NOTES AND CORRESPONDENCE

Comment on "Terrain-Following Coordinates and the Hydrostatic Approximation"

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ABSTRACT

By examining chain rule transformation and tensor transformation results, it is shown that the small slope assumption mentioned in Pielke and Martin (1981) is not required for the validity of the hydrostatic equation in terrain-following coordinates.

1. Introduction

In a recent paper by Pielke and Martin (1981), the equations for a numerical model using terrain-following coordinates were derived by applying tensor analysis. A major conclusion reached was that applying the chain rule to the hydrostatic equation before coordinate transformation yields a different set of equations than if the hydrostatic approximation is implemented after the tensor transformation is made. The results are the same only when the slope of the terrain is much smaller than 45°. It was then concluded that the set of hydrostatic model equations used by Mahrer and Pielke (1975) and others was not valid unless the slope of the terrain was small. It appears from the present analysis that the small-slope requirement stated in Pielke and Martin (1981) and also in Pielke (1981) is a result of the assumption that the acceleration of the transformed vertical velocity $\tilde{u}^3$, rather than that of the original Cartesian vertical velocity w, is negligible in the hydrostatic limit. The application of the chain rule and tensor analysis yield exactly the same result, and the existence of a small terrain slope is not a requirement for the validity of the equation for hydrostatic equilibrium, which results from negligible vertical acceleration in the rectangular Cartesian frame.

2. Coordinate transformation and the hydrostatic approximation

Following Pielke and Martin (1981), the contravariant form of the equations of motion, in a generalized coordinate system, obtained from the rectangular Cartesian x–y–z system is represented by

$$\frac{\partial \tilde{u}^i}{\partial t} + \tilde{u}^j \tilde{u}^i_{;j} = -\tilde{G}^{ij}\theta \frac{\partial \pi}{\partial \tilde{x}^j} - \frac{\partial \tilde{x}^i}{\partial z} g - 2\tilde{\epsilon}^{ijk}\tilde{\Omega}_j \tilde{u}_k, \quad (1)$$

where $\tilde{u}^i$ is the ith contravariant velocity component, $\tilde{u}_i$ the ith covariant velocity component, $\tilde{G}^{ij}$ the conjugate tensor, $\tilde{\epsilon}^{ijk} = \epsilon_{ijk}\tilde{G}^{-1/2}$, ϵ_{ijk} the permutation tensor, $\tilde{G}^{1/2}$ the Jacobian,

$$\tilde{u}^i_{;j} = \frac{\partial \tilde{u}^i}{\partial \tilde{x}^j} + \tilde{\Gamma}^i_{jl}\tilde{u}^l$$

is the covariant derivative and

$$\pi = c_p\left(\frac{p}{p_0}\right)^{R/c_p}$$

is the Exner function. The summation convention is used, and the indices may take on any of the values 1, 2 or 3.

The coordinate transformation in question is

$$\left.\begin{aligned} \tilde{x}^1 &= x \\ \tilde{x}^2 &= y \\ \tilde{x}^3 &= \sigma(x, y, z, t) = S\left(\frac{z - Z_G}{S - Z_G}\right) \end{aligned}\right\}, \quad (2)$$

where $\tilde{x}^i$, $i = 1, 2, 3$ are the transformed terrain-following coordinates, x, y, z are the rectangular Cartesian coordinates, S is the height of the model domain, and $Z_G = Z_G(x, y, t)$ is the terrain height. The forms of $\tilde{G}^{ij}$ and $\tilde{\Gamma}^i_{jl}$ for the above transformation are listed in Pielke and Martin (1981).

For the purpose of illustration, we consider the simple case of a two-dimensional (x, z) system where the Coriolis terms are negligible in comparison with the other terms in the equations. It can be shown that the vertical component of the tensor-transformed equation in the terrain-following coordinates, i.e.,

$$\frac{\partial \tilde{u}^3}{\partial t} + \tilde{u}^1 \frac{\partial \tilde{u}^3}{\partial \tilde{x}^1} + \tilde{u}^1\tilde{\Gamma}^3_{13}\tilde{u}^3 + \tilde{u}^3 \frac{\partial \tilde{u}^3}{\partial \tilde{x}^3} + \tilde{u}^3\tilde{\Gamma}^3_{31}\tilde{u}^1 + \tilde{u}^1\tilde{\Gamma}^3_{11}\tilde{u}^1 = -\tilde{G}^{31}\theta\frac{\partial \pi}{\partial \tilde{x}^1} - \tilde{G}^{33}\theta\frac{\partial \pi}{\partial \tilde{x}^3} - g\frac{\partial \tilde{x}^3}{\partial z} \quad (3)$$

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is equivalent to

$$\frac{\partial \tilde{u}^3}{\partial t} = \tilde{G}^{13}\left(-u\frac{\partial u}{\partial \tilde{x}^1} - u\tilde{G}^{13}\frac{\partial u}{\partial \tilde{x}^3} - \frac{w}{G^{1/2}}\frac{\partial u}{\partial \tilde{x}^3} - \theta\frac{\partial \pi}{\partial \tilde{x}^1} - \theta\tilde{G}^{13}\frac{\partial \pi}{\partial \tilde{x}^3}\right) + \frac{1}{\tilde{G}^{1/2}}\left(-u\frac{\partial w}{\partial \tilde{x}^1} - u\tilde{G}^{13}\frac{\partial w}{\partial \tilde{x}^3} - \frac{w}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3} - \frac{\theta}{\tilde{G}^{1/2}}\frac{\partial \pi}{\partial \tilde{x}^3} - g\right), \quad (4)$$

where $u = dx/dt$ and $w = dz/dt$. This involves using the velocity relationships

$$\left.\begin{aligned} \tilde{u}^1 &= u \\ \tilde{u}^3 &= \tilde{G}^{13}u + \frac{w}{\tilde{G}^{1/2}} \end{aligned}\right\}, \quad (5)$$

with

$$\tilde{G}^{13} = \frac{\tilde{x}^3 - S}{S - Z_G}\frac{\partial Z_G}{\partial x},$$

$$\tilde{G}^{1/2} = \frac{S - Z_G}{S},$$

$$\tilde{\Gamma}^3_{31} = \tilde{\Gamma}^3_{13} = -\frac{1}{S - Z_G}\frac{\partial Z_G}{\partial x},$$

$$\tilde{\Gamma}^3_{11} = \frac{S - \tilde{x}^3}{S - Z_G}\frac{\partial^2 Z_G}{\partial x^2},$$

$$Z_G = Z_G(x),$$

in a term-by-term substitution into (3) (see Appendix). Alternatively, Eq. (4) can be obtained directly by applying the chain rule relations

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \tilde{x}^1}{\partial x}\cdot\frac{\partial \phi}{\partial \tilde{x}^1} + \frac{\partial \tilde{x}^3}{\partial x}\cdot\frac{\partial \phi}{\partial \tilde{x}^3} = \frac{\partial \phi}{\partial \tilde{x}^1} + \tilde{G}^{13}\frac{\partial \phi}{\partial \tilde{x}^3} \\ \frac{\partial \phi}{\partial z} &= \frac{\partial \tilde{x}^3}{\partial z}\cdot\frac{\partial \phi}{\partial \tilde{x}^3} = \frac{1}{\tilde{G}^{1/2}}\frac{\partial \phi}{\partial \tilde{x}^3} \end{aligned}\right\}, \quad (6)$$

where ϕ may be π, u or w, to the equations of motion and considering

$$\frac{\partial \tilde{u}^3}{\partial t} = \tilde{G}^{13}\frac{\partial u}{\partial t} + \frac{1}{\tilde{G}^{1/2}}\frac{\partial w}{\partial t}. \quad (7)$$

The above relation (7) is obtained by simply taking the local derivative of (5) and assuming "diastrophic" terms (Deaven, 1976) negligible for physical realism, i.e., the topography is not varying in time, with

$$\frac{\partial \tilde{G}^{13}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\tilde{G}^{1/2}}\right) = 0.$$

Hence, one obtains, from either tensor analysis or chain rule application,

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial \tilde{x}^1} - u\tilde{G}^{13}\frac{\partial w}{\partial \tilde{x}^3} - \frac{w}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3} - \frac{\theta}{\tilde{G}^{1/2}}\frac{\partial \pi}{\partial \tilde{x}^3} - g \quad (8)$$

or

$$\frac{dw}{dt} = -\frac{\theta}{\tilde{G}^{1/2}}\frac{\partial \pi}{\partial \tilde{x}^3} - g,$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \tilde{u}^1\frac{\partial}{\partial \tilde{x}^1} + \tilde{u}^3\frac{\partial}{\partial \tilde{x}^3}.$$

Using the chain rule, Eq. (8) is obtained by direct application of (6) to the vertical equation of motion

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \theta\frac{\partial \pi}{\partial z} - g.$$

Using tensor analysis, Eq. (8) is obtained by comparing (4) and (7) and considering (A2). When the hydrostatic approximation is based on w acceleration being negligible, the hydrostatic equation in the terrain-following coordinates is

$$\theta\frac{\partial \pi}{\partial \tilde{x}^3} = -g\tilde{G}^{1/2}. \quad (9)$$

If the assumption is $\tilde{u}^3$ acceleration being small, then using (3) one obtains, after considering $\tilde{G}^{31} = \tilde{G}^{13}$ and $\tilde{G}^{33} = (\tilde{G}^{13})^2 + (\tilde{G}^{-1/2})^2$,

$$\theta\left\{\tilde{G}^{13}\frac{\partial \pi}{\partial \tilde{x}^1} + [(\tilde{G}^{13})^2 + (\tilde{G}^{-1/2})^2]\frac{\partial \pi}{\partial \tilde{x}^3}\right\} = -g\tilde{G}^{-1/2}. \quad (10)$$

When $\partial Z_G/\partial x$ is small, $\tilde{G}^{13}$ is negligible and (10) is reduced to (9).

By chain rule analysis, Eq. (9) is obtained whether the chain rule is applied to the hydrostatic equation, or the hydrostatic approximation is made after the chain rule is applied. Therefore, to say that application of the chain rule to the hydrostatic equation produces a different result from the hydrostatic approximation made after tensor transformation, implies that chain rule and tensor analysis produce different results in transforming the original equations, which appears to be not so in the present case.

3. Conclusion

With tensor transformation and chain rule transformation producing identical results, the difference between (9) and (10) must be due to the difference between neglecting $\tilde{u}^3$ acceleration and neglecting w acceleration.

In fact, since (8) is obtainable from tensor analysis, it can be concluded that any difference between (9) and (10) is entirely due to the difference between negligible $\tilde{u}^3$ acceleration and negligible w acceleration.

Evidently the conclusion does not depend on the use of the chain rule.

Generally, appreciable terrain gradient implies that possibly $w/u \approx \partial Z_G/\partial x$ is not negligible in layers close to ground and that the ratio of vertical to horizontal length scale is not sufficiently small to justify the application of the hydrostatic approximation. However, exceptions to this broad statement are not difficult to find. For example, in Smith's (1979) review of Queney's treatise, it is shown that mountain wave patterns cannot be described correctly by a hydrostatic set of equations even with very small slope. On the other hand, in quasi-steady small valley wind circulation on clear nights with light winds and strong inversion conditions, negligible w acceleration can occur without the gradient of terrain being particularly small. In any case, the hydrostatic approximation should involve only negligible w acceleration, as represented by (9). Neglecting $\tilde{u}^3$ acceleration, as in (10), gives only an approximation of (9), when $\partial Z_G/\partial x$ is small.

It is therefore concluded that (9) is valid whenever the hydrostatic approximation is valid. Although under certain conditions, terrain gradient may affect w acceleration to the extent that the hydrostatic approximation cannot be assumed to hold, the small slope assumption suggested by Pielke and Martin (1981) is required only for (10) to approximate (9) and not for the validity of (9).

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APPENDIX

Derivations of Eqs. (4) and (8) from Tensor Analysis Results

Given (5) and the definitions of $\tilde{G}^{13}$, $\tilde{G}^{1/2}$, $\tilde{\Gamma}^3_{31}$, $\tilde{\Gamma}^3_{13}$ and $\tilde{\Gamma}^3_{11}$, the terms in (3) become

$$\tilde{u}^1\frac{\partial \tilde{u}^3}{\partial \tilde{x}^1} = u\frac{\partial}{\partial \tilde{x}^1}\left(\tilde{G}^{13}u + \frac{w}{\tilde{G}^{1/2}}\right) = u\tilde{G}^{13}\frac{\partial u}{\partial \tilde{x}^1} + \frac{u}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^1} + u^2\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^1} + uw\frac{\partial}{\partial \tilde{x}^1}\left(\frac{1}{\tilde{G}^{1/2}}\right),$$

$$\begin{aligned}\tilde{u}^3\frac{\partial \tilde{u}^3}{\partial \tilde{x}^3} &= u\tilde{G}^{13}\frac{\partial}{\partial \tilde{x}^3}\left(\tilde{G}^{13}u + \frac{w}{\tilde{G}^{1/2}}\right) + \frac{w}{\tilde{G}^{1/2}}\frac{\partial}{\partial \tilde{x}^3}\left(\tilde{G}^{13}u + \frac{w}{\tilde{G}^{1/2}}\right)\\ &= u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} + u(\tilde{G}^{13})^2\frac{\partial u}{\partial \tilde{x}^3} + uw\tilde{G}^{13}\frac{\partial}{\partial \tilde{x}^3}\left(\frac{1}{\tilde{G}^{1/2}}\right) + u\frac{\tilde{G}^{13}}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3}\\ &\quad + \frac{wu}{\tilde{G}^{1/2}}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} + \frac{w\tilde{G}^{13}}{\tilde{G}^{1/2}}\frac{\partial u}{\partial \tilde{x}^3} + \frac{w}{\tilde{G}}\frac{\partial w}{\partial \tilde{x}^3} + \frac{w^2}{\tilde{G}^{1/2}}\frac{\partial}{\partial \tilde{x}^3}\left(\frac{1}{\tilde{G}^{1/2}}\right),\end{aligned}$$

$$\tilde{u}^1\tilde{\Gamma}^3_{13}\tilde{u}^3 = -u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} - \frac{uw}{\tilde{G}^{1/2}}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3},$$

$$\tilde{u}^3\tilde{\Gamma}^3_{31}\tilde{u}^1 = \tilde{u}^1\tilde{\Gamma}^3_{13}\tilde{u}^3,$$

$$\tilde{u}^1\tilde{\Gamma}^3_{11}\tilde{u}^1 = \frac{S-\tilde{x}^3}{S-Z_G}\cdot\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)\cdot u^2,$$

$$\tilde{G}^{31}\theta\frac{\partial \pi}{\partial \tilde{x}^1} = \tilde{G}^{13}\theta\frac{\partial \pi}{\partial \tilde{x}^1},$$

$$\tilde{G}^{33}\theta\frac{\partial \pi}{\partial \tilde{x}^3} = (\tilde{G}^{13})^2\theta\frac{\partial \pi}{\partial \tilde{x}^3} + \frac{\theta}{\tilde{G}}\frac{\partial \pi}{\partial \tilde{x}^3},$$

$$g\frac{\partial \tilde{x}^3}{\partial z} = \frac{g}{\tilde{G}^{1/2}}.$$

Note that in this case

$$\frac{\partial Z_G}{\partial \tilde{x}^1} = \frac{\partial Z_G}{\partial x}, \quad \frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right) = \frac{\partial^2 Z_G}{\partial x^2}.$$

After rearranging,

$$\begin{aligned}\frac{\partial \tilde{u}^3}{\partial t} = &-\frac{u}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^1} - u^2\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^1} - uw\frac{\partial}{\partial \tilde{x}^1}\left(\frac{1}{\tilde{G}^{1/2}}\right) - u\frac{\tilde{G}^{13}}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3} - \frac{w}{\tilde{G}}\frac{\partial w}{\partial \tilde{x}^3} + u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} + \frac{uw}{\tilde{G}^{1/2}}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} - \frac{S-\tilde{x}^3}{S-Z_G}\\ &\times\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)\cdot u^2 - \frac{\theta}{\tilde{G}}\frac{\partial \pi}{\partial \tilde{x}^3} - \frac{g}{\tilde{G}^{1/2}} - \underline{u\tilde{G}^{13}\frac{\partial u}{\partial \tilde{x}^1}} - \underline{u(\tilde{G}^{13})^2\frac{\partial u}{\partial \tilde{x}^3}} - \underline{w\frac{\tilde{G}^{13}}{\tilde{G}^{1/2}}\frac{\partial u}{\partial \tilde{x}^3}} - \underline{\tilde{G}^{13}\theta\frac{\partial \pi}{\partial \tilde{x}^1}} - \underline{(\tilde{G}^{13})^2\theta\frac{\partial \pi}{\partial \tilde{x}^3}}\\ &- u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} - \frac{uw}{\tilde{G}^{1/2}}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} + u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} + \frac{uw}{\tilde{G}^{1/2}}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3} - uw\tilde{G}^{13}\frac{\partial}{\partial \tilde{x}^3}\left(\frac{1}{\tilde{G}^{1/2}}\right) - \frac{w^2}{\tilde{G}^{1/2}}\frac{\partial}{\partial \tilde{x}^3}\left(\frac{1}{\tilde{G}^{1/2}}\right).\end{aligned} \tag{A1}$$

A total of 21 terms in Cartesian velocities and terrain-following coordinates appear on the right-hand side of the above equation. The last six terms disappear through cancelation or because $\tilde{G}^{1/2}$ is not a function of $\tilde{x}^3$. The five underlined terms may be written as

$$\tilde{G}^{13}\left[-u\left(\frac{\partial u}{\partial \tilde{x}^1}+\tilde{G}^{13}\frac{\partial u}{\partial \tilde{x}^3}\right)-\frac{w}{\tilde{G}^{1/2}}\frac{\partial u}{\partial \tilde{x}^3}-\theta\left(\frac{\partial \pi}{\partial \tilde{x}^1}+\tilde{G}^{13}\frac{\partial \pi}{\partial \tilde{x}^3}\right)\right].$$

The terms within the brackets are equivalent to $\partial u/\partial t$. This can be shown by direct application of the chain rule relations (6) to the horizontal equation of motion

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-w\frac{\partial u}{\partial z}-\theta\frac{\partial \pi}{\partial x},$$

or by using the tensor equation for the horizontal component

$$\frac{\partial \tilde{u}^1}{\partial t}+\tilde{u}^j\tilde{u}^1_{;j}=-\tilde{G}^{1j}\theta\frac{\partial \pi}{\partial \tilde{x}^j}-g\frac{\partial \tilde{x}^1}{\partial z}$$

[from (3) in Pielke and Martin (1981)],

which leads to

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial \tilde{x}^1}-\tilde{G}^{13}u\frac{\partial u}{\partial \tilde{x}^3}-\frac{w}{\tilde{G}^{1/2}}\frac{\partial u}{\partial \tilde{x}^3}-\theta\frac{\partial \pi}{\partial \tilde{x}^1}-\theta\tilde{G}^{13}\frac{\partial \pi}{\partial \tilde{x}^3}, \quad \text{(A2)}$$

because of the velocity relations (5) and the fact that $\tilde{G}^{11}=1$ and $\tilde{\Gamma}^1_{jl}=0$. In any case, according to (7), the remaining terms of (A1) should yield $\partial w/\partial t$ after multiplying through by $\tilde{G}^{1/2}$, the Jacobian of the transformation.

Jacobian weighting the remaining terms gives, after rearranging,

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial \tilde{x}^1}-u\tilde{G}^{13}\frac{\partial w}{\partial \tilde{x}^3}-\frac{w}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3}-\frac{\theta}{\tilde{G}^{1/2}}\frac{\partial \pi}{\partial \tilde{x}^3}-g$$

$$\underbrace{-\tilde{G}^{1/2}u^2\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^1}}_{(1)}\underbrace{-\tilde{G}^{1/2}uw\frac{\partial}{\partial \tilde{x}^1}\left(\frac{1}{\tilde{G}^{1/2}}\right)}_{(2)}$$

$$\underbrace{+\tilde{G}^{1/2}u^2\tilde{G}^{13}\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3}}_{(3)}\underbrace{+uw\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3}}_{(4)}$$

$$\underbrace{-\tilde{G}^{1/2}\cdot\frac{S-\tilde{x}^3}{S-Z_G}\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)\cdot u^2}_{(5)}.$$

The numbered terms cancel out since

(1) $$-\tilde{G}^{1/2}u^2\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^1}=-\frac{u^2(\tilde{x}^3-S)}{S}\cdot\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)-\frac{u^2}{S}\cdot\frac{\tilde{x}^3-S}{S-Z_G}\cdot\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)^2,$$

(2) $$-\tilde{G}^{1/2}uw\frac{\partial}{\partial \tilde{x}^1}\left(\frac{1}{\tilde{G}^{1/2}}\right)=-\frac{uw}{S-Z_G}\cdot\frac{\partial Z_G}{\partial \tilde{x}^1},$$

(3) $$\tilde{G}^{1/2}\tilde{G}^{13}u^2\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3}=\frac{u^2}{S}\cdot\frac{\tilde{x}^3-S}{S-Z_G}\cdot\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)^2,$$

(4) $$uw\frac{\partial \tilde{G}^{13}}{\partial \tilde{x}^3}=\frac{uw}{S-Z_G}\frac{\partial Z_G}{\partial \tilde{x}^1},$$

(5) $$-\tilde{G}^{1/2}\cdot u^2\cdot\frac{S-\tilde{x}^3}{S-Z_G}\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right)=\frac{u^2(\tilde{x}^3-S)}{S}\frac{\partial}{\partial \tilde{x}^1}\left(\frac{\partial Z_G}{\partial \tilde{x}^1}\right),$$

leaving

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial \tilde{x}^1}-u\tilde{G}^{13}\frac{\partial w}{\partial \tilde{x}^3}-\frac{w}{\tilde{G}^{1/2}}\frac{\partial w}{\partial \tilde{x}^3}-\frac{\theta}{\tilde{G}^{1/2}}\frac{\partial \pi}{\partial \tilde{x}^3}-g,$$

identical to the result of transforming the Cartesian vertical equation of motion by the chain rule.

Hence, it is shown that (4) and (8) can be derived from (3) and that chain rule transformation and tensor analysis yield the same results. The only difference between the two approaches is that tensor analysis gives equations with the transformed velocity components as dependent variables, whereas chain rule application retains the original Cartesian velocity components as dependent variables.

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