential vorticity is not proportional to the EP flux divergence, in the geostrophic approximation. However, a modified potential vorticity can be defined with the property that its meridional flux is equal to the form of the EP flux divergence as it appears in the zonal mean momentum equation. The resulting eddy potential vorticity equation is identical to that given by Matsuno (1970, 1971).

Secondly, the quantity that exactly describes the refraction of either group velocity paths or EP flux trajectories for planetary-scale motion is Matsuno's index of refraction squared divided by the sine of latitude squared. A consequence of this form of refractive index is that in low latitudes, EP flux trajectories should have little or no vertical component, even if the zonal mean wind is locally westerly.

Furthermore, the equation that expresses the refraction of the EP flux trajectories in the WKBJ limit cannot easily be written in a coordinate invariant form and there is a unique coordinate grid in the meridional plane (which differs from a latitude, height grid) on which EP flux trajectories are straight lines in the absence of a gradient in refractive index. One consequence of this is that on a standard latitude, height grid it is possible for EP flux trajectories to look curved yet not be refracted. In practice this coordinate distortion is not important because, in general, refraction effects will be dominant.

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0022-4928/82/050992-06$05.50

2. EP flux divergence and potential vorticity flux

With beta-plane geometry, the quasi-geostrophic EP flux, $\mathbf{F}$, is given in the (y–z) plane by

$$\mathbf{F} = e^{-z/H}\left(-\overline{u'v'}, f_0 \frac{\overline{v'\theta'}}{\bar{\theta}_z}\right).$$

Overbars and primes denote zonal means and departures from zonal means. The coordinate y represents northward distance, and z denotes a log-pressure coordinate with constant scale height H. Zonal and meridional velocity are given by u and v, f is the Coriolis parameter, equal here to a constant midlatitude value f_0, and θ is potential temperature. Static stability is given by $\bar{\theta}_z$ where subscripts x, y or z denote partial differentiation with respect to x, y or z.

Defining the quasi-geostrophic eddy potential vorticity flux on the beta plane as

$$q'_{(\beta)} = v'_x - u'_y + f_0\left(\frac{\theta'}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H}, \tag{2.1}$$

then

$$\begin{aligned}\overline{v'q'_{(\beta)}} &= -(\overline{u'v'})_y + \overline{u'v'_y} + \left(f_0 \frac{\overline{v'\theta'}}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H} - \frac{f_0}{\bar{\theta}_z}\overline{\theta' v'_z} \\ &= \nabla\cdot\mathbf{F}\, e^{z/H},\end{aligned}$$

since on a beta plane the geostrophic wind is nondivergent, and, from the thermal wind relationship, $v'_z \propto \theta'_x$. Here x is the zonal coordinate.

In spherical geometry the EP flux becomes

$$\mathbf{F} = r\cos\phi e^{-z/H}\left(-\overline{u'v'}, f\frac{\overline{v'\theta'}}{\bar{\theta}_z}\right), \tag{2.2}$$

where r is the radius of the earth and ϕ is latitude. With eddy potential vorticity on the sphere defined (as in Edmon *et al.*, 1980) by replacing Cartesian derivatives with spherical derivatives in (2.1), then

$$q' = v'_x - \frac{1}{\cos\phi}(\cos\phi u')_y + f\left(\frac{\theta'}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H}, \tag{2.3}$$

where

$$\frac{\partial}{\partial x} = \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}, \qquad \frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\phi},$$

and λ is longitude. Substituting for the geostrophic wind in (2.3), then for planetary-scale motions $\overline{v'q'}$ is no longer proportional to the EP flux divergence. The reason for this is simply the fact that for such scales the geostrophic wind is horizontally divergent.

If, on the other hand, a quantity $q'_{(M)}$ is defined by

$$q'_{(M)} = v'_x - \frac{f}{\cos\phi}\left(\frac{\cos\phi}{f}u'\right)_y + f\left(\frac{\theta'}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H}, \tag{2.4}$$

then

$$\begin{aligned}\overline{v'q'_{(M)}} &= -\frac{1}{\cos^2\phi}(\cos^2\phi\overline{u'v'})_y + \frac{\overline{(\cos\phi f v')_y u'}}{(\cos\phi f)} \\ &\quad + \left(f\frac{\overline{v'\theta'}}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H} - \frac{f}{\bar{\theta}_z}\overline{\theta' v'_z}, \\ &= \nabla\cdot\mathbf{F}/(r\cos\phi e^{-z/H})\end{aligned} \tag{2.5}$$

since $(\cos\phi f v')_y = -\cos\phi f u'_x$ for the geostrophic wind, and the thermal wind relationship $v'_z \propto \theta'_x$ still holds. The quantity on the right-hand side of (2.5) is simply the total eddy-induced forcing on the zonal mean circulation as given by the transformed Eulerian-mean zonal momentum equation in log-pressure coordinates (see Palmer, 1981a)

$$\frac{\partial\bar{u}}{\partial t} - f\bar{v}^* = \nabla\cdot\mathbf{F}/(r\cos\phi e^{-z/H}).$$

Here $\bar{v}^*$ is the meridional component of the residual circulation (see Edmon *et al.*, 1980).

The quantity $q'_{(M)}$ can be considered as a modified potential vorticity since the equation

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q'_{(M)} + \bar{q}_y v' = 0, \tag{2.6}$$

with

$$\bar{q}_y = \frac{2\Omega\cos\phi}{r} - \left\{\frac{1}{\cos\phi}(\bar{u}\cos\phi)_y\right\}_y + f\left(\frac{\bar{\theta}_y}{\bar{\theta}_z} e^{-z/H}\right)_z e^{z/H},$$

is identical to the linearized potential vorticity equation used by Matsuno (1970, 1971) for studying both planetary-wave propagation and the interaction of planetary waves with a zonal mean flow. The difference between $q'_{(M)}$ and q' is equal to $u' d\ln f/dy$, and arises because the meridional component of wind which advects planetary vorticity must, by scaling arguments, include not only the geostrophic wind, but also the isallobaric component

$$v'_I = \frac{1}{f}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)u'$$

(see Matsuno, 1970, Section 2; and Matsuno, 1971, Section 3a, for details). Hence

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q'_{(M)} = \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + v'_I\frac{df}{dy}.$$

Matsuno argued that the form (2.6) was necessary to be consistent with Lorenz's (1960) arguments on the energetics of approximate systems of equations. Equivalently, if (2.6) is multiplied by $q'_{(M)}$ and zonally averaged, then for linear conservative waves on a steady zonal flow we have the local conservation equation

$$\partial A_{(M)}/\partial t + \nabla\cdot\mathbf{F} = 0, \tag{2.7}$$

where

$$A_{(M)} = \frac{1}{2} r \cos\phi e^{-z/H} \overline{q'^2_{(M)}}/\bar{q}_y . \tag{2.8}$$

is an exactly conservable measure of local quasi-geostrophic planetary scale activity. In contrast, the quantity

$$A = \frac{1}{2} r \cos\phi e^{-z/H} \overline{q'^2}/\bar{q}_y ,$$

is only an approximate conservable density. Following the nomenclature of Edmon *et al.* (1980), $A_{(M)}$ may be referred to as the density of EP planetary scale wave activity.

Finally, we may define a velocity

$$\mathbf{C}_{(M)} = \mathbf{F}/A_{(M)} \tag{2.9}$$

which, from (2.7), advects linear conservative EP planetary scale wave activity in the meridional plane.

3. EP flux and group velocity

Consider a steady wave with zonal wavenumber k, frequency ω, and geopotential Φ. If ψ is defined by

$$\Phi = e^{z/2H}\,\mathrm{Re}\{\psi e^{i(\omega t - k\lambda)}\},$$

then (2.6) can be written as the second-order differential equation

$$\left(\omega - \frac{\bar{u}k}{r\cos\phi}\right)\left\{\mu^2 \frac{\partial^2\psi}{\partial Y^2} - \frac{k^2}{r^2\cos^2\phi}\psi + \frac{f^2}{N^2}\left|\nu^2\frac{\partial^2\psi}{\partial Z^2} - \frac{1}{4H^2}\psi\right|\right\} - \frac{k\bar{q}_y}{r\cos\phi}\psi = 0, \tag{3.1}$$

where $N^2 = N^2(z)$ is the Brunt-Vaisalla frequency,

$$\mu = \sin^2\phi/\cos\phi, \quad \nu = (N/\Omega)^2,$$

Ω is an arbitrary normalizing constant, and Y and Z are coordinates defined by the transformations

$$dY = \mu dy, \quad dZ = \nu dz,$$

which are introduced so that no first-order derivatives occur in (3.1). Putting

$$\psi = |\psi| e^{i\zeta}, \tag{3.2}$$

we can obtain locally wavelike (or WKBJ) solutions to (3.1) in the form

$$\zeta = lY + mZ, \tag{3.3}$$

where $|\psi|$, l and m are slowly varying functions (see below).

The dispersion relation for such a solution is, from (3.1),

$$\omega = \left(\frac{\bar{u}k}{r\cos\phi} - \frac{k\bar{q}_y}{r\cos\phi}\right)\Bigg/ \left\{\mu^2 l^2 + \frac{f^2}{N^2}\nu^2 m^2 + \frac{k^2}{r^2\cos^2\phi} + \frac{f^2}{4H^2N^2}\right\}, \tag{3.4}$$

The components of group velocity in the meridional plane are defined in the (Y, Z) coordinate system by

$$C_{(g)}{}^Y = \frac{\partial\omega}{\partial l} = 2\mu^2 k\bar{q}_y l \Bigg/ \left\{r\cos\phi\left(\mu^2 l^2 + \frac{f^2}{N^2}\nu^2 m^2 + \frac{k^2}{r^2\cos^2\phi} + \frac{f^2}{4H^2N^2}\right)^2\right\}, \tag{3.5}$$

$$C_{(g)}{}^Z = \frac{\partial\omega}{\partial m} = 2\nu^2 k\bar{q}_y m f^2 \Bigg/ \left\{r\cos\phi N^2 \times\left(\mu^2 l^2 + \frac{f^2}{N^2}\nu^2 m^2 + \frac{k^2}{r^2\cos^2\phi} + \frac{f^2}{4H^2N^2}\right)^2\right\} \tag{3.6}$$

(see, for example, Whitham, 1974). These expressions may be simplified by using the equality

$$f^2\overline{q'^2_{(M)}} = \tfrac{1}{2} e^{z/H}|\psi|^2\left(\mu^2 l^2 + \frac{f^2}{N^2}\nu^2 m^2 + \frac{k^2}{r^2\cos^2\phi} + \frac{f^2}{4H^2N^2}\right)^2.$$

Now since $\mu l = \zeta_y$, $\nu m = \zeta_z$, then from (2.8)

$$A_{(M)} = \frac{1}{4}\frac{r\cos\phi|\psi|^2}{\bar{q}_y f^2}\left\{(\zeta_y)^2 + \frac{f^2}{N^2}(\zeta_z)^2 + \frac{k^2}{r^2\cos^2\phi} + \frac{f^2}{4H^2N^2}\right\}^2, \tag{3.7}$$

$$C_{(g)}{}^Y = \frac{k|\psi|^2}{2f^2 A_{(M)}}\mu\zeta_y ,$$

$$C_{(g)}{}^Z = \frac{k|\psi|^2}{2N^2 A_{(M)}}\nu\zeta_z .$$

Hence, defining the pair $(C_{(g)}{}^Y, C_{(g)}{}^Z)$ to transform as a vector under a general coordinate transformation in the meridional plane, so that

$$C_{(g)}{}^y = \frac{dy}{dY} C_{(g)}{}^Y,$$

$$C_{(g)}{}^z = \frac{dz}{dZ} C_{(g)}{}^Z,$$

then the group velocity in (y, z) coordinates can be written as

$$C_g{}^y = \frac{k|\psi|^2}{2f^2 A_{(M)}}\zeta_y , \tag{3.8}$$

$$C_g{}^z = \frac{k|\psi|^2}{2N^2 A_{(M)}}\zeta_z . \tag{3.9}$$

Finally, writing (2.2) in terms of $|\psi|$ and ζ,

$$\mathbf{F} = \tfrac{1}{2}k|\psi|^2\left(\frac{1}{f^2}\zeta_y, \frac{1}{N^2}\zeta_z\right) \tag{3.10}$$

(a result which does not require ζ to have a locally-plane wave form). Hence combining (3.8)–(3.10),

$$\mathbf{F} = A_{(M)}\mathbf{C}_{(g)}. \tag{3.11}$$

If the EP flux is defined to transform as a vector from (y, z) coordinates to any other coordinate system, then (3.11) is a manifestly coordinate invariant result, and $\mathbf{F}$ is always parallel to $\mathbf{C}_{(g)}$. Furthermore, from (2.9), $\mathbf{C}_{(M)} = \mathbf{C}_{(g)}$ in the WKBJ limit.

From (3.7), (3.8) and (3.9), $A_{(M)}$, $C_{(g)}^y$ and $C_{(g)}^z$ do not depend on μ and ν. Hence, within the WKBJ approximation, the EP flux is also parallel to the group velocity derived from the eddy potential vorticity equation

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + \bar{q}_y v' = 0.$$

For WKBJ theory to apply, the phase function ζ must vary rapidly in space compared with any factor connected with the sphericity of the earth. On the other hand, for planetary-scale waves, the slowly varying functions ζ_y, ζ_z and $|\psi|$ will, by definition, have comparable variation with such spherical factors.

4. EP flux and refractive index

For simplicity consider a stationary wave so that the dispersion relation (3.4) can be written in the form

$$(\zeta_y)^2 + \frac{f^2}{N^2}(\zeta_z)^2 = \frac{Q}{r^2}, \tag{4.1}$$

where

$$Q = r^2\bar{q}_y/\bar{u} - k^2/\cos^2\phi - f^2r^2/4H^2N^2,$$

is Matsuno's refractive index squared. Now there is a unique coordinate system $(\tilde{y}, \tilde{z})$ in which (4.1) is essentially isotropic in horizontal and vertical derivatives. This is given by the transformation

$$d\tilde{y} = \frac{f}{\Omega}dy,$$

$$d\tilde{z} = \frac{N}{\Omega}dz,$$

whence (4.1) becomes

$$(\zeta_{\tilde{y}})^2 + (\zeta_{\tilde{z}})^2 = \tilde{Q}/4r^2, \tag{4.2}$$

where

$$\tilde{Q} = Q/\sin^2\phi.$$

It is important to note that the requirement that $d\tilde{y}$ and $d\tilde{z}$ be exact differentials, a property which will be used below, uniquely fixes this isotropic coordinate system. Notice also that each term in (4.2) is slowly varying, i.e., has variation comparable with the earth's sphericity.

The form (4.2) of the dispersion relation gives rise to a simple yet exact relation between the curvature of the integral curves of the EP flux (or EP flux trajectories) and the gradient of $\tilde{Q}$. This has been discussed briefly in Palmer (1981b) and used as a model diagnostic by Butchart *et al.* (1982); however, in view of the importance of this relation it is worthwhile giving a more extensive account here, emphasizing in particular the difficulty in expressing this result in a coordinate invariant form.

Differentiating (4.2) with respect to $\tilde{y}$ and $\tilde{z}$ gives

$$\zeta_{\tilde{y}}\zeta_{\tilde{y}\tilde{y}} + \zeta_{\tilde{z}}\zeta_{\tilde{z}\tilde{y}} = \frac{1}{8r^2}\tilde{Q}_{\tilde{y}}, \tag{4.3}$$

$$\zeta_{\tilde{y}}\zeta_{\tilde{y}\tilde{z}} + \zeta_{\tilde{z}}\zeta_{\tilde{z}\tilde{z}} = \frac{1}{8r^2}\tilde{Q}_{\tilde{z}}. \tag{4.4}$$

Now defining a vector whose components in $(\tilde{y}, \tilde{z})$ coordinates are

$$\mathbf{P} = (\zeta_{\tilde{y}}, \zeta_{\tilde{z}}),$$

then using the identity

$$\zeta_{\tilde{y}\tilde{z}} = \zeta_{\tilde{z}\tilde{y}},$$

(which, of course, only holds if $d\tilde{y}$ and $d\tilde{z}$ are exact differentials), (4.3) and (4.4) can be written as

$$(\mathbf{P}\cdot\nabla)\mathbf{P} = \frac{1}{8r^2}\nabla\tilde{Q}, \tag{4.5}$$

where ∇ stands for the gradient operator $(\partial/\partial\tilde{y}, \partial/\partial\tilde{z})$.

Now, since we have defined the EP flux to transform as a vector, its components in the $(\tilde{y}, \tilde{z})$ coordinate system are given by

$$F^{\tilde{y}} = \frac{d\tilde{y}}{dy}F^y = \frac{1}{2\Omega^2}k|\psi|^2\zeta_{\tilde{y}}, \tag{4.6}$$

$$F^{\tilde{z}} = \frac{d\tilde{z}}{dz}F^z = \frac{1}{2\Omega^2}k|\psi|^2\zeta_{\tilde{z}}. \tag{4.7}$$

Hence, in the $(\tilde{y}, \tilde{z})$ coordinate system (and therefore in all coordinate systems),

$$\mathbf{F} = \frac{1}{2\Omega^2}k|\psi|^2\mathbf{P},$$

and $\mathbf{F}$ is parallel to $\mathbf{P}$.

Now the integral curves $\mathbf{x}(t)$ of $\mathbf{P}$ (and therefore $\mathbf{F}$) are, by definition, solutions of the equation

$$\mathbf{P} = d\mathbf{x}(t)/dt \tag{4.8}$$

for some parameter t along the integral curves. Substituting (4.8) into (4.5) we have

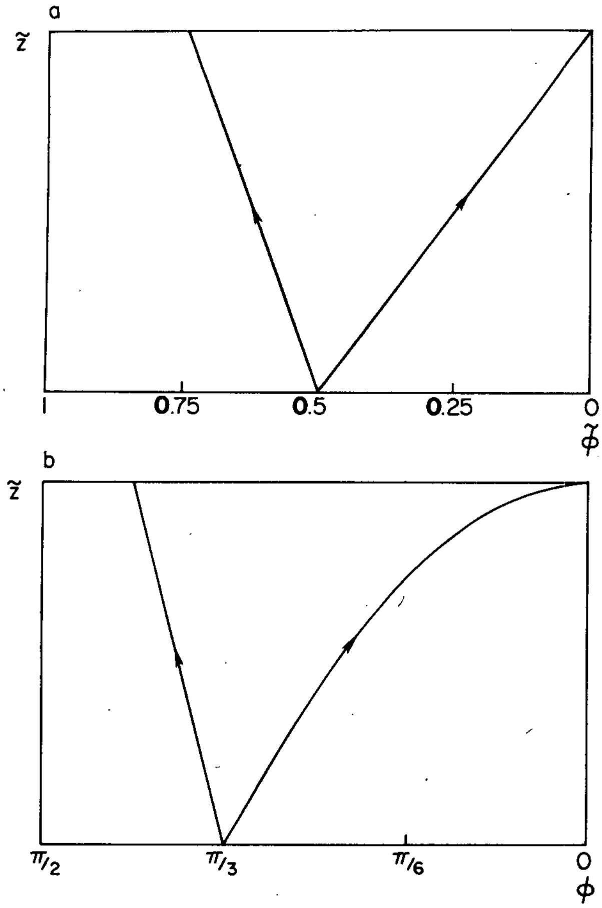

FIG. 1. (a) Two (hypothetical) EP flux trajectories on a $(\tilde{\phi}, \tilde{z})$ grid, where $\tilde{\phi} = (1 - \cos\phi)$ (so that $d\tilde{\phi} = \sin\phi d\phi$). Since the trajectories are straight the refractive index gradient is zero. (b) The same trajectories in $(\phi, \tilde{z})$ coordinates. There is no refraction, yet the trajectories are curved.

$$\frac{d^2\mathbf{x}(t)}{dt^2} = \frac{1}{8r^2}\nabla\tilde{Q}.$$

This equation shows that $\tilde{Q}$ acts as a potential function, curving the trajectories $\mathbf{x}(t)$ up its gradient.

Another way of seeing the role of $\tilde{Q}$ as a potential function is to consider the angle Θ that the EP flux (or $\mathbf{P}$, or $\mathbf{C}_{(g)}$) makes with the horizontal in the $(\tilde{y}, \tilde{z})$ coordinate system. From (4.6) and (4.7)

$$\tan\Theta = \zeta_{\tilde{z}}/\zeta_{\tilde{y}}\,.$$

The rate of change of Θ with respect to the parameter t is given by

$$\sec^2\Theta\,\frac{d\Theta}{dt} = \left(\frac{d\zeta_{\tilde{z}}}{dt}\,\zeta_{\tilde{y}} - \frac{d\zeta_{\tilde{y}}}{dt}\,\zeta_{\tilde{z}}\right)\Big/(\zeta_{\tilde{y}})^2,$$

which, rearranging and using (4.5) becomes

$$\frac{d\Theta}{dt} = \frac{1}{(\zeta_{\tilde{y}})^2 + (\zeta_{\tilde{z}})^2}\,[\tilde{Q}_{\tilde{z}}\zeta_{\tilde{y}} - \tilde{Q}_{\tilde{y}}\zeta_{\tilde{z}}]\,\frac{1}{8r^2}\,. \quad (4.9)$$

If we write

$$\nabla\tilde{Q} = \mathbf{i}(\nabla\tilde{Q})^{\parallel} + \mathbf{j}(\nabla\tilde{Q})^{\perp},$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors parallel and perpendicular to $\mathbf{P}$ respectively, then (4.9) can be written as

$$\left|\frac{d\Theta}{dt}\right| = \frac{|(\nabla\tilde{Q})^{\perp}|}{8\|\mathbf{P}\|r^2}, \quad (4.10)$$

where

$$\|\mathbf{P}\|^2 = (\zeta_{\tilde{y}})^2 + (\zeta_{\tilde{z}})^2.$$

Eq. (4.10) makes it clear that the integral curves of $\mathbf{F}$, $\mathbf{P}$, or $\mathbf{C}_{(g)}$ are refracted by the component of the gradient of $\tilde{Q}$ normal to these curves. For comparison, the form of refractive index used by Karoly and Hoskins (1982) is equal to $Q\cos^2\phi$, and the form used by O'Neill and Youngblut (1982) is essentially equal to Q. Both of these forms are only approximate for planetary-scale motions. For example, Karoly and Hoskins define a Mercator coordinate y^* such that $dy^* = \sec\phi\, dy$, and a vertical coordinate z^* such that $dz^* = N\sec\phi/f dz$. In terms of (y^*, z^*), (4.1) becomes

$$(\zeta_{y^*})^2 + (\zeta_{z^*})^2 = Q\cos^2\phi/r^2.$$

However, the analysis following (4.1) fails if it is applied to the above equation because

$$\zeta_{y^*z^*} \neq \zeta_{z^*y^*}\,,$$

i.e., dz^* is not an exact differential [alternatively, there is no function $z^*(y, z)$ satisfying $dz^* = N\sec\phi/fdz$]. The effect of allowing full variation of f is illustrated in Fig. 2 of Karoly and Hoskins (1982). Reference to that figure shows that ray paths calculated using full variation of f are more strongly refracted in low latitudes than if f is held at a constant midlatitude value. This is consistent with the rapid growth of $1/\sin^2\phi$ as $\phi \rightarrow 0$. One practical consequence of the singularity of $1/\sin^2\phi$ at $\phi = 0$ is that when refractive index theory is appropriate, EP fluxes should have little or no vertical component in low latitudes, irrespective of the position of the subtropical zero wind line, i.e., irrespective of whether Q itself becomes large in the tropics. This appears to be borne out by observation (Palmer, 1981a). Hence the dynamics embodied in the governing equations require that in the tropics the meridional heat flux of forced planetary waves should be very small.

The equations (4.5) and (4.10) only strictly hold in the $(\tilde{y}, \tilde{z})$ coordinate system. In practice, however, EP fluxes are plotted on a standard latitude–height grid (usually scaled by f_0^2/N_0^2, where N_0^2 is a typical value of N). With this grid EP flux trajectories may be curved both because the refractive index gradient is non-zero, and, independently, because of the distortion of the grid relative to $(\tilde{y}, \tilde{z})$ coordinates. In the stratosphere, where the Brunt-Vaisalla frequency is nearly constant, this variation is due almost entirely to the effect of the variation of the Coriolis parameter.

Fig. 1 illustrates the effect of this coordinate distortion. We assume that N^2 is constant so that the vertical scales in Fig. 1a and b are identical (and arbitrary). Fig. 1a shows two hypothetical EP flux trajectories under (unlikely) circumstances where the gradient of $\tilde{Q}$ is zero. Fig. 1b shows how this trajectory looks curved on a regular latitude–height diagram, even though there is no refraction. In high latitudes this curvature is not noticeable; south of ~45°N it becomes more noticeable. In practice, however, variations in the low-latitude refractive index (see above) would generally be dominant over the effects of this coordinate distortion.

The technical reason that the curvature of EP flux trajectories is coordinate-dependent is that in (4.5) the quantity $(\mathbf{P}\cdot\nabla)\mathbf{P}$ does not transform as a vector so that (4.5) is not, as written, a coordinate invariant equation. The machinery necessary to describe the law of refraction of the EP flux, or group velocity, in a manifestly coordinate invariant manner is well known in other branches of physics (see, for example, Sommerfeld, 1964) requiring the use of affine connection coefficients and the associated covariant derivative.

Acknowledgments. My thanks to Dr. A. O'Neill for many helpful discussions, and Dr. N. Butchart for correspondence. This research was supported by the National Aeronautics and Space Administration Grant NAG2-66.

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