An Investigation of a Three-Dimensional Asymmetric Vortex

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ABSTRACT

A three-dimensional numerical simulation is presented for the asymmetric vortex motion which occurs in a Ward-type vortex chamber. The initial state is taken to be one of axisymmetric irrotational flow where the flow enters through the sides at the bottom and exits through the top of the chamber. As tangential velocity is added to the inflowing fluid, the structure of the flow in the meridional plane is modified from a ‘one-celled’ flow (updraft everywhere) to a ‘two-celled’ flow (updraft surrounding a central downdraft). Asymmetric vortices develop in the location of maximum vorticity of the ‘two-celled’ vortex which, it is shown, must be in the gradient between the updraft and the downdraft (but in updraft). Structural features of these asymmetric vortices, such as the tilt with height and propagation rate, are examined. Although the laboratory model upon which the present numerical calculations are based lacks the ability to simulate some important aspects of atmospheric flow, several significant features are shown to resemble the structure of observed tornados and mesocyclones.

1. Introduction

The present numerical model is based upon Ward’s (1972) laboratory model of a mesocyclone and may be considered the extension to three dimensions of the earlier axisymmetric calculations by the author (Rotunno, 1977; 1979). The basic physical model upon which Ward’s model is based has been discussed by Davies-Jones (1976), Church et al. (1979) and most recently by Snow (1982). Fig. 1 is a schematic diagram of the Purdue vortex generator (based on Ward’s design). There is a cylindrical chamber bounded above by a baffle over which suction is created by an exhaust fan. The rotating screen imparts angular momentum to the air as it enters the chamber. Among the various nondimensional numbers which might govern the flow, the swirl ratio

\[ S = \frac{R \Gamma_R}{2Q} \]  

(1)

has been found the most important (Davies-Jones, 1973). In this definition, \( R \) is the radius of the updraft hole, \( 2 \pi \Gamma_R \) is the circulation about the central axis at the outer radius of the convergence region, and \( 2 \pi Q \) is the rate of volume flow through the chamber. Thus, the laboratory device is a model of a thunderstorm updraft with rotation imposed on the inflow. Although Ward’s model is a coarse representation of an actual mesocyclone, lacking the effects of buoyancy and mean wind shear, I believe a number of relevant physical processes occur in this model. To the extent that the numerical simulation is successful, a complete data set is thus available to facilitate the physical interpretations of these processes. As discussed by Snow (1982), the type of vortex produced in the chamber varies significantly with \( S \); small values of \( S \) produce thin columnar axisymmetric vortices while large values of \( S \) produce large “two-celled” (a central downdraft surrounded by a rotating updraft) vortices. The latter type may, for larger \( S \), metamorphose into a system of secondary vortices existing near the boundary between the inner and outer cell; it is with this regime that the present work is concerned.

After the description of the three-dimensional numerical model (Section 2), the results of the model integrations are presented and analyzed in the following two sections. Section 3 concerns the evolution of the axisymmetric vortex from a state of no rotation. Of particular interest is the transition from a “one-celled” vortex (\( \omega > 0 \) everywhere) to a “two-celled” vortex (central downdraft surrounded by updraft), which occurs as rotation is added. Because, as it happens, the two-celled axisymmetric vortex is unstable to three-dimensional perturbations, the integration shows that after a transition period, a new asymmetric quasi-steady state is reached. Section 4 describes this transition and presents analysis of the structural features of the secondary vortices which are the most prominent feature of the asymmetric flow. Similarities with certain measurements made in the laboratory are discussed. Section 5 contains a discussion of the relation of the model results to observations and suggestions for future work.

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2. The model

The flow in the inner cylindrical region bounded by the radius ($R$) and the height of the baffle ($H'$) is studied herein (see Fig. 1). The equations governing the flow are the momentum equations, \( r \)-direction:

\[
\frac{du}{dt} + \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{\text{Re}} \left( \frac{\partial \eta}{\partial z} - \frac{1}{r} \frac{\partial \xi}{\partial r} \right),
\]

\( \theta \)-direction:

\[
\frac{dv}{dt} + \frac{uv}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{\text{Re}} \left( \frac{\partial \xi}{\partial r} - \frac{\partial \eta}{\partial r} \right),
\]

\( z \)-direction:

\[
\frac{dw}{dt} = -\frac{\partial p}{\partial z} + \frac{1}{\text{Re}} \left( \frac{1}{r} \frac{\partial \xi}{\partial r} - \frac{1}{r} \frac{\partial \eta}{\partial \theta} \right),
\]

and the continuity equation,

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{\text{Re}} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,
\]

where

\[
(u, v, w) = \frac{1}{u_R} (u', v', w'), \quad p = \frac{p'}{\rho u_R^2},
\]

\[
(\xi, \eta, \zeta) = \frac{R}{u_R} (\xi', \eta', \zeta')
\]

\[
(z, r) = \frac{1}{R} (z', r'), \quad t = \frac{u_R}{R} t'.
\]

Here \( u', v' \) and \( w' \) are the dimensional components of the velocity vector in the \( r', \theta, \) and \( z' \) directions, respectively; \( p' \) is the pressure, \( \rho \) the density and \( u_R \) is the magnitude of the radial velocity at the edge of the updraft hole. The quantities \( \xi', \eta' \) and \( \zeta' \) are the dimensional components of the vorticity vector in the \( r', \theta \) and \( z' \) directions, respectively. The lengths \( H' \) and \( h' \) are nondimensionalized by \( R; H = H'R^{-1} \) and \( h = h'R^{-1} \). The Reynolds number is defined as

\[
\text{Re} = \frac{u_R R}{\nu},
\]

where \( \nu \) is the kinematic viscosity.

The general approach toward the numerical solution of (2)-(5) is to specify \( u, v, w, \) and \( p \) at an initial time and then to 'march' (2)-(4) forward in time to obtain new values for \( u, v, w \). The divergence of (2)-(4) yields a Poisson equation for \( p \) which involves no time derivatives of the dependent variables, hence the new values of \( u, v, w \) may be used to calculate the corresponding new value of \( p \). The details of the numerical procedures are described by Williams (1969) and appropriate modifications for the present case are discussed in the Appendix. The computational domain and boundary conditions are displayed in Fig. 2; a discussion of these conditions follows.

1) \( z = 0 \). At the lower surface it is required that no flow leave or enter the domain, hence

\[
w_{|z=0} = 0.
\]
It will be demonstrated here that most observable features of the secondary vortices can be obtained using the free-slip conditions,

$$\frac{\partial u}{\partial z} \bigg|_{z=0} = \frac{\partial v}{\partial z} \bigg|_{z=0} = 0. \quad (9)$$

A condition for $p$ is obtained by evaluating (4) at $z = 0$, which yields $\partial p / \partial z \big|_{z=0} = 0$.

2) $z = H$. Fig. 1 shows there is a baffle through which the air must flow before leaving the lower chamber. I believe that an effect of the baffle is to eliminate horizontal air motion, that is

$$u_{l|z=H} = v_{l|z=H} = 0. \quad (10)$$

The condition on $v$ used here is different than that used in the previous studies by the author (Rotunno, 1977; 1979) where it was required that $\partial u / \partial z \big|_{z=H} = 0$. The reason for using (10) here is that it is a better approximation to the flow at the baffle. However, these boundary conditions are the cause of numerical reflections which must be suppressed by applying a filter to the uppermost four grid points (see the Appendix). A condition for $w$ is obtained by evaluating (5) at $z = H - \Delta z/2$ (see the Appendix). A condition for $p$ is obtained by integrating (2) from $a$ to $r$ at $z = H$ using (10); it is

$$p_{l|z=H} = p_{r=a} \bigg|_{z=H} - \int_a^r \left[ \frac{w}{\partial z} - \frac{1}{Re} \frac{\partial^2 u}{\partial z^2} \bigg|_{z=H} \right] dr, \quad (11)$$

where $a$ is the radius of a small cylinder placed on the central axis (for reasons discussed below). Because the absolute value of pressure is irrelevant,

$$p_{r=a} \bigg|_{z=H} = 0,$$

is arbitrarily specified.

3) $r = 1$. The edge of the updraft hole is modeled as follows. The radial velocity is specified constant over the depth of the convergence region ($h$). A fictitious wall is placed between $h$ and $H$ where the radial velocity is set to zero, thus

$$u_{l|r=1} = \begin{cases} -1, & 0 \leq z < h \\ 0, & h \leq z \leq H. \end{cases} \quad (12)$$

A condition on $w$ is obtained by requiring the azimuthal component of vorticity to be zero at $r = 1$, i.e.,

$$\frac{\partial w}{\partial r} \bigg|_{r=1} = \frac{\partial u}{\partial z} \bigg|_{r=1}. \quad (13)$$

This boundary condition is justified since the incoming air possesses zero azimuthal vorticity as it enters the device and encounters no sources in the convergence region upstream of the updraft hole save for the thin boundary layers on the upper and lower plates. The weaker approximation here is that embodied in (12).

Recent laboratory measurements made by Baker (1981, p. 96) show that in the region $(0 \leq z \leq h)$ the radial velocity becomes more negative with height before dropping to small values for $h \leq z \leq H$. Wilson (1981, p. 50) was able to obtain this behavior by solving Eq. (20) [which assumes zero azimuthal vorticity and is thus consistent with (13)] for an L-shaped region similar to that obtained by including the convergence region, the confluence region and the convective region (again with a fictitious wall at $r = 1$ extending upwards to the baffle). Wilson performed several experiments with the condition (12) and the improved condition and found little difference in the results in the interior. Since the present study was already at an advanced stage, the condition (12) was deemed satisfactory.

The azimuthal velocity is specified at $r = 1$ as

$$v_{l|r=1} = \frac{v_R}{u_R} \left[ 1 - \exp(-2i) \right], \quad 0 \leq z < h, \quad (14)$$

to simulate the spin-up of the rotating screen to a constant rotation rate. Again, the inflow profile of $v$ at $r = 1$ is taken to be independent of height; this agrees well with the laboratory measurements (see Baker, 1981, p. 96). That the condition (14) fixes the swirl ratio for the simulation may be seen by noting that $Q = R u_R h$ and $\Gamma_R = R v_R$ so that

$$S = \frac{R}{2h} \frac{v_R}{u_R} \quad (15)$$

Along the fictitious wall $(h \leq z \leq H)$ at $r = 1$, it is assumed that the vertical component of the vorticity is zero, i.e.,

$$\frac{\partial}{\partial r} (rv) \bigg|_{r=1} = 0. \quad (16)$$

This condition is found to be more compatible with the internal flow than a zero-stress condition because the latter requires $v \propto r$ (instead of $r^{-1}$ which is more consistent with the nearly irrotational flow in this part of the domain). The boundary condition on $\partial p / \partial r \bigg|_{r=1}$ is obtained from (2).

4) $r = 0$. There is in principle no need to require anything of the flow in the middle of the domain. However certain practical considerations dictate the use of a small inner cylinder which does not exist in the laboratory. The system of cylindrical coordinates, in which the code is written, has ever decreasing azimuthal grid lengths, $r \Delta \theta$, as $r \rightarrow 0$. The condition for viscous stability that $\Delta t < \text{Re}(r \Delta \theta)^2/8$ is thus very expensive to satisfy. Because most of the interesting dynamics take place in this problem about halfway between $r = 0$ and $r = 1$ and I desired to keep the code as simple as possible, the placement of a small inner cylinder (centered at $r = 0$ with radius $a$) was deemed a reasonable compromise. The conditions at this cylinder are that there be zero radial motion

$$u_{l|r=a} = 0, \quad (17)$$
zero shear stress in the azimuthal direction,
\[
\frac{\partial}{\partial r} \left( \frac{v}{r} \right) \bigg|_{r=a} = 0, \quad (18)
\]
and zero shear stress in the vertical direction,
\[
\frac{\partial w}{\partial r} \bigg|_{r=a} = 0. \quad (19)
\]
The condition on \( \partial p/\partial r \big|_{r=a} \) is then obtained from (2).

The sensitivity of the model results to the size of the inner cylinder was tested by letting \( a = 0.1 \) (0.02 was used in the present case); there was no significant alteration of the basic results. (Furthermore, Prof. J. T. Snow (personal communication, 1983) placed a length of garden hose on the center axis in a laboratory multiple-vortex experiment and detected no significant changes.)

The parameter values for this experiment are \( S = 1.25, \ Re = 150, \ H = 2, \ h = 0.4, \ a = 0.02 \). The Reynolds number is to be interpreted as a turbulent one based on an eddy viscosity, and so it is much lower than the values given by Church et al. (1979) based on the kinematic viscosity of air at room temperature. The decision not to implement a more complicated representation of the effects of turbulence will undoubtedly impair the simulation to some extent. However, because the laboratory data consist mainly of flow visualization and surface pressure measurements and the model can simulate the observed features, the extra costs of computer time and complexity that go along with such a representation were thought too high.

3. The axisymmetric flow

a. Time evolution

The philosophy of the present numerical experiment is to begin the simulation with a steady axisymmetric ‘vacuum-cleaner’-type flow without rotation. Then, rotation is added axisymmetrically at \( r = 1, 0 \leq z < h \) to simulate the effect of the rotating screen. This addition of rotation modifies the vacuum-cleaner flow into one which is unstable to asymmetric perturbations. In the course of this study, it was found that the time for this modification is much shorter than that for the development of the instability of the modified flow. Hence, to economize on computer time, an axisymmetric version of the model described in Section 2 is used over the initial period of integration (\( t = 0, 10 \)) and a new steady-state flow with rotation is thus obtained.

The flow at \( t = 0 \) is determined by the solution to the system
\[
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} = 0, \quad (20)
\]
\[
\psi_{z=0} = \psi_{z=-H} = \partial \psi/\partial z |_{z=-H} = 0; \quad \psi |_{r=1} = \int_0^z u(r,z) \, dz.
\]

The lhs of (20) is the azimuthal vorticity and \( \psi(r, z) \) is the nondimensional streamfunction. The solution to (20) then yields a state of axisymmetric irrotational flow as obtained by Rotunno (1977, Fig. 4.1). Fig. 3 displays the velocity and pressure fields associated with the solution to (20). The contour plot of the radial velocity indicates radial inflow monotonically increasing to zero at the center axis \( (r = a) \) and at the top \( z = 2 \). The vertical velocity increases upward (for \( 0 \leq z \leq 0.4 \)) as demanded by continuity given the radial variation of \( u \). There is a maximum in \( w \) in the vicinity of \( (r = 1, z = 0.4) \) which is the edge of the updraft hole. This behavior is demanded by the conditions (12) and (13) which require \( \partial w/\partial r \) to be large and positive (actually singular in the limit of perfect resolution) at \( r = 1, z = 0.4 \) and agrees with Baker's
measurement. The pressure then, by Bernoulli's theorem for irrotational flow, reflects the velocity field—there is low pressure near \( r = 1, z = 0.4 \) and a stagnation high at \( r = a, z = 0 \). This solution is, of course, a steady solution to the fully nonlinear system (2)–(5).

For \( t > 0 \), azimuthal velocity is introduced at \( r = 1 \) and is transported inward and upward. Fig. 4 displays the velocity and pressure fields in a stage of their evolution toward a steady state. The most prominent change, caused by the introduction of azimuthal velocity, is that the flow in the \( r-z \) plane has undergone a transition from a 'one-celled' field of vertical motion \( (w > 0 \text{ everywhere}) \) to a 'two-celled' field where a central downdraft is surrounded by updraft. This transition was explained by Rotunno (1977) as a consequence of the negative azimuthal vorticity produced by having a circulation \((\Omega)\) which decreases with height. But, because most analyses of the dynamics of tornado-like vortices (e.g., Morton, 1969; Smith and Leslie, 1979) and tornadic thunderstorms (Klemp and Rotunno, 1983) are in terms of velocity and pressure, it will be useful to perform a similar analysis here.

Why does the updraft turn to downdraft near the center axis as rotation is added? Fig. 4 indicates a somewhat complicated pressure distribution in the lower portion of the domain owing to the inflow layer, so consider for the moment the flow near \( z = 1 \), that is, about halfway to the top of the domain. To answer the question posed in a quantitative way, one must appeal to the vertical momentum equation; at \( r = a \) it is

\[
\frac{\partial w}{\partial t} = -\frac{\partial w^2}{\partial z} - \frac{\partial \rho}{\partial z} + \text{friction.} \tag{21}
\]

In the initial state, the first two terms on the rhs sum to zero, frictional effects are absent and, as already mentioned, there exists a steady state. After rotation is added and near \( z = 1 \), the radial motion field, its gradient and time derivative are small; hence the radial momentum equation becomes

\[
\frac{\partial \rho}{\partial r} = \frac{v^2}{r}, \tag{22}
\]

the equation of cyclostrophic balance. Integrating (22) from \( r = a \) to \( r = 1 \) obtains

\[
\rho(a, z, t) = \rho(1, z, t) - \int_a^1 \frac{v^2(r, z, t)}{r} dr. \tag{23}
\]

The vertical pressure gradient along the outer wall, near \( z = 1 \), is, to an excellent approximation,

\[
\left. \frac{\partial \rho}{\partial z} \right|_{r=1} = -\frac{\partial}{\partial t} w|_{r=1} - \frac{\partial}{\partial z} \left( w^2 |_{r=1} - w^2 |_{r=a} \right). \tag{24}
\]

Using this after substituting Eq. (23) into (21), I obtain

\[
\frac{\partial}{\partial t} w|_{r=a} - \frac{\partial}{\partial t} w|_{r=1} = \frac{1}{2} \frac{\partial}{\partial z} \left( w^2 |_{r=1} - w^2 |_{r=a} \right) + \frac{\partial}{\partial z} \int_a^1 \frac{v^2}{r} dr + \text{friction}, \tag{24}
\]

which is the time-dependent version of the equation derived by Hall (1966, p. 69). In the initial state, the three terms on the rhs of (24) are identically zero. For \( t > 0 \), rotation is added at low levels and hence the second term must be negative. The first term remains small until the added rotation can change the field of vertical motion from its initial state (in which \( w|_{r=1} \approx w|_{r=a} \)). Finally, the model results indicate that

\[
\left. \frac{\partial w}{\partial t} \right|_{r=1} \ll \left. \frac{\partial w}{\partial t} \right|_{r=a}. \tag{24}
\]
Hence, it may be concluded that the introduction of azimuthal motion at low levels leads to a central pressure that increases with height; this in turn is responsible for the negative acceleration on the center axis which leads to the central downdraft.

In deriving (24), attention is restricted to the middle of the domain away from the complicated features of the inflow and outflow regions. To shed some light on the general problem, the behavior of \( \int_0^2 w_{l-a} dz \) is considered. The vertical momentum equation (4) at \( r = a \) may be integrated vertically to obtain

\[
\frac{\partial}{\partial t} \int_0^2 w_{l-a} dz = -\frac{w^2(a, 2, t)}{2} + p(a, 0, t) + F(t),
\]  

(25)

where the conditions that \( w(a, 0, t) = p(a, 2, t) = 0 \) have been used and \( F(t) \) represents the frictional effect. Fig. 5a displays \( \int_0^2 w_{l-a} dz \) and (\( \partial/\partial t \)(\( \int_0^2 w_{l-a} dz \))) versus time. The latter quantity begins with a value of zero (because \( F(0) = 0 \) and \( p(a, 0, 0) = w^2(a, 2, 0)/2 \)) and becomes negative as time goes by. Accordingly, the integrated vertical velocity decreases from its initial positive value through zero toward negative values. Fig. 5b displays the three terms on the rhs of (25), and it is clear from this that the most important contributor to the negative tendency of the integrated vertical velocity is the drop in pressure at \( z = 0 \). To determine the cause of the drop in central pressure, the radial momentum equation is integrated along \( z = 0 \) from \( r = a \) to 1, the vertical momentum equation is integrated along \( r = 1 \) from \( z = 0 \) to 2 and the radial momentum equation is integrated along \( z = 2 \) from \( r = 1 \) to \( a \). Using the boundary conditions and noting that the tendency and frictional terms are small obtains

\[
p(a, 0, t) = \frac{w^2(1, 2, t)}{2} - \int_a^1 \frac{v^2(r, 0, t)}{r} dr.
\]  

(26)

The two terms on the rhs of (26) are displayed in Fig. 5c, which shows that the cause of the lowering central pressure (which induces the central downdraft) is the introduction of azimuthal motion at low levels.

b. The steady state

The steady-state flow is displayed in Fig. 6. Note that the radial velocity along the lower surface goes to zero at an intermediate radius indicating the separation of the inflow region off the lower surface. This radius is termed the core radius, \( r_c \). Davies-Jones (1973), in reanalyzing Ward’s (1972) laboratory data, found that \( r_c \) is a function of \( S \) only. The axisymmetric numerical model reported on by the author (Rotunno, 1977) also produced the same result, and furthermore, reproduced the observed functional dependence \( r_c(S) \). Examination of Fig. 5 reveals that the flow reaches a steady state at approximately \( t = 7 \). Substituting (26) into (25) and setting the time derivative equal to zero obtains for the steady-state flow

\[
\frac{w^2(1, 2)}{2} - \frac{w^2(a, 2)}{2} - \int_a^1 \frac{v^2(r, 0)}{r} dr + F = 0.
\]  

(27)

A complete theory for the core size must take all these terms into account (see e.g., Baker and Church, 1979; and Gall, 1982); however, a basic understanding can be obtained by acknowledging that the major balance in (27) is between the first and third terms (see Fig. 5 for \( t \geq 7 \)). For a set value of \( S \), there is a value of \( r_c \) which allows the central-pressure-raising Bernoulli effect [first term on the rhs of (26)] to balance the central-pressure-lowering cyclostrophic effect [second term on the rhs of (26)]. A similar physical argument is given by Lewellen (1971, p. 15).

The vorticity field will be of concern later in the discussion of the asymmetric vortex motion, so consider Fig. 7 which contains \( \xi, \eta, \) and \( \zeta \) in the \( r-z \) plane for the steady-state solution. Recall that the components of the vorticity vector are related to the components of the velocity vector for axisymmetric flow by the equations

\[
\xi = -\frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \quad \zeta = -\frac{1}{r} \frac{\partial}{\partial r}(ru).
\]  

(28a,b,c)
The vorticity can be roughly characterized as being small everywhere except in the cylindrical layer where \( w \) and \( v \) increase rapidly with radius and in the baffle layer near \( z = 2 \) where \( v \) has strong variation with \( z \). The discussion of vorticity distribution will be continued in Section 4, but for now I'd like to focus on the vertical component of vorticity. I will refer to \( \zeta \) simply as the 'vorticity' and investigate its behavior near \( z = 0 \). Comparison of Figs. 6 and 7 indicates the maximum vorticity, \( \zeta_{\text{max}} \), occurs almost exactly at \( r = r_c \). That \( \zeta_{\text{max}} \) should occur exactly at \( r = r_c \) can be proved as follows. It is easy to show that for axisymmetric steady flow, (3) may be written at \( z = 0 \), neglecting vertical diffusion, as

\[
\frac{\partial \zeta}{\partial r} = \text{Re} u \zeta. \tag{29}
\]

It is evident that \( \zeta \) cannot vanish for any \( r \) within the interval \( (a, 1) \) otherwise (29) would demand it vanish everywhere. Thus, (29) shows that \( \zeta \) takes an extreme value where \( u(r, 0) = 0 \) which is, by definition, \( r_c \). Since \( \zeta \) is positive it must remain positive because it cannot vanish anywhere. Hence, evaluating the derivative of (29) at \( r = r_c \), viz.,

\[
\frac{\partial^2 \zeta}{\partial r^2} \bigg|_{r=r_c} = \text{Re} \left( \frac{\partial u}{\partial r} \right) \bigg|_{r=r_c} \tag{30}
\]

shows that the extremum must be a maximum because \( \frac{\partial u}{\partial r} \bigg|_{r=r_c} > 0 \) and \( \frac{\partial u}{\partial r} \bigg|_{r=r_c} < 0 \). Close to the lower surface, the vertical velocity may be approximated, using (5) as

\[
w(r, z) \approx -z \frac{\partial}{\partial r} (ru(r, 0)), \tag{31}
\]

which shows that \( w(r_c, z) \approx -z \frac{\partial u}{\partial r} \bigg|_{r=r_c} > 0 \)—the vertical velocity is positive where \( \zeta = \zeta_{\text{max}} \). These relations are summarized in Fig. 8, where \( u(r, 0), \zeta(r, 0)/\zeta_{\text{max}}(0), \) and \( w(r, \Delta z)/w_{\text{max}}(\Delta z) \) are plotted. So over the lower portion of the domain, the maximum vorticity occurs at \( r = r_c \) (where \( u \) vanishes) and both \( w \) and \( \partial w/\partial r \) are positive.

4. The asymmetric flow

a. The instability

The feature of paramount importance in regard to the development of the secondary vortices is the sharp radial shear evident in the solutions for \( v \) and \( w \) (Fig. 6). Ward (1972) hypothesized that secondary vortices form as a result of the instability of the cylindrical shear layer associated with the radial variation of \( v \) (see his Fig. 20). At this stage, this hypothesis can be investigated by adding a small amount of random perturbations to the steady, axisymmetric flow and by then integrating the full three-dimensional equations.

The results of the test are affirmative; the flow depicted in Fig. 6 is unstable and asymmetries begin to develop shortly after the addition of random perturbations. After a time, the azimuthal wavenumber \( m = 2 \) appears to dominate; the unstable wave grows to finite amplitude while modifying the mean flow and subsequently reaches an equilibrium state where the finite amplitude wave may coexist with a modified mean flow (see Figs. 9 and 10).

Analytical results concerning the linear stability of flows \([u(r, z), v(r, z), w(r, z)]\), such as that represented by Fig. 6 (referred to as the base flow) are nonexistent. Some analytical results are available for flows \([0, v(r), w(r)]\); however, very few general statements can be made (see Howard and Gupta, 1962). A few general conditions concerning simpler flows and perturbations thereof are briefly reviewed here.
Rayleigh (1880) showed that the flow \([0, v(r), 0]\) is stable to two-dimensional \((r-\theta)\) perturbations if the vorticity \(r^{-2}\partial v/\partial r\) is a monotonic function of \(r\). Later Rayleigh (1916) found the flow \([0, v(r), 0]\) is stable to axisymmetric \((r-z)\) perturbations if \(r^{-3}\partial (r^2 v^2)/\partial r > 0\). Howard and Gupta (1962) generalized this criterion for flows \([0, v(r), w(r)]\); they found the flow is stable to axisymmetric perturbations if \(r^{-3}\partial (r^2 v^2)/\partial r (\partial w/\partial r)^{-2} > 1/4\). No such general criteria exist for three-dimensional perturbations to flows \([0, v(r), w(r)]\) even with \(w(r) = 0\). Hence the stability of such flows has been investigated on a case-by-case basis. Rather than examining the linear stability of some flow \([0, v(r), w(r)]\) which would anyway be a crude approximation to the base flow (owing to the lack of radial motion and variation with \(z\)), I describe some special numerical calculations made to investigate the stability of the base flow.

The stability of the base flow to axisymmetric disturbances is examined by adding random noise to the base flow and integrating forward with the axisymmetric version of the three-dimensional model described in Section 2. Although the Howard and Gupta (1962) criterion is violated at most horizontal levels, the numerical integrations reveal that the base flow is stable with respect to axisymmetric perturbations.

The idea that secondary vortices arise as an instability of a cylindrical layer of shear in the azimuthal velocity to two-dimensional (independent of \(z\)) perturbations has been proposed by Ward (1972), Davies-Jones and Kessler (1974), Snow (1978), and Staley and Gall (1979). Snow (1978) analyzed the linear stability of a cylindrical vortex sheet of finite thickness and found that no matter how the parameters (shear layer thickness, etc.) were varied, disturbances with azimuthal wavenumber \(m = 1, 2\) are stable. Rotunno (1978) came to the same conclusion for an infinitely thin vortex sheet. Hence it is impossible for the two secondary vortices to be a direct consequence of the instability of any flow \([0, v(r), 0]\) similar to the base flow. Now, it may be that in a nonlinear two-dimensional model, azimuthal wavenumber \(m = 2\) could gain energy through nonlinear transfer. To test this idea, a two-dimensional version of the three-dimensional model described in Section 2 using the flow \([0, v(r), 0]\) where \(v(r)\) is taken from the base flow at several
Different levels) as the initial condition was run. Again, after the addition of random noise, the ensuing integration gave no evidence of instability.

Hence, the inevitable conclusion to be drawn from these experiments is that the instability is an essentially three-dimensional phenomenon.

Although the base flow in unstable to three-dimensional disturbances, the early development of the instability does not show any clear-cut dominance of energy in disturbances with $m = 2$. Instead the energy in $m = 1, 2$ and 3 appears to dominate. Later, while the mean flow is being modified, the $m = 2$ gains the most energy while $m = 1, 3$ slowly recede.

b. The asymmetric steady state

The horizontal flow pattern on Fig. 9 clearly indicates two smaller centers of rotation within the larger vortex; the lowest pressures occur in association with the smaller vortices. These smaller vortices are identified here with the secondary vortices observed in the laboratory. To view the solutions with a similar perspective to that in the laboratory, Fig. 11 displays the horizontal vectors calculated by subtracting out the rotation rate of the secondary vortex pattern. This is because in the laboratory, one fixes attention on the secondary vortex and follows it as it travels around the center axis. Also in Fig. 11 is the vertical velocity field superimposed on the horizontal vectors. The small centers of rotation are clearly in the region where both $w$ and $\partial w/\partial r$ are positive, and this behavior, in light of the discussion which accompanies Fig. 8, is reasonable.

Above the lowest level, the axes of the secondary vortices tilt in the clockwise direction with height, although they are embedded in counterclockwise flow. This is illustrated in Fig. 12 where a three-dimensional contour representation of the pressure field is displayed. This behavior is similar to that observed in the laboratory; Fig. 13 contains a photograph of a flow visualization experiment conducted in the Purdue Tornado Vortex Simulator (Fig. 10 of Church and Snow, 1979). Here, two secondary vortices are present and they tilt clockwise with height and become nearly horizontal close to the upper baffle.

To gain an understanding of why secondary vortices
tilt in this manner, consider the vorticity distribution of the steady axisymmetric flow once again (Fig. 7). Recall that a vortex line is that curve which is everywhere tangent to the vorticity field. Hence, based on the characterization of the vorticity distribution given in Section 3, the vortex lines of the cylindrical shear layer are helices of negative pitch angle which expand slightly with height. Some of the vortex lines that intersect the maximum value of $\zeta$ at $z = 0$ are shown in Fig. 14. These lines represent integrations of the equations

$$\frac{d\nu}{dz} = \frac{\xi(r, z)}{\zeta(r, z)}, \quad \frac{d\theta}{dz} = \frac{1}{r} \frac{\eta(r, z)}{\zeta(r, z)},$$

where $\xi$, $\eta$, and $\zeta$ are obtained from Fig. 7. That the lines intersect the lower boundary at right angles is a consequence of the lower boundary conditions, (8) and (9), which imply $\xi(r, 0) = \eta(r, 0) = 0$. The lines bend in the negative azimuthal direction and slightly outward with increasing height until the baffle layer is reached. At this level, $\zeta$ decreases rapidly to zero [because of the upper conditions, (10)], and thus the lines bend over to the horizontal.

My hypothesis is that three-dimensional instabilities take the shape of these vortex lines as will the ultimate finite amplitude motion which is the secondary vortex.

The secondary vortices propagate at an angular velocity of approximately 1.9 rad per unit of time which is approximately one half the azimuthally averaged maximum rotation rate of the flow at $z = 0$. Thus, the secondary vortex pattern is in retrograde motion with respect to the mean flow. Ward (1972), in reference to the secondary vortices he observed in the laboratory apparatus, states “They (the secondary vortices) are located on opposite sides of the parent vortex near the radius of maximum tangential wind speed and the pair rotates around the central axis at about half that speed.”

Although I have no complete theory to explain this, I think it noteworthy that the angular propagation rate is close to that predicted by linear, two-dimensional
A steady state may exist only in the laboratory, the analysis of the eddy fluxes is omitted here, but the interested reader is advised to see Rotunno and Lilly (1981).

The maximum value of tangential velocity achieved in the asymmetric state (from Fig. 9, $v_{\text{max}} \approx 2.0^*$) is substantially greater than in the base flow ($v_{\text{max}} \approx 1.60$) or the azimuthally averaged flow at $t = 40\langle v_{\text{max}} \rangle \approx 1.77$. The intensity of the secondary vortices is also evinced by the fact the lowest pressures are found in association with them (see Fig. 9). The actual peak speed within a secondary vortex is probably greater than predicted by the present model owing to the coarse resolution and relatively low Reynolds number. Pauley et al. (1982) found in surface pressure measurements of a three-vortex system that the pressure drop was on the order of 2–3 times the pressure drop on the center axis. In the present model the pressure drop associated with the secondary vortex is approximately 1.5 times that of the center and so is a little on the low side of the observations, but still within the range of observed variability (Pauley, personal communication, 1983).

5. Relation of the model to observations
a. Relation of the model to the mesocyclone

Consider a strong updraft on a broad scale (diameter $\sim 6–8$ km). Suppose low-level rotation is introduced; the lower cyclostrophic pressure at low levels produces an adverse pressure gradient which acts on the updraft (essentially the 'vortex-valve' effect used to explain declining updraft strength by Lemon et al., 1975; see Section 3a). The response of the updraft does not have to be one of a uniform decrease, but rather, can be one of decrease only in the middle of the initially broad updraft; this leads to a two-celled vortex (downdraft surrounded by updraft). This vortex has the important property that the vertical component of vorticity is largest not where the vertical velocity is largest, but rather, close to the boundary between updraft and downdraft (but within updraft; Section 3b). This vortex is unstable to three-dimensional perturbations (Section 4a), which grow into finite amplitude asymmetric vortexes (Section 4b) propagating along the boundary between updraft and downdraft.

I believe an example of this sort of behavior in an actual mesocyclone is observed in the Doppler-radar study of Brandes (1978). Figs. 15 and 16 [Fig. 15 is composed of Brandes' (1978) Figs. 5 and 10] contain the azimuthally averaged flow with respect to the mesocyclone center both before (1543 CST) and during (1553 CST) the time a tornado is on the ground. Before the tornado is on the ground, there is a broad region of strong updraft in rotation. By the time the tornado is on the ground, however, there is a marked change in the flow pattern; the rotation at low levels has in-
creased and the vertical velocity field exhibits a maximum off the center axis with a slight indication of a central downdraft. Similar behavior is reported in another multiple Doppler-radar study of a different tornadic thunderstorm (Klemp et al., 1981). Comparison of Fig. 11 with the observed low-level flow in Fig. 16 or Klemp et al.'s Fig. 6a indicates similarity in the qualitative flow structure. There are in both studies elliptically shaped mesocyclones with strong vorticity at the foci when the tornado is most intense. The vorticity maxima (Fig. 8 of Brandes, 1978) are located in updraft but not in the location of maximum updraft as resolved by the multi-Doppler radar analysis. Rather, they are located in the gradient between the updraft and the downdraft. Further, there is an indication that these two vorticity maxima are displaced clockwise with height between the observation levels 0.3 and 1.3 km.

If one took $R = 3$–$4$ km (Brandes, 1978, Table 1), the scale of the smaller subvortex in Fig. 9 is roughly 1 km which is too large to be considered the scale of a visible funnel [$O(100$ m$)]$. Similarly, the smaller circulation at the foci of the elliptically shaped mesocyclone of Brandes (Fig. 16) is too large to be directly identified with the tornado. Indeed, Lewellen and Sheng (1981) take the boundary condition for their
Fig. 16. Horizontal storm relative winds (with radar reflectivity (dBZ) superimposed, at (a) 1543 CST (appearance of incipient tornado vortex) and (c) 1553 CST (tornado on the ground) and contour plot of the vertical velocity field at (b) 1543 CST and (d) 1553 CST at \( z = 300 \) m above ground level. The heavy dot indicates the position of a strong shear anomaly in the raw radar data and the letter D indicates downdraft. (From Figs. 4 and 5 of Brandes, 1978.)
axisymmetric numerical model (the height and radius of which is 1 km) from one such of these smaller vortices. Hence, if \( R \approx 3-4 \) km, the hypothesis is that the asymmetric vortices which appear in the present numerical simulation are akin to the asymmetric vortices found in Brandes (1978) and these are, in turn, the “parent circulations” of tornadoes (Agee et al., 1976; Lewellen and Sheng, 1981).

b. Relation of the model to the tornado

As mentioned in the Introduction, the laboratory vortex chamber produces narrow, axisymmetric vortices for small \( S \) and vortices of the ‘two-celled’ variety for larger \( S \). If one considers \( R \) the radius of the tornado’s parent circulation (nominally 1 km), the type of vortex motion that results will again depend on the effective \( S \) of the smaller domain. Laboratory experiments show (Baker, 1981) that for \( 0.1 < S < 0.5 \) very intense narrow core vortices are produced. These vortices exhibit a strong vertical velocity on the central axis which decays rapidly with distance away from the origin (essentially a “vortex jet”; see Morton, 1969), and there is a high degree of amplification of the azimuthal velocity over inflow values. My guess is that when the effective \( S \) of the parent circulation is in this range, the tornado, although within a parent circulation which is asymmetrically located with respect to the mesocyclone, will have an axisymmetric character. On the other hand, if the effective \( S \) of the parent circulation is \( O(1) \) or larger, the tornado will be one containing multiple vortices.

Multiple vortices were discovered by Fujita (1971); an example of this phenomenon comes from the film of the multiple-vortex tornado which occurred near Orienta, Oklahoma on 2 May 1979. Fig. 17 contains a time sequence of frames reproduced from the film. Here, the view is to the south and the whole system is moving toward the east (left). The sequence shows that the eastern vortex \( a \) and western vortex \( b \) approach each other in a fashion where \( a \) is in the foreground and \( b \) in the background. Funnel \( b \) clearly tilts clockwise with height (a does too, but less obviously so; the tilt is more apparent in the motion picture) although embedded in a counterclockwise flow.

It is not possible to simulate the low-\( S \) case using a free-slip lower boundary condition as employed in the present study; Rotunno (1979) has shown the narrow jet-like vortex is a direct consequence of boundary layer convergence which is absent when one uses a free-slip condition. Owing to the coarse grid resolution and lack of a more sophisticated turbulence parameterization, it was not possible to include this effect in the present study. However, it is my view that an important next step must be to investigate flows with both boundary-layer processes and asymmetric flow conditions. The existence of intense asymmetric jets of inflow into tornadoes (see Golden and Purcell, 1978)

![Fig. 17. Photograph of a multiple vortex tornado near Orienta, Oklahoma, 2 May 1979. The view is to the south and the system is moving to the east (left). (From the NSSL motion picture by K. Emanuel.)](image)

and the position of the visible funnel very close to downdrafts (Lemon and Doswell, 1979) suggest these factors could be crucial.

6. Summary

A three-dimensional numerical simulation of the asymmetric, secondary vortices which occur in a Ward (1972) type vortex chamber has been discussed. Principal findings of this study are as follows.

Starting with an axisymmetric irrotational flow where the flow is upward at all levels, a central downdraft is induced by the introduction of swirling motion at low levels. The analysis of the model data reveals that this downdraft is affected by the lowered pressure
on the central axis associated with the introduction of tangential motion at low levels. An axisymmetric steady state is reached where there is a weakly swirling central downdraft surrounded by a strongly swirling updraft; all three components of vorticity are large on this boundary. This flow is unstable to three-dimensional perturbations which grow to finite amplitude and persist in a quasi-steady equilibrium with the azimuthally averaged flow. These secondary vortices are identified with those in the laboratory and evince several important similarities. The relation between the model results (and ipso facto the laboratory model results) to observations of an observed mesocyclone and tornado is discussed.

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APPENDIX

The Numerical Model

The numerical integration procedure follows that of Williams (1969) except that the nonlinear advective terms are put into the finite-difference form suggested by Piacsek and Williams (1970).

Because the upper boundary condition on \( u \) and \( v \) is reflective, a spatial filter is used over the uppermost four grid points to damp the "2-D" waves produced by the application of this condition. Physically, the upper boundary will induce turbulent mixing as the swirling air streams to pass through. One could use a height-dependent eddy viscosity, but the values needed are large enough that our (already small) step time would have to be reduced further to satisfy the diffusive stability criterion. Hence the following procedure was thought a reasonable compromise. On the staggered mesh the upper boundary lies between \( u_{N+2} \) and \( u_{N+1} \); zero velocity at this point requires that \( u_{N+2} = -u_{N+1} \). Following Shapiro (1970), the smoothing operation

\[ \tilde{u}_j = \frac{u_{j-1} + u_{j+1}}{2}, \quad j = N - 1, N + 1, \]

is applied to the interior points at each time step. To insure that \( \tilde{u} = 0 \) at \( z = H \), \( \tilde{u}_{N+2} = -\tilde{u}_{N+1} \). Similar comments apply to \( v \). With \( u \), \( v \) and \( w \) known at all the interior grid points, \( w_{N+1} \) (defined at \( z = H \)) is computed by applying the continuity equation at \( z = H - \Delta z/2 \) (see Roache, 1976, p. 196).

The time step is chosen so as to satisfy the CFL condition and the diffusive stability criteria, viz.,

\[ \Delta t < \min \frac{1}{8} \left( \frac{r^2 \Delta \theta^2}{u_{\text{max}}}, \frac{\Delta z^2}{w_{\text{max}}}, \frac{\Delta r^2}{u_{\text{max}}} \right). \]

In this study, \( \Delta t = 0.001, M = 30, N = 30, L = 32 \), and integrations leading to the results in Figs. 9 and 10 were carried out 40 000 time steps.

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