# Fluctuations and Dissipation in a Barotropic Flow Field[1]

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ABSTRACT

Herein we present the first systematic derivation of stochastic mode rate equations for a geophysical hydrodynamic system. Coarse graining concepts from nonequilibrium statistical mechanics are applied to the vorticity equations for barotropic motion on a $\beta$-plane. The projection of the initial field equations onto a restricted subspace yields nonlinear stochastic mode rate equations for the physical observables with completely determined statistical properties. It is shown that the usual assumptions of ergodicity and Markovicity are valid only under some very restrictive conditions.

## 1. Introduction

A dominant characteristic of meteorological data is its extreme variability. As discussed by Monin (1972), it is this broadband response that makes the predictability of weather patterns from deterministic primitive equations so uncertain. The status of this indeterminacy was the topic of a recent conference (Holloway and West, 1983). The fluctuations in atmospheric flow fields that give rise to the indeterminacy have been associated with small-scale turbulence. The scales of these fluctuations are unresolved in global circulation models even though their effects are manifest on these large scales. One technique for treating these fluctuations analytically is to replace the deterministic primitive equations by stochastic equations (e.g., Thompson, 1957; Landau and Lifshitz, 1959; Lorenz, 1969; Leith, 1971; Hasselmann, 1976; Holloway and Hendershott, 1977; Egger, 1982). Such replacements have in the past been purely phenomenological, but more recent studies (Lindenberg and Seshadri, 1981; West, 1982; West and Lindenberg, 1983) suggest how one may proceed more systematically from the deterministic to the stochastic representation.

In writing phenomenological stochastic equations of motion, it has generally been recognized that the stochasticity is induced by the unresolved degrees of freedom in the flow field. The stochastic equations are understood to be a coarse-grained representation of the underlying primitive equations as discussed, for example, by Hasselmann (1976). Although it is appreciated that such relationships exist, quantitative models with explicit functional relationships have not previously been developed. Our purpose in the present paper is to construct such a model and to test the assumptions, which are usually justified by qualitative arguments in the phenomenological constructs.

The strategy we adopt is to coarse grain (Lax, 1966; Zwanzig, 1973; Lindenberg and Seshadri, 1981; West, 1982) an atmospheric flow field and explicitly determine the statistical properties of the resulting projected description. Specifically, we investigate barotropic motion on a $\beta$-plane subject to doubly periodic boundary conditions. Although simple, the hydrodynamic content of this model is sufficiently rich that we are able to develop our arguments fully. Using this model we are also able to show that the coarse-grained description indeed yields a stochastic equation in which the unresolved degrees of freedom show up in fluctuating as well as in systematic ("dissipative") terms. These terms are related to one another via "fluctuation–dissipation relations" that have not previously arisen in this context and that are crucial to ensure the proper balance of the fluxes through the system. The stochastic equations that we obtain take the form of the usual phenomenological models only under some very restrictive conditions. In general, the properties of ergodicity and Markovicity so often assumed in the phenomenological context are shown here not to be valid. Even when the conditions for ergodicity and Markovicity are satisfied, the fluctuations are delta-correlated in time only if additional constraints are also satisfied.

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In Section 2 we describe the coarse graining procedure and illustrate its application using a general linear system. In Section 3 we then apply the method to the (nonlinear) barotropic equations and derive a stochastic description of the observable modes of oscillation of the atmosphere. In Section 4 we discuss conditions under which these general equations take the form of the more familiar phenomenological equations, as well as some limitations of the method.

## 2. Linear stochastic model

In this section we present the arguments that enable us to replace a set of deterministic equations by an *equivalent* set of stochastic differential equations. Following our earlier work (West and Lindenberg, 1983), we assume that the state of the dynamic system can be described by $N$ variables $\{A_\alpha(t)\}$, $\alpha = 1, 2, \ldots, N$, where, for example, the state vector $\mathbf{A}(t)$ might represent the mode amplitudes of velocity, density, temperature, etc. in a general flow field. These amplitudes may be fixed grid points in a large-scale simulation or they may be the time-dependent amplitudes for the expansion of the physical observables in a series of eigenfunctions, where the limit $N \to \infty$ can be taken. The evolution of the flow field can be described by a deterministic trajectory $\Gamma_t(\hat{\mathbf{a}})$ in an $N$-dimensional phase space with the set of axes $\hat{\mathbf{a}} = \{\hat{a}_\alpha\}$ corresponding to the full set $\{A_\alpha(t)\}$ of dynamic variables. The curve $\Gamma_t(\hat{\mathbf{a}})$ is indexed by the time $t$, which is a parameter in the phase space. A trajectory begins at a specified point $\hat{\mathbf{a}}_0 = \{A_\alpha(0)\}$ and describes the evolution of the flow toward its final state $\{A_\alpha(\infty)\}$. In practical calculations, only a small number $M$ of variables ($M \ll N$) is used to represent the flow field. Mathematically this means that one is interested only in the projected trajectory

$$\Gamma_t(\mathbf{a}) = P\Gamma_t(\hat{\mathbf{a}}), \tag{2.1}$$

where $\mathbf{a} = \{a_\alpha\}$ with $\alpha = 1, 2, \ldots, M$ and where $P$ denotes an appropriate integration over the variables $a_{M+1}, a_{M+2}, \ldots, a_N$. If we consider two trajectories with identical initial values $A_1(0), A_2(0), \ldots, A_M(0)$, but with different initial values $A_{M+1}(0), \ldots, A_N(0)$, the two projected trajectories may be different. Thus, two trajectories ostensibly initiated from the same state $A_1(0), \ldots, A_M(0)$ in the reduced phase space can follow different phase space orbits in the reduced space. We refer to the instantaneous differences between two such trajectories as fluctuations and presume that they admit of a statistical description. The statistics enter through the specification of the initial conditions of the eliminated degrees of freedom; since these are not observable, one can specify a distribution of initial conditions $A_{M+1}(0), \ldots, A_N(0)$ consistent with the "macroscopic" initial state of flow.

For clarity we restrict the discussion in this section to a linear model system. This system is not hydrodynamiclike and is chosen for purely pedagogical reasons. We separate the dynamic variables $\{A_\alpha(t)\}$ into the vectors $\mathbf{X} = [A_1(t), \ldots, A_M(t)]$ and $\mathbf{Y} = [A_{M+1}(t), \ldots, A_N(t)]$ and write the dynamic equations as

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{AX}(t) + \mathbf{BY}(t), \tag{2.2}$$

$$\frac{d}{dt}\mathbf{Y}(t) = \mathbf{CY}(t) + \mathbf{EX}(t), \tag{2.3}$$

where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{E}$ have known time-independent elements. We refer to $\mathbf{X}(t)$ as the vector of physical observables and to $\mathbf{Y}(t)$ as the vector of unresolved or unobserved degrees of freedom.

To eliminate the unresolved motion from the dynamic description of the physical observables, we first formally integrate (2.3) to obtain

$$\mathbf{Y}(t) + \mathbf{C}^{-1}\mathbf{EX}(t) = e^{\mathbf{C}t}(\mathbf{Y}_0 + \mathbf{C}^{-1}\mathbf{EX}_0) + \int_0^t d\tau e^{\mathbf{C}(t-\tau)}\mathbf{C}^{-1}\mathbf{E}\dot{\mathbf{X}}(\tau) \tag{2.4}$$

in terms of the initial values $\mathbf{Y}_0 \equiv \mathbf{Y}(0)$ and $\mathbf{X}_0 \equiv \mathbf{X}(0)$ and the time rate of change $\dot{\mathbf{X}}(\tau)$ in the physical observables in the time interval $(0, t)$. The evolution of unresolved motion away from the initial state of the system is exactly taken into account by (2.4), *including its modulation by the physical observables*. Substituting (2.4) into (2.2) yields the rate equation for the physical observables:

$$\dot{\mathbf{X}}(t) = (\mathbf{A} - \mathbf{BC}^{-1}\mathbf{E})\mathbf{X}(t) + \mathbf{f}(t) - \int_0^t d\tau \mathbf{K}(t-\tau)\dot{\mathbf{X}}(\tau), \tag{2.5}$$

where

$$\mathbf{f}(t) \equiv \mathbf{B}e^{\mathbf{C}t}(\mathbf{Y}_0 + \mathbf{C}^{-1}\mathbf{EX}_0), \tag{2.6}$$

$$\mathbf{K}(t-\tau) \equiv -\mathbf{B}e^{\mathbf{C}(t-\tau)}\mathbf{C}^{-1}\mathbf{E}. \tag{2.7}$$

Thus, the only vestiges of the dependence of the physical observables on the unresolved degrees of freedom in (2.5) are through the initial conditions in $\mathbf{f}(t)$ and the coupling matrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{E}$.

Because the small-scale variations are not resolved in the above description of the motion, there is a fundamental uncertainty in the specification of the initial state $\mathbf{Y}_0$. As mentioned above, for an apparently unique specification of the observed initial state $\mathbf{X}_0$ of the system there is an ensemble of initial conditions for the unobserved degrees of freedom. Therefore, we specify $\mathbf{Y}_0$ by an ensemble distribution function and interpret $\mathbf{f}(t)$ as a fluctuating driving force in (2.5). Consider an ensemble of initial states in which the physical observables $\mathbf{X}_0$ are held fixed and the initial $\mathbf{Y}$'s are drawn from the distribution (Zwanzig, 1973; Haken, 1975; Lindenberg and Seshadri, 1981):

$$P(\mathbb{Y}_0|\mathbb{X}_0) = N \exp\Big[ -\frac{1}{2} (\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0)^+ \mathbb{Q} (\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0) \Big], \quad (2.8)$$

where $N$ is the normalization constant and where $A^+$ is the hermitian conjugate of $A$. Using (2.8) we obtain for the average initial state

$$\langle \mathbb{Y}_0 \rangle = -\mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0, \quad (2.9)$$

and for the second moments

$$\langle (\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0)(\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0)^+ \rangle = \mathbb{Q}^{-1}, \quad (2.10)$$

so that the distribution of initial states is a zero-centered Gaussian in the shifted set of variables $\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0$ with covariance matrix $\mathbb{Q}^{-1}$. We note that this covariance matrix is in general Hermitian. Because of the linear form of (2.6), the fluctuations $\mathbb{f}(t)$ are then also Gaussian.

Consider the correlation of the fluctuations at time $t$ [cf. (2.6)] with those at time $t'$:

$$\langle \mathbb{f}(t)\mathbb{f}^+(t') \rangle = \mathbb{B}e^{\mathbb{C}t} \times \langle (\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0)(\mathbb{Y}_0 + \mathbb{C}^{-1}\mathbb{E}\mathbb{X}_0)^+ \rangle e^{\mathbb{C}^+t'}\mathbb{B}^+. \quad (2.11)$$

Using the second moment results (2.10), this becomes

$$\langle \mathbb{f}(t)\mathbb{f}^+(t') \rangle = \mathbb{B}e^{\mathbb{C}t}\mathbb{Q}^{-1}e^{\mathbb{C}^+t'}\mathbb{B}^+. \quad (2.12)$$

If the matrix $\mathbb{C}$ is antihermitian then the hermiticity of $\mathbb{Q}$ leads to the relation

$$\mathbb{Q}^{-1}\mathbb{C}^+ = -\mathbb{C}\mathbb{Q}^{-1}. \quad (2.13)$$

This allows us to write (2.12) as

$$\langle \mathbb{f}(t)\mathbb{f}^+(t') \rangle = -\mathbb{B}e^{\mathbb{C}(t-t')}\mathbb{Q}^{-1}\mathbb{B}^+, \quad (2.14)$$

and the fluctuations are therefore stationary. Furthermore, by comparing (2.14) and (2.7), if

$$\mathbb{C}^{-1}\mathbb{E} = \beta\mathbb{Q}^{-1}\mathbb{B}^+, \quad (2.15)$$

then the system obeys the fluctuation–dissipation relation (Callen and Welton, 1951; Onsager and Machlup, 1953; Reichl, 1980) such that

$$\langle \mathbb{f}(t)\mathbb{f}^+(t') \rangle = \frac{1}{\beta}\mathbb{K}(t-t'), \quad (2.16)$$

where $1/\beta$ is the "temperature," i.e., a relative measure of the excitation of the unresolved degrees of freedom. In this case, (2.5) has the form of a set of generalized Langevin equations (Zwanzig, 1973; Haken, 1975; Lindenberg and Seshadri, 1981; West, 1982), i.e., the elements of $\mathbb{K}$ are positive definite quantities and therefore the last term is a dissipative current. The fluctuation–dissipation relation then ensures the balance between the influx due to fluctuations $\mathbb{f}(t)$ and the efflux due to the dissipation. In general, (2.13) may not be satisfied, and, as a result, the system (2.5) has a richer variety of dynamic responses.

We have thus succeeded in projecting a set of deterministic equations, (2.1), onto a reduced subspace in which the equations are stochastic (2.5). The stochastic equations contain three distinct manifestations of the coupling to the eliminated degrees of freedom. First, they contain the additive Gaussian fluctuations (usually added onto deterministic equations in an *ad hoc* fashion, see, e.g., Thompson, 1972). In phenomenological models it is usually assumed that these fluctuations are delta correlated. From (2.14) we can establish the conditions under which this is a valid assumption, as done by Ford *et al.* (1965) in a different context. The second consequence of the eliminated variables is the memory kernel $\mathbb{K}(t-\tau)$ in (2.5). This term is the average interaction between the physical observables and the unobservables. To relate this memory kernel to the usual dissipative term in phenomenological models, we integrate the last term in (2.5) by parts to obtain

$$\dot{\mathbb{X}}(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{f}(t) + \int_0^t d\tau \dot{\mathbb{K}}(t-\tau)\mathbb{X}(\tau). \quad (2.17)$$

It is then assumed (implicitly or explicitly) that $\dot{\mathbb{K}}(t-\tau) = -\mathbb{A}\delta(t-\tau)$ where $\mathbb{A}$ is the dissipative matrix (see, e.g., Haken, 1978). We note that this association is inconsistent with the Rayleigh dissipation function, which associates the dissipation with a quadratic term in the *velocity* in the Lagrangian derivation of the equations of motion (Lindenberg and West, 1984). The correct memoryless limit, i.e., Markovian limit, of (2.5) is the choice $\mathbb{K}(t-\tau) = -\mathbb{A}\delta(t-\tau)$. The third consequence of the eliminated variables is the modification of the interaction coefficients of the physical observables among themselves, i.e., $\mathbb{A} \to (\mathbb{A} - \mathbb{B}\mathbb{C}^{-1}\mathbb{E})$, an effect which is ostensibly absent in (2.17) due to the integration by parts.

Finally we note that if (2.1) constitutes a thermodynamically isolated physical system, then the reduced system (2.5) must achieve a steady-state distribution characterized by the same temperature as that of its surroundings as $t \to \infty$. This physical requirement imposes the constraint (2.15) on the interaction and covariance matrices.

## 3. Stochastic barotropic motion

### a. The deterministic system

In the preceding section we investigated a linear system and showed that a projection onto a restricted subspace of physical observables leads to a stochastic differential equation. We now follow that same elimination procedure to study barotropic motion on a $\beta$-plane satisfying doubly periodic boundary conditions. The equation of motion is

$$\frac{\partial}{\partial t}(\nabla^2 + \alpha^2)\psi + \beta\frac{\partial}{\partial x}\psi + J[\psi, \nabla^2\psi] = 0, \quad (3.1)$$

(see, e.g., Pedlosky, 1979) where $\psi$ is the velocity streamfunction, $\nabla^2$ the horizontal Laplacian, $\alpha^{-1} = \sqrt{gH}/f$ the effective radius of deformation, $H$ the effective depth of the atmosphere, $\beta$ the meridional derivative of the Coriolis parameter $f(\equiv 2\Omega \sin\phi)$, $x$ east, $y$ north and $J(\cdot)$ the Jacobian determinant. Note that (3.1) is free from any *ad hoc* forcing or dissipative terms.

Imposing the doubly periodic boundary conditions we decompose $\psi(\mathbf{x}, t)$ into the Fourier components

$$\psi(\mathbf{x}, t) = \sum_{\mathbf{k}} \psi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.2}$$

so that (3.1) can be written

$$\dot{\psi}_{\mathbf{k}}(t) + i\omega_{\mathbf{k}}\psi_{\mathbf{k}}(t) = \sum_{\mathbf{l}+\mathbf{m}=\mathbf{k}} \Gamma^{\mathbf{k}}_{\mathbf{lm}}\psi_{\mathbf{l}}(t)\psi_{\mathbf{m}}(t), \tag{3.3}$$

where the frequency is

$$\omega_{\mathbf{k}} \equiv -\beta \frac{k_x}{k^2 + \alpha^2}, \tag{3.4a}$$

and the coupling coefficients are

$$\Gamma^{\mathbf{k}}_{\mathbf{lm}} \equiv \frac{1}{2} \frac{(l_x m_y - l_y m_x)(m^2 - l^2)}{(k^2 + \alpha^2)}. \tag{3.4b}$$

A more symmetric set of equations can be obtained in terms of the mode amplitudes $A_{\mathbf{k}} \equiv (k^2 + \alpha^2)^{1/2}\psi_{\mathbf{k}}$ yielding from (3.3)

$$\dot{A}_{\mathbf{k}}(t) + i\omega_{\mathbf{k}} A_{\mathbf{k}}(t) = \sum_{\mathbf{l}+\mathbf{m}=\mathbf{k}} V_{\mathbf{klm}} A_{\mathbf{l}}(t) A_{\mathbf{m}}(t), \tag{3.5}$$

where the coupling coefficients here have the more symmetric form

$$V_{\mathbf{klm}} = \frac{1}{2} \frac{(l_x m_y - l_y m_x)(m^2 - l^2)}{[(k^2 + \alpha^2)(l^2 + \alpha^2)(m^2 + \alpha^2)]^{1/2}}. \tag{3.6}$$

The symmetry properties of these coefficients are by construction (with $\mathbf{k} = \mathbf{l} + \mathbf{m}$),

$$V_{\mathbf{klm}} + V_{\mathbf{lmk}} + V_{\mathbf{mkl}} = 0, \tag{3.7a}$$

$$k^2 V_{\mathbf{klm}} + l^2 V_{\mathbf{lmk}} + m^2 V_{\mathbf{mkl}} = 0, \tag{3.7b}$$

$$V_{\mathbf{klm}} = V_{\mathbf{kml}}, \tag{3.7c}$$

$$V_{\mathbf{kll}} = V_{\mathbf{kkm}} = V_{\mathbf{klk}} = 0. \tag{3.7d}$$

The symmetry properties (3.7) indicate that the nonlinear interactions in (3.5) conserve both the wave energy $|A_{\mathbf{k}}|^2 = (k^2 + \alpha^2)|\psi_{\mathbf{k}}|^2$ and the potential enstrophy $k^2|A_{\mathbf{k}}|^2 = k^2(k^2 + \alpha^2)|\psi_{\mathbf{k}}|^2$ (see, e.g., Kraichnan, 1975).

Note that (3.5) defines a system of deterministic coupled mode rate equations. From the point of view of statistical mechanics, all the small-scale local variations in mass, energy and momentum density have been averaged in writing the original equations (3.1) so that no small-scale fluctuations remain.

Traditionally, truly microscopic degrees of freedom are manifest through the molecular dissipation term in the Navier-Stokes equations. It is always assumed that the fluctuations induced by these molecular motions are totally negligible in geophysical flow fields. On the barotropic scale of motion, molecular dissipation is also considered negligible. Other sources of short-scale variations in the dynamics of the flow field have been introduced via *ad hoc* forcing functions. Such a function has been proposed by Landau and Lifshitz (1959) to model stress fluctuations in the hydrodynamic equations. Another has been used by Thompson (1972), among others, for modeling two-dimensional flow fields in the atmosphere. We are not concerned here with processes that would modify (3.1). Rather, we are concerned with effects implicit in (3.1) and that arise from the notion that coarse graining the description of the vorticity field results in equations that are stochastic and dissipative. The procedure followed in the preceding section reveals that the dissipation, as well as the fluctuations, are manifestations of the degrees of freedom eliminated from the description in the subspace of the physical observables. Thus, if the dynamics of the vorticity field are fully characterized by $2N$ modes $\{A_{\mathbf{k}}(t)\}$ in (3.5), then a possible projection is a reduction to a set of $2M$ modes $(M \ll N)$ to describe the observable atmospheric effects. These latter degrees of freedom are the lowest order modes of the atmosphere. It should be noted that the range of applicability of the primitive equation (3.1) restricts the spectral range of the modal decomposition.

### b. *The stochastic subsystem*

As in the linear case we again separate the dynamic variables into two groups. The first group, denoted by $[B_{\mathbf{k}}(t)]$, is that of the physical observables and the second group, denoted by $\{C_{\mathbf{k}}(t)\}$, contains the unresolved modes. To simplify the notation we assume that the region of the flow field over which we have assumed periodic boundary conditions is of length L and width W such that $(\mathrm{L/W})^2$ is irrational. This implies that the wave vector $\mathbf{k} = (2\pi n_x/L, 2\pi n_y/W)$ is nondegenerate, i.e., there is no value of the integers $n_x$ and $n_y$ for which $n_x/n_y = \mathrm{L/W}$. This fact allows us to index each mode by a single quantity. We index the observable modes by a Greek letter that takes on the values $\gamma = -M, -M+1, \ldots, M-1, M$ with the value $\gamma = 0$ excluded. The unobservable modes are indexed with a Roman letter $l$ such that $|l|$ takes on all integer values in the interval $(N - M + 1, N)$. Thus we replace $\{B_{\mathbf{k}}(t), C_{\mathbf{k}}(t)\}$ by $\{B_\gamma(t), C_l(t)\}$ and rewrite (3.5) as the coupled set of equations

$$\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t) = \epsilon \sum_{\mu,l} V_{\gamma\mu l} C_l(t) B_\mu(t) + \epsilon G_\gamma(t) + \epsilon F_\gamma(t), \tag{3.8}$$

$$\dot{C}_l(t) + i\omega_l C_l(t) = \epsilon \sum_{\mu,m} V_{lm\mu} C_m(t) B_\mu(t) + \epsilon G_l(t) + \epsilon F_l(t), \quad (3.9)$$

where

$$G_{\cdot}(t) = \sum_{\mu,\nu} V_{\cdot\mu\nu} B_\mu(t) B_\nu(t), \quad (3.10a)$$

$$F_{\cdot}(t) = \sum_{l,m} V_{\cdot lm} C_l(t) C_m(t). \quad (3.10b)$$

We have introduced a parameter $\epsilon$ that will enable us to keep track of how many interaction potentials occur explicitly in a given term. At the end of the calculation this parameter will be set to unity. In each of the sums in (3.8)–(3.10), as well as in subsequent sums, the wave–vector conservation conditions [cf. (3.7)] and the resultant restriction on the indices of the summands are to be understood even when not indicated explicitly.

In each set of equations, (3.8) and (3.9), there are three types of nonlinear interactions. The first is the interaction between the resolved and unresolved flow fields having the product form $C_l(t)B_\mu(t)$. The coupling coefficients $V$ vanish when two indices are equal so that this term does not introduce $B_\gamma$ in the equation for $\dot{B}_\gamma$. Rather, the unresolved flow field $C_l$ couples the other modes in the subspace to $B_\gamma$. The second type of interaction is embodied in $G(t)$ which contains the interactions among the physical observables only. The final type of interaction is contained in $F(t)$ and involves the unresolved modes only.

To formally integrate (3.9) we define the $2(N - M)$ component vectors $\mathbb{C}(t)$, $\mathbb{G}(t)$ and $\mathbb{F}(t)$ in the following general way:

$$\mathrm{g}^+(t) \equiv [g^*_{-N}(t), \ldots, g^*_{-M-1}(t), g^*_{M+1}(t), \ldots, g^*_N(t)], \quad (3.11)$$

where + denotes the Hermitian conjugate of the column vector $\mathrm{g}(t)$, the elements of the $2(N - M) \times 2(N - M)$ matrix $\mathbb{D}$ by

$$\epsilon D_{lm}(t) \equiv \epsilon \sum_{\mu} V_{lm\mu} B_\mu(t), \quad (3.12)$$

and the diagonal matrix $\Omega$ by

$$\Omega = \begin{bmatrix} -\omega_N & & & & & 0 \\ & \ddots & & & & \\ & & -\omega_{M-1} & & & \\ & & & \omega_{M+1} & & \\ & & & & \ddots & \\ 0 & & & & & \omega_N \end{bmatrix}, \quad (3.13)$$

where $\omega_{-j} = -\omega_j$ [cf. (3.4a)] so that (3.9) can be written

$$\dot{\mathbb{C}}(t) + i\Omega\mathbb{C}(t) = \epsilon\mathbb{D}(t)\mathbb{C}(t) + \epsilon\mathbb{G}(t) + \epsilon\mathbb{F}(t). \quad (3.14)$$

By introducing the transformed quantity

$$\hat{\mathbb{D}}(t) = e^{i\Omega t}\mathbb{D}(t)e^{-i\Omega t}, \quad (3.15)$$

the formal solution to (3.14) can be written

$$\mathbb{C}(t) = e^{-i\Omega t}\left[ e_T^{\int_0^t \hat{\mathbb{D}}(\tau)d\tau}\mathbb{C}(0) + \epsilon \int_0^t e_T^{\epsilon \int_\tau^t \hat{\mathbb{D}}(\tau')d\tau'} e^{i\Omega\tau}\{\mathbb{G}(\tau) + \mathbb{F}(\tau)\}d\tau \right], \quad (3.16)$$

where the subscript $T$ denotes time ordering (van Kampen, 1981). The solution (3.16) is a function of the initial values of the microscopic degrees of freedom $\mathbb{C}(0)$, a function of the physical observables $\{B_\gamma(\tau)\}$ through $\hat{\mathbb{D}}$ and $\mathbb{G}$ and a nonlinear function of the set $\{C_l(\tau)\}$ itself through $\mathbb{F}$.

An explicit perturbative solution of (3.9) can be obtained from the integral equation (3.16) by expanding the right side in powers of $\epsilon$. To first order in $\epsilon$ we obtain

$$C_l(t) = C_l^{(0)}(t) + \epsilon \int_0^t d\tau e^{-i\omega_l(t-\tau)}\Big[\sum_{\mu,m} V_{l\mu m} B_\mu(\tau) C_m^{(0)}(\tau) + \sum_{m,n} V_{lmn} C_m^{(0)}(\tau) C_n^{(0)}(\tau) + G_l(\tau)\Big], \quad (3.17)$$

where $\mathbb{C}^{(0)}(t)$ is the zeroth order solution

$$\mathbb{C}^{(0)}(t) = e^{-i\Omega t}\mathbb{C}(0). \quad (3.17a)$$

Using the results of the linear analysis as a guide, we observe that (3.17) has a structure corresponding to the solution of (2.17). This association, as pointed out earlier, leads to an inconsistent interpretation of fluctuations and dissipation. In the spirit of (2.5), which provides a consistent view, we reexpress all terms in (3.17) containing $B(\tau)$ in terms of $\dot{B}(\tau)$ via integration by parts to obtain

$$\begin{aligned} C_l(t) + \frac{i\epsilon}{\omega_l} G_l(t) + \frac{i\epsilon}{\omega_l} \sum_{\mu,m} V_{l\mu m} B_\mu(t) C_m^{(0)}(t) &= C_l^{(0)}(t) + \frac{i\epsilon}{\omega_l} e^{-i\omega_l t} G_l(0) + \frac{i\epsilon}{\omega_l} e^{-i\omega_l t} \sum_{\mu,m} V_{l\mu m} B_\mu(0) C_m(0) \\ &+ \epsilon \int_0^t d\tau e^{-i\omega_l(t-\tau)}\Big[ i \sum_{\mu,m} \frac{V_{l\mu m}}{\omega_l} \dot{B}_\mu(\tau) C_m^{(0)}(\tau) + \sum_{m,n} V_{lmn} C_m^{(0)}(\tau) C_n^{(0)}(\tau) + \frac{i}{\omega_l}\dot{G}_l(\tau)\Big] \\ &+ \epsilon \int_0^t d\tau e^{-i\omega_l(t-\tau)} \sum_{\mu,m} \frac{\omega_m V_{l\mu m}}{\omega_l} B_\mu(\tau) C_m^{(0)}(\tau). \end{aligned} \quad (3.18)$$

Substitution of (3.18) into (3.8) yields the following equation of motion for the observables, containing all terms up to and including $O(\epsilon^2)$:

$$\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t) = \epsilon G_\gamma(t) + \epsilon \sum_{\mu,l} V_{\gamma\mu l} B_\mu(t) C_l^{(0)}(t)$$
$$+ \epsilon \sum_{l,m} V_{\gamma lm} C_l^{(0)}(t) C_m^{(0)}(t)$$
$$+ i\epsilon^2 \sum_{\mu,l} \frac{V_{\gamma\mu l}}{\omega_l} B_\mu(t) \int_0^t d\tau e^{-i\omega_l(t-\tau)} \dot{G}_l(\tau)$$
$$+ 2i\epsilon^2 \sum_{l,m,n,\mu} \frac{V_{\gamma lm} V_{m\mu n}}{\omega_m} C_l^{(0)}(t) \int_0^t d\tau e^{-i\omega_m(t-\tau)}$$
$$\times \dot{B}_\mu(\tau) C_n^{(0)}(\tau) + \epsilon^2 R_\gamma(t). \quad (3.19)$$

The "remainder" $R_\gamma(t)$ is given by the collection of terms

$$R_\gamma(t) = i \sum_l [e^{-i\omega_l t} \bar{G}_l(0) - \bar{G}_l(t)]$$
$$\times \left[ \sum_\mu \frac{V_{\gamma\mu l}}{\omega_l} B_\mu(t) + 2 \sum_m \frac{V_{\gamma ml}}{\omega_l} C_m^{(0)}(t) \right]$$
$$+ \sum_{\mu,l,m,\nu} V_{\gamma\mu l} V_{l\nu m} B_\mu(t) \int_0^t d\tau e^{-i\omega_l(t-\tau)} B_\nu(\tau) C_m^{(0)}(\tau)$$
$$+ \sum_{\mu,l,m,n} V_{\gamma\mu l} V_{lmn} B_\mu(t) \int_0^t d\tau e^{-i\omega_l(t-\tau)} C_m^{(0)}(\tau) \dot{C}_n^{(0)}(\tau)$$
$$+ 2 \sum_{j,l,m,n} V_{\gamma jl} V_{lmn} C_j^{(0)}(t) \int_0^t d\tau e^{-i\omega_l(t-\tau)} C_m^{(0)}(\tau) C_n^{(0)}(\tau)$$
$$+ 2 \sum_{l,m} V_{\gamma lm} C_l^{(0)}(t) \int_0^t d\tau e^{-i\omega_m(t-\tau)} \dot{G}_m(\tau)$$
$$+ \sum_{\nu,l,\mu,m} \frac{\omega_m V_{\gamma\nu l} V_{l\mu m}}{\omega_l} B_\nu(t) \int_0^t d\tau e^{-i\omega_l(t-\tau)} B_\mu(\tau) C_m^{(0)}(\tau), \quad (3.20)$$

where

$$\bar{G}_l(t) \equiv G_l(t) + \sum_{\nu,m} V_{l\nu m} B_\nu(t) C_m^{(0)}(t). \quad (3.20a)$$

We close this subsection with a number of observations. We have replaced the set of deterministic equations (3.5) with a reduced set of equations (3.19) for the macroscopic observables $\{B_\gamma(t)\}$ in which the unobservable degrees of freedom appear only through their initial values in $\{C_l^{(0)}(t)\}$. Since we intend to characterize these initial values of $C$ statistically, all terms in which they appear are stochastic, which, of course, implies that (3.19) is a stochastic differential equation. We note that similar equations have been *postulated* in flow fields in the past, see e.g., Leith (1971), Hasselmann (1976), Egger (1982), as well as Landau and Lifschitz (1959), but the above argument is to our knowledge the first *derivation* of such equations in which fluctuations arise from the uncertainty in the initial conditions of the eliminated modes. A crucial difference between the present dynamic equations and those usually postulated lies in the appearance of $\dot{B}(\tau)$ rather than $B(\tau)$ in the generalized Langevin equation (3.19). We expect the distinction between these two types of contributions to be no less important in this nonlinear system than it is in the linear system discussed previously. Some consequences of this distinction will be considered in Section 4. We show in the next section that as a consequence of the statistical hypothesis the terms $\epsilon^2 R_\gamma(t)$ must be neglected in order to obtain a self-consistent theory to $O(\epsilon^2)$. The reason for this will be given in due course.

### c. *Fluctuation–dissipation relations*

The physical interpretation of the stochastic equation (3.19) cannot be completed until we specify the statistical properties of the initial values $\{C_l(0)\}$. These properties are determined by the choice of distribution of the initial state for the unresolved flow field. We select the distribution suggested by the fluid equilibrium studies of Onsager (1949) and Kraichnan (1975) for interacting point vortices and by the simple quasi-geostrophic models of Salmon *et al.* (1976):

$$P_{eq}(\mathbf{C}) = Z^{-1} \exp(-\sum_l \Lambda_l c_l c_{-l}). \quad (3.21)$$

Here $\Lambda_l$ is a state-dependent "thermodynamic potential" first introduced in this context by Kraichnan (1967), $Z$ is the partition function and $c_l$ is the phase space variable corresponding to the initial value $C_l(0)$. We emphasize that the distribution (3.21) describes only the *initial* state of the unresolved degrees of freedom. The dynamic equations determine the subsequent evolution of these degrees of freedom away from the initial state. This evolution might be characterized by an enstrophy cascade or by any other motion consistent with the dynamic equations.

To aid us further in the interpretation of (3.19) we find it convenient to rewrite it as

$$\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t)$$
$$= \epsilon G_\gamma(t) + \epsilon f_\gamma(t) + \epsilon^2 \sum_\mu \int_0^t d\tau \mathcal{K}_{\gamma\mu}(t,\tau) B_\mu(\tau)$$
$$+ \epsilon \sum_\mu g_{\gamma\mu}(t) B_\mu(t) - \epsilon^2 \sum_{\mu,\eta,\nu} \int_0^t d\tau M_{\gamma\mu\eta\nu}(t-\tau) B_\mu(t)$$
$$\times \frac{d}{d\tau} [B_\eta(\tau) B_\nu(\tau)] + \epsilon^2 R_\gamma(\tau), \quad (3.22)$$

where we have introduced the following functions:

$$f_\gamma(t) = \sum_{l,m} V_{\gamma lm} C_l^{(0)}(t) C_m^{(0)}(t), \quad (3.23a)$$

$$\mathcal{K}_{\gamma\mu}(t,\tau) = 2i \sum_{l,m,n} \frac{V_{\gamma lm}V_{m\mu n}}{\omega_m} C_l^{(0)}(t) e^{-i\omega_m(t-\tau)} C_n^{(0)}(\tau), \tag{3.23b}$$

$$g_{\gamma\mu}(t) = \sum_l V_{\gamma\mu l} C_l^{(0)}(t), \tag{3.24a}$$

$$M_{\gamma\mu\eta\nu}(t-\tau) = -i \sum_l \frac{V_{\gamma\mu l}V_{l\eta\nu}}{\omega_l} e^{-i\omega_l(t-\tau)}. \tag{3.24b}$$

The first three of these, i.e., $f$, $\mathcal{K}$, and $g$, are stochastic functions. One must distinguish between the meaning of "stochastic" as applied to the functions $f$ and $\mathcal{K}$ and as applied to $g$. In the latter case the wave vector restriction on the interaction coefficients reduces the sum to a single term ($\gamma = \mu + l$). Thus, the statistical properties of $g_{\gamma\mu}$ are completely determined by the distribution (3.21). In any given realization selected from this ensemble, $g_{\gamma\mu}(t)$ is *periodic* in time, not *fluctuating* in time. Each member has the same frequency $\omega_{\gamma-\mu}$ but has a different amplitude. This behavior is quite different from that of either $f$ or $\mathcal{K}$. Consider the form of $f$: taking into account the wave vector restriction still leaves a free sum over the entire unobservable subspace. Hence, regardless of the initial values $\mathbb{C}(0)$, i.e., for any given realization of $C$, $f_\gamma(t)$ varies erratically in time due to the dephasing of the individual contributions to the sum. We anticipate that the statistics of $\mathbf{f}$ are insensitive to the particular choice of distribution of initial conditions. Consequently, any particular realization of $\mathbb{f}(t)$ is expected to be representative of the ensemble of all realizations, i.e., $\mathbb{f}(t)$ is essentially ergodic. A similar argument applies to $\mathcal{K}$ but not to $g$.

We begin with the determination of the statistical properties of the functions $\{f_\gamma(t)\}$. Denoting averages over (3.21) by an angle bracket we immediately see that

$$\begin{aligned}\langle f_\gamma(t)\rangle &= \sum_{l,m} V_{\gamma lm}\langle C_l^{(0)}(t) C_m^{(0)}(t)\rangle \\ &= \sum_{l,m} V_{\gamma lm} e^{-i(\omega_l+\omega_m)t}\langle C_l(0)C_m(0)\rangle.\end{aligned} \tag{3.25}$$

Since $\langle C_l(0)C_m(0)\rangle = \langle C_l(0)C_{-l}(0)\rangle\delta_{l,-m}$ and since $V_{\gamma,l,\pm l} = 0$ it follows that

$$\langle f_\gamma(t)\rangle = 0. \tag{3.26}$$

The two-time correlation function is

$$\begin{aligned}\langle f_\gamma(t) f_\mu(t')\rangle = \sum_{l,m,j,n} & V_{\gamma lm}V_{\mu jn} e^{-i(\omega_l+\omega_m)t} \\ & \times e^{-i(\omega_j+\omega_n)t'}\langle C_l(0)C_m(0)C_j(0)C_n(0)\rangle.\end{aligned} \tag{3.27}$$

The fourth-order moment in (3.27) partitions into products of second-order moments because the distribution (3.21) is a multivariate Gaussian for a set of uncorrelated modes. The partitioning that associates $l$ with $m$ and $j$ with $n$ has vanishing interaction coefficients and therefore does not contribute to (3.27). The surviving terms require $l = -j$ and $m = -n$, or $l = -n$ and $m = -j$. They both contribute equally, due to the symmetry of the coupling, and lead to

$$\langle f_\gamma(t) f_\mu(t')\rangle = \delta_{\mu\gamma} \sum_l \phi_\gamma^l(t-t') \tag{3.28}$$

with

$$\phi_\gamma^l(t-t') = \frac{2V^2_{\gamma,l,\gamma-l}}{\Lambda_l\Lambda_{\gamma-l}} e^{-i(\omega_l+\omega_{\gamma-l})(t-t')}. \tag{3.29}$$

Furthermore, since the number of unresolved degrees of freedom is assumed to be large, $f_\gamma(t)$ is a sum of a large number of identically distributed random variables with finite second moments. Then, the central limit theorem tell us that each element of $\mathbb{f}(t)$ is approximately a *Gaussian* random variable and, hence, is approximately specified by its first two moments. The dependence on higher moments is determined by the spectral width of the unresolved degrees of freedom. Furthermore, each element of $\mathbb{f}(t)$ is statistically independent of all the other elements. We have thus succeeded in *deriving* the statistics of the fluctuations $f_\gamma(t)$ in (3.22).

Next we investigate the statistical properties of the stochastic kernel $\mathcal{K}_{\gamma\mu}(t,\tau)$. Its average value is given by

$$\begin{aligned}-K_{\gamma\mu}(t-\tau) &\equiv \langle \mathcal{K}_{\gamma\mu}(t,\tau)\rangle \\ &= 2i \sum_{l,m,n} \frac{V_{\gamma lm}V_{m\mu n}}{\omega_m} e^{-i(\omega_l+\omega_m)t} e^{-i(\omega_n-\omega_m)\tau}\langle C_l(0)C_n(0)\rangle \\ &= 2i \sum_{l,m} \frac{V_{\gamma lm}V_{m\mu l}}{\omega_m\Lambda_l} e^{-i(\omega_l+\omega_m)(t-\tau)},\end{aligned} \tag{3.30}$$

i.e., using Eq. (3.29)

$$K_{\gamma\mu}(t-\tau) = i\delta_{\gamma\mu} \sum_l \frac{\Lambda_{\gamma-l}}{\omega_{\gamma-l}} \phi_\gamma^l(t-\tau). \tag{3.31}$$

We have thus related the average of the kernel $\mathcal{K}$ to the correlation function of the additive fluctuations $f_\gamma(t)$. Equation (3.31) constitutes *a generalized fluctuation–dissipation relation* (see, e.g., Reichl, 1980). One would anticipate the existence of such a relation in a thermodynamically closed system, e.g, the system of Section 2 subject to the constraint (2.15), or in a system that achieves local equilibrium (i.e., a steady state in which each spatial point can be characterized by a local thermodynamic potential).

The two fluctuating terms that have so far been discussed, $f$ and $\mathcal{K}$, both arise from the same source in (3.8), i.e., from the nonlinear coupling of the unresolved degrees of freedom in $\mathbb{F}(t)$. The above discussion indicates that we can write the stochastic kernel as its mean $-K_{\gamma\mu} \equiv -K_\gamma\delta_{\gamma\mu}$ plus a fluctuating portion:

$$\mathcal{K}_{\gamma\mu} = -K_\gamma \delta_{\gamma\mu} + \delta\mathcal{K}_{\gamma\mu}. \quad (3.32)$$

The mean kernel $K_\gamma$ appears multiplied by $\epsilon^2$ in (3.22) and is related to the correlation function of the O($\epsilon$) fluctuations via the relation (3.31). This mean kernel embodies the average interaction between the resolved modes and the eliminated degrees of freedom implicit in $\mathbf{F}(t)$ because of the back reaction of the unobservable modes to the observable ones. The remaining fluctuations caused by $\mathbf{F}(t)$ [aside from those represented by $f_\gamma(t)$] are of O($\epsilon^2$) and will be subsumed into $R_\gamma$:

$$R_\gamma(t) \rightarrow R_\gamma(t) + \sum_\mu \int_0^t d\tau \delta\mathcal{K}_{\gamma\mu}(t, \tau) B_\mu(\tau). \quad (3.33)$$

We thus rewrite (3.22) as

$$\begin{aligned} &\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t) \\ &= \epsilon G_\gamma(t) + \epsilon f_\gamma(t) - \epsilon^2 \int_0^t d\tau K_\gamma(t-\tau)\dot{B}_\gamma(\tau) \\ &\quad + \epsilon \sum_\mu g_{\gamma\mu}(t) B_\mu(t) - \epsilon^2 \sum_{\mu,\eta,\nu} \int_0^t d\tau M_{\gamma\mu\eta\nu}(t-\tau) \\ &\quad \times B_\mu(t) \frac{d}{d\tau}[B_\eta(\tau)B_\mu(\tau)] + \epsilon^2 R_\gamma(t). \quad (3.34) \end{aligned}$$

Now let us examine the remaining O($\epsilon$) stochastic term. First we note that $g_{\gamma\mu}(t)$ appears as a coefficient of the dependent variable $B_\mu(t)$ and, consequently, we shall refer to it as a *multiplicative stochastic parameter*. It should be emphasized that the variations appearing in this way can be amplified or suppressed depending on the state of the system. This behavior is quite different from that of additive fluctuations, which contribute the same level of excitation regardless of the state of the system. We note that parametric excitations of this kind have not heretofore been discussed in the atmospheric context even though they occur in the *same order* as do the state-independent fluctuations.

The statistics of the parametric elements of $\mathbf{g}(t)$ are completely determined by the first two moments. The statistics are Gaussian because (3.24a) is a linear form in the $C_l(0)$, which are themselves Gaussian. The average value of each element of $\mathbf{g}(t)$ vanishes because $\mathbf{C}(0)$ is zero-centered. The correlation functions between elements of $\mathbf{g}(t)$ are

$$\begin{aligned} \langle g_{\gamma\mu}(t) g_{\eta\nu}(t')\rangle &= \sum_{l,m} V_{\gamma\mu l} V_{\eta\nu m} e^{-i(\omega_l t + \omega_m t')} \langle C_l(0) C_m(0)\rangle \\ &= \Phi_{\gamma\mu\eta}(t-t')\delta_{\nu,\gamma+\eta-\mu}, \quad (3.35) \end{aligned}$$

where

$$\Phi_{\gamma\mu\eta}(t-t') = -\frac{V_{\gamma,\mu,\gamma-\mu} V_{\eta,\gamma+\eta-\mu,\gamma-\mu}}{\Lambda_{\gamma-\mu}} e^{-i\omega_{\gamma-\mu}(t-t')}. \quad (3.36)$$

This correlation function is related to the kernel $M$ in (3.34) by the relation

$$M_{\gamma\mu\eta\nu}(t-\tau) = i\delta_{\nu,\gamma+\eta-\mu} \frac{\Lambda_{\gamma-\mu}}{\omega_{\gamma-\mu}} \Phi_{\gamma\mu\eta}(t-\tau) \quad (3.37)$$

[cf. (3.24b)].

Equation (3.37) has the formal structure of a fluctuation–dissipation relation. However, this interpretation is inappropriate because the elements of $\mathbf{g}(t)$ are regular (periodic) functions of time, as are the elements of $\mathbf{M}$. The correlation (3.35) merely eliminates the dependence on the stochastic parameters but remains periodic in time because it contains only a single frequency. Thus, just as the elements of $\mathbf{g}$ are not fluctuations in time, those of $\mathbf{M}$ are not dissipative. Nevertheless, one can still interpret the term in (3.34), which is cubic in $B$, as an average interaction and the term containing $\mathbf{g}$ as the variations around this average interaction. Equation (3.37) then balances the average interaction against the correlation of these variations in the same formal way as fluctuations are balanced by dissipative interactions. It is the recognition of this balance between the two terms that motivates the retention of the cubic term.

We have thus identified each explicit term in (3.34) arising from the unresolved degrees of freedom as either a "fluctuation" or its associated "dissipation." Further, since $\langle \mathbf{f}(t)\mathbf{g}(\tau)\rangle = 0$, these sets of stochastic variations are mutually orthogonal. Note again that the fluctuations and dissipation are of course not of the same order in $\epsilon$ since the latter is related quadratically to the former. The first term in $\mathbf{R}(t)$ is a reversible shift in the potential due to the back reaction of the unresolved degrees of freedom. Since this term is of third order in the physical observables we do not include it in this analysis. The remaining terms in $\mathbf{R}(t)$ [cf. (3.20) and (3.33)] all represent zero-centered fluctuations of O($\epsilon^2$). The dissipative terms that balance these fluctuations must be at least of O($\epsilon^3$). If we omit these dissipative terms, then it would be inconsistent to retain $\mathbf{R}(t)$. We thus omit $\mathbf{R}(t)$ as well. These remarks pertain to a system in which we wish to insure the existence of a thermodynamic steady state.

Thus, in schematic form we have *derived* the set of nonlinear stochastic differential equations for the physical observables $\mathbf{B}(t)$:

$$\begin{aligned} \dot{\mathbf{B}}(t) + i\Omega\mathbf{B}(t) &= \mathbf{G}(t) + \mathbf{f}(t) - \int_0^t d\tau \mathbf{K}(t-\tau)\dot{\mathbf{B}}(\tau) \\ &\quad + \mathbf{g}(t)\mathbf{B}(t) - \int_0^t d\tau \mathbf{M}(t-\tau)\mathbf{B}(t)\frac{d}{d\tau}[\mathbf{B}(\tau)\mathbf{B}(\tau)], \quad (3.38) \end{aligned}$$

and we have now reset $\epsilon$ to unity. This equation together with the generalized fluctuation–dissipation relations (3.31) and (3.37) constitute the main results of this paper.

## 4. Discussion and conclusions

In this paper we have presented the first *systematic derivation* of stochastic mode rate equations for an atmospheric system. We have used concepts from nonequilibrium statistical mechanics and applied them to the vorticity equations for barotropic motion on a beta plane. The projection of the initial field equations onto a restricted subspace for the longest wavelength modes of the system has been shown to yield nonlinear stochastic mode rate equations with completely determined statistical properties.

The first point we wish to emphasize is that the projection of a deterministic conservative field onto a restricted subspace can yield a set of *dissipative* stochastic equations in this subspace. If the primitive equations describe a thermodynamically isolated field, then the restricted subspace is thermodynamically closed. The connection between the physical observables constituting the restricted subspace and the unresolved degrees of freedom is maintained through the fluctuation–dissipation relations that emerge naturally in the derivation. Thus, one is not free to specify the statistical properties of the stochastic elements in an *ad hoc* manner. Rather, these properties are connected to the dissipative properties of the system. To our knowledge this connection, manifest in the fluctuation–dissipation relations, has, with few exceptions, not been acknowledged in phenomenological models of atmospheric fluctuations (cf. Leith, 1971, 1975; Hasselmann, 1976; Holloway and Hendershott, 1977).

A procedure that has been used in the past (Hasselmann, 1976) is to simply replace $C_l(t)$ by its first-order solution $C_l^{(0)(t)}$ in (3.8). This technique has been used to motivate the term $\mathbb{f}(t)$ (i.e., the stochastic terms) but neglects the corresponding "dissipative" terms. Certain higher order terms in the direct interaction approximation have been discussed by Leith (1971). Terms of the form $\mathbb{g}(t)\mathbb{B}(t)$ have not previously been recognized even within these approximations. We have shown that even though such a simple replacement exhausts the $O(\epsilon)$ contributions, retention of appropriate $[O(\epsilon^2)]$ terms is necessary to achieve thermodynamic consistency. We stress that a simple perturbative approach would lead one to retain all terms of a given order in $\epsilon$. The above thermodynamic considerations reveal that all terms of a given order should not be treated equally. One must associate terms of different orders to ensure the proper balance of the fluxes through the system. An alternative interpretation of the higher order terms that must be retained in the expansion of $C_l(t)$ is that these terms determine the reaction of the unresolved degrees of freedom to the physical observables. This "back-reaction" describes the evolution of the unresolved degrees of freedom from their initial state. The fluctuations and dissipation in the physical observables have their corresponding effects on the unobservables. Although the unobservables can be eliminated, their evolution must be properly taken into account.

An assumption that is often made in the stochastic description of hydrodynamic systems is that of Markovicity, i.e., that $\dot{\mathbb{B}}(t)$ depends only on $\mathbb{B}(t)$ and *not* on the previous history of the flow (see, e.g., Landau and Lifshitz, 1959). Our derivation makes it possible to investigate the conditions under which this assumption is valid. From the discussion in Section 3, since $\mathbb{M}$ is periodic the Markov approximation can never be made if the cubic term in (3.38) is nonnegligible. There is no *a priori* justification for neglecting this term even though in specific instances it may be negligible. If for the sake of discussion we do neglect this term, then for consistency we must also neglect the $\mathbb{g}\mathbb{B}$ term.

The traditional equations of motion that occur in most phenomenological treatments are the Markov limit of the following generalized Langevin equation (cf. Leith, 1971 and Thompson, 1972):

$$\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t) = \sum_{\mu,\nu} V_{\gamma\mu\nu}B_\mu(t)B_\nu(t) + f_\gamma(t) - \int_0^t d\tau \bar{K}_\gamma(t-\tau)B_\gamma(\tau). \quad (4.1)$$

The Markov approximation in (4.1) consists of replacing the memory kernel $\bar{K}_\gamma$ by a delta function in time:

$$\bar{K}_\gamma(t-\tau) = 2\bar{\lambda}_\gamma\delta(t-\tau). \quad (4.2)$$

If (4.2) were valid then the evolution of the observables would be given by

$$\dot{B}_\gamma(t) + (\bar{\lambda}_\gamma + i\omega_\gamma)B_\gamma(t) = \sum_{\mu,\nu} V_{\gamma\mu\nu}B_\mu(t)B_\nu(t) + f_\gamma(t). \quad (4.3)$$

Equation (4.3) has a form that is often encountered as a phenomenological primitive equation in which $\bar{\lambda}_\gamma$ and $f_\gamma(t)$ are usually unrelated and are motivated by heuristic arguments (see, e.g., Thompson, 1972; Kraichnan, 1975). Our derivation provides an explicit relation between the memory kernel and the microscopic parameters $V$ and $\omega$:

$$\bar{K}_\gamma(t-\tau) = 2\sum_l \frac{V^2_{\gamma,l,\gamma-l}}{\Lambda_l} e^{-i(\omega_l+\omega_{\gamma-l})(t-\tau)}. \quad (4.4)$$

One can formulate smoothness conditions on the coefficients $V$ and $\Lambda$ and on the density of frequencies in the neighborhood of a given wave vector such that

(4.3) is satisfied (Ford *et al.*, 1965). Comparison of (4.4) and the fluctuation covariance function

$$\langle f_\gamma(t) f_\mu(\tau) \rangle = 2\delta_{\mu\gamma} \sum_l \frac{V^2_{\gamma,l,\gamma-l}}{\Lambda_l \Lambda_{\gamma-l}} e^{-i(\omega_l+\omega_{\gamma-l})(t-\tau)} \quad (4.5)$$

immediately shows that the convergence of (4.4) leads to the divergence of (4.5) if $\Lambda_{\gamma-l} \to 0$ for any $\gamma - l$. In general, there are no physically reasonable conditions on $\Lambda$ that would cause (4.5) to be sharply peaked if (4.4) is sharply peaked. In particular, proportionality of $\bar{K}$ and $\langle ff \rangle$ would require the thermodynamic potential $\Lambda_l$ to be independent of $l$, i.e., $\Lambda_l \equiv \Lambda$ for all $l$. This choice is inconsistent with the equilibrium spectrum found by Kraichnan (1967) for the energy and enstrophy of a stochastically driven vorticity field. We conclude again that (4.3) does not provide a faithful description of a dissipative Markovian system.

We stress that our point of view is that $\bar{K}$ in (4.1) is not a dissipative kernel. Rather, we replace (4.3) with the appropriate terms from (3.38) to obtain

$$(1 + \lambda_\gamma)\dot{B}_\gamma(t) + i\omega_\gamma B_\gamma(t) = \sum_{\mu,\nu} V_{\gamma\mu\nu} B_\mu(t) B_\nu(t) + f_\gamma(t), \quad (4.6)$$

where we have set

$$K_\gamma(t - \tau) = 2\lambda_\gamma \delta(t - \tau). \quad (4.7)$$

One can again specify the appropriate conditions on the parameters in the expression [cf. (3.31)]

$$K_\gamma(t - \tau) = 2i \sum_l \frac{\Lambda_{\gamma-l}}{\omega_{\gamma-l}} \left[ \frac{V^2_{\gamma,l,\gamma-l}}{\Lambda_l \Lambda_{\gamma-l}} e^{-i(\omega_l+\omega_{\gamma-l})(t-\tau)} \right] \quad (4.8)$$

to obtain (4.7). It is clear in the present formulation that if $\Lambda_{\gamma-l}$ is proportional to $\omega_{\gamma-l}$, i.e., if,

$$\Lambda_{\gamma-l} = \beta\omega_{\gamma-l}, \quad (4.9)$$

then comparing (4.8) and (4.5) we have

$$\langle f_\gamma(t) f_\gamma(\tau) \rangle = \beta^{-1} \mathcal{K}_\gamma(t - \tau). \quad (4.10)$$

This has the form of a classical fluctuation–dissipation relation for a thermodynamically closed system of temperature $T = (k_B\beta)^{-1}$, where $k_B$ is Boltzmann's constant. In this case a Markovian dissipation and delta-correlated fluctuations are simultaneously realizable.

If (4.9) is not satisfied, then of course the time scales of the covariance function and of the dissipative kernel can be quite different. In particular, a Markovian dissipation does not imply delta-correlated fluctuations, usual phenomenological assumptions notwithstanding (see also Lindenberg and West, 1984, where the same effect appears in a different context). Thus, the traditional measures of predictability, such as those developed in Lorenz (1969), Monin (1972) and more recently in Holloway and West (1983), must be reexamined to properly take into account the possibly correlated fluctuations driving the physical observables in (4.6).

Some final comments are in order regarding the multiplicative fluctuations and nonlinear dissipation terms omitted from the discussion immediately preceding. It is known that such contributions in other systems can dominate the behavior of the physical observables (Horsthemke and Lefever, 1984). Their effect in the present context has not as yet been determined, since their existence was heretofore unknown.

We end this discussion by pointing out some limitations of our presentation, limitations we hope can be corrected in the future. First, it has not been established whether the expansion in $\epsilon$ that we have introduced is convergent. Second, the effect of a finite correlation time in the additive fluctuations is not known for this system. Third, we do not know the effects of the multiplicative fluctuations and nonlinear dissipation. Testing of the second and third of these points may constitute an extensive procedure that requires preliminary analysis in somewhat simpler model systems. Finally, nothing is known about the higher order terms that have been omitted in this development. We view the present work as an introduction of an approach whose detailed applicability to specific atmospheric models and phenomena requires much further study.

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