Reflection of Hydrostatic Gravity Waves in a Stratified Shear Flow. Part I: Theory

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ABSTRACT

Continuous partial reflection of linear hydrostatic gravity waves that propagate through a stratified shear flow is examined. The complex reflection coefficient R satisfies a Riccati equation, which is a first-order nonlinear differential equation. It is shown that $|R| < 1$ since critical levels and overreflection are not considered. In this case the conservation of wave action flux may be expressed as a relationship between $|R|$ and $\mathscr{E}l^{-1}$, where $\mathscr{E}$ is the wave energy and l a characteristic inverse vertical length scale of the background state.

It is demonstrated that R for a layered model represents a limiting solution of the Riccati equation. A general solution is also derived, under the assumption that the characteristic scale l is directly proportional to the inverse scale height of the characteristic impedance associated with a stratified shear flow. It is shown that the vanishing of $|R|$ at a specific level is analogous to the vanishing of $|R|$ in a three layer model, when the characteristic impedances in the top and the bottom layers satisfy a matching condition. Finally, various properties of the reflection coefficient are displayed for a particular background state. The extension of the theory to encompass other types of wave motion is indicated.

1. Introduction

The reflection and transmission of waves propagating through a nonhomogeneous medium has attracted considerable attention since the pioneering work of Rayleigh (1880). The transmission of waves from tropospheric sources to upper atmospheric sinks of wave energy (e.g., Hines and Reddy, 1967), and the possible enhancement of surface winds by partially trapped mountain waves (e.g., Klemp and Lilly, 1975) represent two applications in the theory of wave reflection.

In these examples, as well as in many other applications, one of two approaches is usually taken. The multilayer approach is exemplified in the work of Hines and Reddy and of Vincent (1969). The atmosphere is divided into a finite number of layers and the atmospheric parameters, Brunt–Väisälä frequency and wind, are specified constants in each layer. The solutions may be expressed in terms of both a reflected and transmitted wave in each layer; the layer solutions are joined by the requirement that both the pressure and displacement produced by the wave be continuous across the interface between two adjoining layers. Solutions for the reflection and transmission coefficients are generally determined by numerical computations when the number of layers exceeds two or three.

The second approach is a variant on the multilayer method. In the simplest case, there is a continuous change in the refractive index in a layer bounded by layers in which the refractive indices are constant. In the lower layer, the solution may be expressed in terms of a transmitted and a reflected wave, while in the upper layer only a transmitted wave is present. A refractive index is chosen for the nonhomogeneous middle layer that permits analytic solutions of the wave equation to be determined. Matching of the solutions at each interface then leads to the determination of how much wave energy is transmitted through the middle layer. Rayleigh used this approach, and many examples that illustrate both acoustic and electromagnetic wave propagation may be found in the treatise by Brekhovskikh (1980).

While both of these techniques are useful, and will continue to be employed, some questions have been raised concerning the accuracy of the results in relation to 1) layer thickness and 2) the discontinuous change in atmospheric parameters from layer to layer. Vincent has illustrated some of these shortcomings in his evaluation of the multilayer approximation.

There is another technique available, to determine the reflective properties of the medium, that has apparently not been exploited in application to either atmospheric or oceanic wave propagation problems. This approach requires the solution of a nonlinear first order equation for the complex reflection coefficient [see (21)]. In fact, this equation for the reflection coefficient, may be derived from the recursive relations of the multilayer approach by allowing the layer depths to become infinitesimally small, e.g., Tolstoy (1955) and Hines and Reddy (1967). The question of whether it is preferable to use the multilayer approach or to solve two coupled first-order equations, to find the modulus and the phase of the reflection coefficient, will not be considered here. The most suitable approach would probably have to be determined on a case by case basis. However, there is a distinct advantage in dealing with the equation for the reflection coefficient. Some inherent properties of wave reflection in a medium in which both static stability and wind vary continuously may be uncovered by analysis of this equation. Moreover, wave solutions are not required, although such solutions can be obtained once the reflection coefficient has been determined.

The present study will consider the reflection of hydrostatic internal gravity waves that propagate vertically through regions of both varying static stability and wind. Although this model may be considered a prototype, applications to real flows are not unwarranted. A particular application to hydrostatic mountain waves will be presented in Part II of this study.

The effects of a nonhydrostatic acceleration, compressibility, planetary rotation, viscosity and the presence of a critical level may be taken into account in development of the theory, but these added features will not be considered here. As a consequence, the reflection coefficient is only a function of the Brunt–Väisälä frequency and the basic state wind distribution. The fundamental equation for the reflection coefficient, a Riccati equation, is derived in Section 2. Some general properties of the reflection coefficient are derived in Section 3, where a comparison is also made with the transmission properties of a two-layer model atmosphere. A general solution of the Riccati equation, that is valid for a particular class of basic states, is presented in Section 4, and a particular solution is examined in Section 5. Some remarks appear in Section 6.

2. Derivation of the Riccati equation for the reflection coefficient

a. *Reflection coefficient*

The model used for the description of hydrostatic internal gravity waves is assumed to be linear, two-dimensional and steady ($\partial/\partial t \equiv 0$). The Boussinesq approximation is employed, planetary rotation and dissipative effects are neglected, and critical levels where $\bar{u} = 0$ are not considered. The model equations are:

$$\bar{u}\partial u_*/\partial x + w_* d\bar{u}/dz = -\partial\pi_*/\partial x, \tag{1}$$

$$0 = -\partial\pi_*/\partial z + g\theta_*/\bar{\theta}_0, \tag{2}$$

$$\partial u_*/\partial x + \partial w_*/\partial z = 0, \tag{3}$$

$$\bar{u}\partial(g\theta_*/\bar{\theta}_0)/\partial x + N^2 w_* = 0, \tag{4}$$

where (u_*, w_*) are the perturbation velocity components along the horizontal and vertical axes (x, y), $\pi_* = p_*/\bar{\rho}_0$ is the perturbation pressure divided by a constant reference density $\bar{\rho}_0 = \bar{\rho}(0)$ and the perturbation potential temperature θ_* is divided by the constant reference value $\bar{\theta}_0 = \bar{\theta}(0)$. The acceleration of gravity per unit mass is denoted by g, and the Brunt–Väisälä frequency $N = (gd\ln\bar{\theta}/dz)^{1/2}$ and the basic flow $\bar{u}$ are both functions of height z. Either the basic state potential temperature $\bar{\theta}(z)$ or the potential density $\bar{\rho}(z)$ may be used to define N.

It is convenient to define a streamline displacement ζ_* from a level z by

$$w_*/\bar{u} = \partial\zeta_*/\partial x. \tag{5}$$

Then (1)–(5) may be represented in Fourier space by setting $\partial/\partial x = i\alpha$, where α denotes the x-wavenumber, and denoting the Fourier components by (u, w, π, θ). Elimination of both u and θ yields the equations

$$d(w/\bar{u})/dz = i\alpha\pi/\bar{u}^2, \tag{6a}$$

$$d\pi/dz = iN^2 w/\alpha\bar{u}. \tag{6b}$$

The first step is to define a reflection coefficient that is an appropriate definition when N and/or $\bar{u}$ vary continuously with z. The rationale to be adopted has been presented by Schelkunoff (1951) and by Tolstoy (1967). If a wave propagates in a medium in which N and $\bar{u}$ are constant, the streamline displacement may be expressed as

$$w/\bar{u} = A(e^{ilz} + Re^{-ilz})e^{i\alpha x}, \tag{7}$$

where $l = N/\bar{u}$, A is an arbitrary constant, and this solution represents the sum of an upward progressing wave and a reflected wave with the complex reflection coefficient denoted by R. The solution for π, obtained from (6a), is

$$\pi = (AS^2/\alpha)(e^{ilz} - Re^{-ilz})e^{i\alpha x}, \tag{8}$$

where

$$S^2 = N\bar{u}. \tag{9}$$

If the incident wave impinges on an inhomogeneous layer, where S^2 varies with z, the reflection coefficient at $z = 0$, say, will be given by

$$\alpha\frac{\pi}{w/\bar{u}} = S^2\left(\frac{1-R}{1+R}\right) \tag{10}$$

or

$$R = \frac{S^2 - \alpha\dfrac{\pi}{w/\bar{u}}}{S^2 + \alpha\dfrac{\pi}{w/\bar{u}}}. \tag{11}$$

The quantity $S^2 = N\bar{u}$ is the *characteristic impedance*, which is assumed to be positive and continuous throughout the depth of the atmosphere. The quantity $\alpha\pi/w\bar{u}^{-1} = -i\pi\zeta^{-1}$ is the *wave impedance*, which must be continuous at $z = 0$ in view of the fact that both the perturbation pressure and the streamline displacement must be continuous across the interface between the two layers. These definitions follow from Schelkunoff's development of the reflection of a plane wave incident on an arbitrary vertically stratified inhomogeneous layer. They are consistent with the concept used in acoustics, for example, that reflection occurs whenever there is an imperfect match between the impedances associated with the fluid medium and the incident plane wave. A complementary interpretation, that involves the flux of wave energy or wave action will be presented in Section 3.

As the wave penetrates into the inhomogeneous layer it is not, in general, possible to decompose the solution unambiguously into upward and downward propagating waves. A unique representation in terms of an incident and a reflected wave is only possible within a layer in which properties that characterize the background medium are either constant or slowly varying functions of position. For example, Schelkunoff has demonstrated that an expression of the form $A(z) \times \exp(i\Phi)$, where A is the wave amplitude and Φ is the phase, does not necessarily represent an upward propagating plane wave. While Blumen and Cox (1984) have shown that when $S(z)$ varies *slowly* within a layer, a unique representation in terms of both an incident and a reflected wave is possible. A similar representation applies when a WKBJ solution is employed, but in both cases wave reflection is negligible within the layer in which slow changes in the background medium occur. Yet it is still possible to define (11) as a reflection coefficient when the characteristic impedance S^2 is not restricted to slow variations because the wave impedance is also required to vary continuously with z throughout any inhomogeneous layer. In this regard the modulus and phase of the reflected wave energy may be represented by

$$R = |R|e^{i\theta}, \tag{12}$$

although a simple description of wave reflection in terms of incident and reflected waves is obscured by a series of such reflections taking place at the boundaries of each infinitesimal layer.

A first-order nonlinear differential equation for the reflection coefficient in a nonhomogeneous medium has been derived by Tolstoy (1955) and Hines and Reddy (1967), among others. Their approach is to find the recursive relation for R that is appropriate for a series of finite homogeneous layers. They then consider the limit as the layer thickness and the velocity and temperature differences between the layers become infinitesimal. Since the stated intention is to examine some properties of continuous reflection, a different approach will prove to be more illuminating for this purpose.

b. Riccati equation

Elimination of $w\bar{u}^{-1}$ between (6a) and (6b) yields

$$\frac{d^2\pi}{dz^2} - \frac{2}{N}\frac{dN}{dz}\frac{d\pi}{dz} + \frac{N^2}{\bar{u}^2}\pi = 0. \tag{13}$$

It is convenient to introduce the stretched variable (Blumen and Dietze, 1982)

$$Z = \int_0^z N(\xi)\bar{u}^{-1}(\xi)d\xi, \tag{14}$$

which transforms (13) into

$$\frac{d^2\pi}{dZ^2} - \frac{1}{S^2}\frac{dS^2}{dZ^2}\frac{d\pi}{dZ} + \pi = 0, \tag{15}$$

where the characteristic impedance S^2 is defined by (9). The general solution of (15), presented by Meyer, (1975) may be written as

$$\pi = S\{A(1-r)he^{iZ} + B(1-r^*)h^*e^{-iZ}\}, \tag{16}$$

where (A, B) are arbitrary constants, the asterisk denotes a complex conjugate,

$$h = \exp\left(-\frac{1}{2}\int_Z^\infty \frac{d\ln S^2}{dZ'}r(Z')dZ'\right), \tag{17}$$

and r satisfies

$$\frac{dr}{dZ} - \frac{1}{2}(1-r^2)\frac{d\ln S^2}{dZ} + 2ir = 0. \tag{18}$$

The quantity r needs to be determined. First, we consider

$$\frac{d\pi}{dz} = i\frac{N}{\bar{u}}\left(\frac{1+r}{1-r}\right)\pi, \tag{19}$$

which may be derived from (16), using (14), (17) and (18). Next (19) is introduced into (6b) to obtain

$$r = \frac{S^2 - \alpha\dfrac{\pi}{w/\bar{u}}}{S^2 + \alpha\dfrac{\pi}{w/\bar{u}}}. \tag{20}$$

Comparison with (11) shows that $r = R$ is the complex reflection coefficient, defined earlier. Consequently, the reflection coefficient satisfies (18), which may be rewritten as

$$\frac{dR}{dz} - \frac{1}{2}(1-R^2)\frac{d\ln S^2}{dz} + 2i\frac{N}{\bar{u}}R = 0 \tag{21}$$

with the aid of (14). This equation (21) for R is a Riccati equation, e.g., Davis, 1962. It was apparently first derived by Walker and Wax (1946), for the voltage reflection coefficient in a nonuniform transmission line. Tolstoy (1955; 1967, §3.2) also presents a derivation of the Riccati equation for a stratified fluid medium but does not include the wind shear, although he does consider the compressibility. There are no general techniques available for the solution of (21), although a solution may be obtained when the coefficients have a particular form (see Section 4).

Since π satisfies a second-order differential equation, it is necessary to solve (21) for R and then to perform the integration in (17) for the determination of h in order to represent the solution as in (16).

3. Some properties of wave reflection

The Riccati equation may be separated into a coupled system of first-order equations by introducing (12) into (21) to obtain

$$\frac{d|R|}{dz} - \frac{1}{2}(1 - |R|^2)\cos\theta \frac{d\ln S^2}{dz} = 0, \qquad (22)$$

$$|R|\frac{d\theta}{dz} + \frac{1}{2}(1 + |R|^2)\sin\theta \frac{d\ln S^2}{dz} + 2\frac{N}{\bar{u}}|R| = 0. \qquad (23)$$

The real and imaginary parts of R are related to $|R|$ and θ by

$$\cos\theta = R_r/|R|, \quad \sin\theta = R_i/|R|. \qquad (24)$$

It will be assumed that S^2 becomes constant at some high level, so that both $|R|$ and θ also become constant. In particular, the upper boundary conditions are

$$\left.\begin{array}{l} |R| \rightarrow 0 \\ \theta \rightarrow \text{constant} \end{array}\right\} \quad z \rightarrow \infty. \qquad (25)$$

a. A bound on the magnitude of $|R|$

Introduction of (24) into (22) yields

$$\frac{d}{dz}\ln(1 - |R|^2) + R_r\frac{d\ln S^2}{dz} = 0. \qquad (26)$$

Integration of (26) and use of (25) provides Meyer's result

$$1 - |R|^2 = \exp\left[\int_z^\infty (R_r d\ln S^2/dz')dz'\right] > 0, \qquad (27)$$

which is not unexpected since critical levels and possible overreflection ($|R| > 1$) are not taken into account.

b. A conservation principle

In order to proceed further, (6a) is used to eliminate π in the expression for the reflection coefficient. Then (11) becomes

$$R = \frac{wS^2 + i\bar{u}^2(\partial w/\partial z - wd\ln\bar{u}/dz)}{wS^2 - i\bar{u}^2(\partial w/\partial z - wd\ln\bar{u}/dz)}. \qquad (28)$$

Multiplication of (28) by R^* yields

$$|R|^2 = \frac{l^2|w|^2 + \left|\frac{\partial w}{\partial z} - w\frac{d\ln\bar{u}}{dz}\right|^2 + il\left(w^*\frac{\partial w}{\partial z} - w\frac{\partial w^*}{\partial z}\right)}{l^2|w|^2 + \left|\frac{\partial w}{\partial z} - w\frac{d\ln\bar{u}}{dz}\right|^2 - il\left(w^*\frac{\partial w}{\partial z} - w\frac{\partial w^*}{\partial z}\right)}, \qquad (29)$$

where $l = N\bar{u}^{-1}$.

It is necessary to return to the Fourier representation of (1)–(5) for an interpretation of the terms appearing in (29). The horizontal perturbation velocity component at the level of the streamline displacement ζ is

$$u(z + \zeta) \approx u(z) + \zeta d\bar{u}/dz = i\alpha^{-1}(\partial w/\partial z - wd\ln\bar{u}/dz) \qquad (30)$$

using (3) and (5), and

$$lw = -i\alpha(g\theta/N\theta_0) \qquad (31)$$

using (4). Eliassen and Palm (1961) have shown that

$$\frac{\overline{\text{Re}p\ \text{Re}w}}{\bar{u}} = -i\frac{\bar{\rho}(0)}{4\alpha}\left(w^*\frac{\partial w}{\partial z} - w\frac{\partial w^*}{\partial z}\right) > 0, \qquad (32)$$

where the bar represents an average over x, and $p = \bar{\rho}(0)\pi$ is the perturbation pressure. Then, making use of (30)–(32), $|R|^2$ may be expressed as

$$|R|^2 = \frac{\mathcal{E}/l - 2\,\overline{\text{Re}p\ \text{Re}w}/\alpha\bar{u}}{\mathcal{E}/l + 2\,\overline{\text{Re}p\ \text{Re}w}/\alpha\bar{u}}, \qquad (33)$$

where

$$\mathcal{E} = \frac{1}{2}\bar{\rho}(0)\left(|u(z + \zeta)|^2 + \left|\frac{g\theta}{N\theta_0}\right|^2\right) \qquad (34)$$

represents the sum of the perturbation kinetic energy and the available potential energy. The kinetic energy associated with the vertical velocity in this hydrostatic model is negligible compared to the contributions retained in (34). The quantity

$$\frac{\overline{\text{Re}p\ \text{Re}w}}{\alpha\bar{u}} = \text{constant} > 0 \qquad (35)$$

is the flux of wave action, which Eliassen and Palm have shown to be *independent of height z*. It then follows that

$$\frac{\mathcal{E}}{l}\left(\frac{1 - |R|^2}{1 + |R|^2}\right) = 2\,\overline{\text{Re}p\ \text{Re}w}/\alpha\bar{u} \qquad (36)$$

is also a conservative property. The constant is determined by application of the upper boundary condition (25). Then the relationship

$$\frac{\frac{\mathcal{E}}{l} - \left.\frac{\mathcal{E}}{l}\right|_\infty}{\left.\frac{\mathcal{E}}{l}\right|_\infty} = \frac{2|R|^2}{1 - |R|^2} \qquad (37)$$

follows from conservation of the wave action flux (35). It also follows, from (27), (33) and (37), that

$$\left.\frac{\mathcal{E}}{l}\right|_\infty = \frac{2\,\overline{\text{Re}p\ \text{Re}w}}{\alpha\bar{u}} \leqslant \frac{\mathcal{E}}{l}. \qquad (38)$$

The quantity $\mathcal{E}/l$ is not a conservative property but is directly related to variations of $|R|$. These results (33)–(38) point up the relationship between energy transfer through a nonhomogeneous medium and partial reflection associated with spatial variations in the background state.

Now consider a medium in which N and $\bar{u}$ are slowly varying functions of height z. The dispersion relation that is appropriate for the time-dependent counterpart of (1)–(4) is

$$\omega = \alpha N/l, \tag{39}$$

where ω is the real frequency. The vertical component of the group velocity is

$$\partial\omega/\partial l = -\omega/l, \tag{40}$$

which reduces to

$$\partial\omega/\partial l = \alpha\bar{u}/l > 0 \tag{41}$$

for steady flow

$$\omega = -\alpha\bar{u}. \tag{42}$$

Considering wave solutions of the form

$$p,\, w \propto e^{i(lz+\alpha x)},$$

to insure that the vertical flux of wave energy is directed upward, it is readily established that

$$2\frac{\overline{\text{Re}p\ \text{Re}w}}{\alpha\bar{u}} = -\frac{\mathscr{E}}{\omega}\frac{\partial\omega}{\partial l} > 0. \tag{43}$$

Substitution of (41) and (42) into (43) yields

$$2\frac{\overline{\text{Re}p\ \text{Re}w}}{\alpha\bar{u}} = \frac{\mathscr{E}}{l}. \tag{44}$$

Consequently, $\mathscr{E}/l$ may be equated with the flux of wave action when $d\ln S^2/dz \approx 0$. Alternatively, a plane wave travels through a nonhomogeneous medium without undergoing reflection when the impedances match or, equivalently, when the wave action fluxes match.

c. Comparison with a two-layer model

The relatively simple two-layer representation of a semi-infinite model atmosphere, corresponding to

$$\left.\begin{array}{ll} N_2 = \text{constant}, & 0 \leqslant z \leqslant \infty \\ N_1 = \text{constant}, & -H \leqslant z \leqslant 0 \\ \bar{u} = \text{constant}, & -H \leqslant z \leqslant \infty \end{array}\right\}, \tag{45}$$

represents a limit of a continuously varying background state. The reflection properties of this model will be compared to the properties that have been extracted from (11) and (21).

The equations to be satisfied in each layer, obtained from (6a, b), are

$$\frac{d^2 w_i}{dz^2} + l_i^2 w_i = 0, \tag{46}$$

where $i = 1, 2$ represent respectively the lower and upper layers, and $l_i = N_i\bar{u}^{-1}$. The solution in layer 1 is provided by (7), while in layer 2 the solution is

$$w_2/\bar{u} = ATe^{i(\alpha x + l_2 z)}, \tag{47}$$

where T denotes the transmission coefficient, and the vertical flux of wave energy is directed upward in accord with (32). The solutions are joined at $z = 0$ by the requirement that the streamline displacement and the pressure be continuous.

Eliassen and Palm (1961) have shown that the reflection coefficient is

$$|R_L| = \left|\frac{l_2 - l_1}{l_2 + l_1}\right| = \left|\frac{N_2 - N_1}{N_2 + N_1}\right|. \tag{48}$$

In addition, the phase change of the reflected wave across the depth of the lower layer is

$$\theta = 2l_1 H \tag{49}$$

which has been obtained from (23).

In order to illustrate the variation of $R(z)$ to its limiting values, provided by (48) and (49), the characteristic impedance is represented by

$$S^2/S_0^2 = N/N_0 = 1 + (\Delta N/N_0)\tanh aZ, \tag{50}$$

where Z is defined in (14), the subscript denotes the value at $z = 0$, $\Delta N = N\infty - N_0 = N_0/3$ and a is a constant. The variation of N/N_0 as a function of Z is displayed in Fig. 1; the corresponding modulus and phase of R, determined from (21), appear in Figs. 2a and 2b. The limiting value of $|R|$ is $|R| = 0.33$, and H has been chosen to achieve a limiting value $\theta = 2\pi$ across the lower layer.

The ratio of the characteristic scales in (21) is, for $\bar{u}$ = constant,

$$\left|\frac{1}{N}\frac{dN}{dz}\Big/\frac{N}{\bar{u}}\right| = \left|\frac{1}{N}\frac{dN}{dZ}\right|.$$

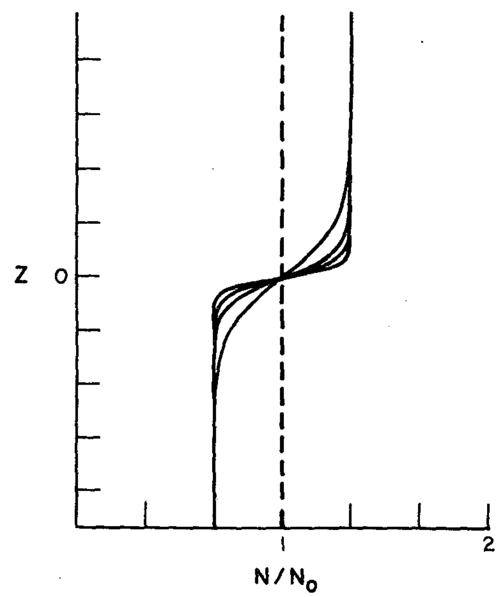
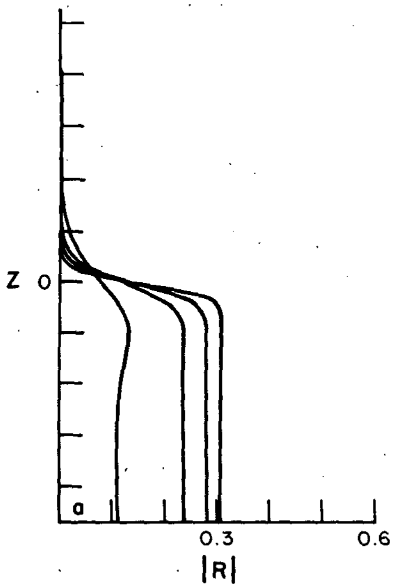


FIG. 1. Distribution of N/N_0, given by (50), for a = 1.0, 2.0, 3.0 and 5.0. A nondimensional height Z = 1 corresponds approximately to 10^3 m for $N_0 = 1.5 \times 10^{-2}$ s^{-1} and $\bar{u}$ = 15 m s^{-1}.



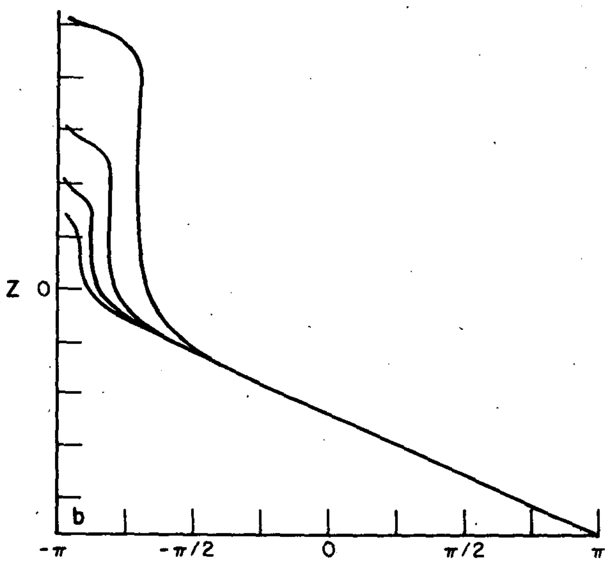


FIG. 2. (a) The modulus $|R|$ and (b) phase θ of the reflection coefficient for the distribution shown in Fig. 1. $|R|$ approaches 0.333, and the phase change across the lower layer approaches 2π.

At $z = 0$, this ratio is $a\Delta N/N_0 = a/3 = a|R_L|$, where $|R_L|$ is the limiting value given by (48). A relatively accurate determination of $|R|$ is attained when $a \gtrsim 3$, i.e., $(a\Delta N/N_0 \gtrsim 1)$.

The property that $|R| < 1$ is assured by (27). The conservation principle may be derived from

$$\frac{d^2w}{dz^2} + l^2w = 0, \tag{51}$$

where $\bar{u}$ = constant and $N = N(z)$. It follows, from (51), that

$$\frac{d\mathscr{E}}{dz} = \frac{1}{2\alpha^2}\bar{\rho}(0)|w|^2\frac{dl^2}{dz}, \tag{52}$$

where $\mathscr{E}$ is defined by (34), (30) and (31). In the limiting case the jump in $\mathscr{E}$, that corresponds to the jump in l^2, may be determined by integration of (52). The result is

$$\mathscr{E}_2 - \mathscr{E}_1 = \frac{1}{2\alpha^2}\bar{\rho}(0)|w_2|^2(l_2{}^2 - l_1{}^2), \tag{53}$$

where w is continuous across the interface so that either w_1 or w_2 may be inserted into (53). It follows from (47) that

$$\bar{\rho}(0)|w_2|^2/\alpha^2 = \mathscr{E}_2/l_2{}^2. \tag{54}$$

Substitution of (54) and (48) into (53) yields, after some manipulation,

$$\frac{\mathscr{E}_1/l_1 - \mathscr{E}_2/l_2}{\mathscr{E}_2/l_2} = 2\frac{|R|^2}{1 - |R|^2}. \tag{55}$$

This result is the two-layer counterpart of (37), which would be the anticipated limit in view of the numerical result displayed in Fig. 2a. Moreover, since $|R|^2 < 1$, it follows that $\mathscr{E}_2/l_2 < \mathscr{E}_1/l_1$.

The matching of impedances to eliminate reflection, at various levels of the atmosphere, is a demonstrable property of continuous partial reflection. The two-layer model ($N_1 \neq N_2$) does not share this property; a three- or multi-layer model representation must be considered. This feature, which appears for example in the results presented by Eliassen and Palm (1961), will be examined in the following section.

4. Solutions for the reflection coefficient

The reflection coefficient cannot be found, in general, from (21) unless numerical methods are employed. However, a general solution of (21) can be obtained for the class of basic flows that satisfy

$$N/\bar{u} = \beta d\ \ln N\bar{u}/dz, \tag{56}$$

where $N\bar{u} = S^2$ and β is a constant. A solution of (56) is displayed in Section 5. Here, attention will be directed to the determination of R as a function of β and S^2.

Substitution of (56) into (21) yields

$$\frac{dR}{dz} - \left[\frac{1}{2}(1 - R)^2 - 2i\beta R\right]\frac{d\ln S^2}{dz} = 0. \tag{57}$$

This equation is separable, and its solution is readily obtained. The integration of (57) provides:

$$\gamma^{-1}\ln\left[\frac{2i\beta + R + \gamma}{2i\beta + R - \gamma}\right] = \ln S^2 + \text{constant}, \tag{58a}$$

when $\gamma^2 = 1 - 4\beta^2 > 0$;

$$(i\gamma)^{-1} \ln\left[\frac{R + i(2\beta + \gamma)}{R + i(2\beta - \gamma)}\right] = \ln S^2 + \text{constant}, \quad (58b)$$

when $\gamma^2 = 4\beta^2 - 1 > 0$;

$$2(i + R)^{-1} = \ln S^2 + \text{constant}, \quad (59c)$$

when $\gamma^2 = 0$, corresponding to $\beta = ½$. The reflection coefficient is $R = |R|(\cos\theta + i \sin\theta)$, and the integration constant is determined by application of the upper boundary condition (25). For present purposes, it will be assumed that $|R| = 0$ and θ = constant when $z \geqslant z_0 > 0$, and that both $|R|$ and θ are continuous functions of z in $0 \leqslant z \leqslant \infty$.

Separation into real and imaginary parts, and denoting $S_0 = S(z_0)$, produces:

$$\left.\begin{aligned} |R|^2 &= \frac{\gamma^{-2}[(S_0{}^{2\gamma} - S^{2\gamma})/(S_0{}^{2\gamma} + S^{2\gamma})]^2}{1 + (2\beta/\gamma)^2[(S_0{}^{2\gamma} - S^{2\gamma})/(S_0{}^{2\gamma} + S^{2\gamma})]^2} \\ \tan\theta &= (2\beta/\gamma)[(S_0{}^{2\gamma} - S^{2\gamma})/(S_0{}^{2\gamma} + S^{2\gamma})] \end{aligned}\right\}, \quad (60a)$$

when $\gamma^2 = 1 - 4\beta^2 > 0$;

$$\left.\begin{aligned} |R|^2 &= \frac{1 - \cos[\gamma \ln(S^2/S_0{}^2)]}{2\gamma^2 + \{1 - \cos[\gamma \ln(S^2/S_0{}^2)]\}} \\ \tan\theta &= -\frac{2\beta}{\gamma}\left[\frac{1 - \cos[\gamma \ln(S^2/S_0{}^2)]}{1 + \cos[\gamma \ln(S^2/S_0{}^2)]}\right]^{1/2} \end{aligned}\right\}, \quad (60b)$$

when $\gamma^2 = 4\beta^2 - 1 > 0$;

$$\left.\begin{aligned} |R|^2 &= \frac{[\ln(S^2/S_0{}^2)]^2}{4 + [\ln(S^2/S_0{}^2)]^2} \\ \tan\theta &= -\frac{1}{2}\ln(S^2/S_0{}^2) \end{aligned}\right\}, \quad (60c)$$

when $\gamma^2 = 0$, corresponding to $\beta = ½$. Both (60a) and (60b) reduce to (60c) as $\beta \to ½$.

A more compact notation, defined by

$$Z' = \beta^{-1} \int_z^{z_0} N\bar{u}^{-1} d\xi = -\ln(S^2/S_0{}^2), \quad (61)$$

may be introduced into (60a, b, c). In particular, (60a) becomes

$$\left.\begin{aligned} |R|^2 &= \frac{\gamma^{-2} \tanh^2\gamma Z'/2}{1 + (2\beta/\gamma)^2 \tanh^2\gamma Z'/2} \\ \tan\theta &= (2\beta/\gamma) \tanh\gamma Z'/2 \end{aligned}\right\}, \quad (62)$$

for $\gamma^2 = 1 - 4\beta^2 > 0$.

The parameter β, defined by (56), may be interpreted as the ratio of two characteristic inverse length scales: the inverse scale height for the characteristic impedance is $s \equiv |d \ln N\bar{u}/dz|$, and the natural inverse length scale is $l = N\bar{u}^{-1}$ ($\lambda = 2\pi/l$ is the characteristic vertical wavelength). When $\beta \ll 1$, a vertically propagating wave encounters a relatively abrupt transition in $S^2 = N\bar{u}$ and partial reflection occurs. The value of β associated with a discontinuity in $S^2 = N\bar{u}$ may be derived from (56), expressed as

$$d(N\bar{u})^{-1}/dz = -(\beta\bar{u}^2)^{-1}. \quad (63)$$

The basic flow $\bar{u}$ is assumed to be continuous, and $\beta = \pm|\beta|$. Integration of (63) yields

$$\left.\frac{1}{N\bar{u}}\right|_{z-\Delta z}^{z+\Delta z} = -\beta^{-1} \int_{z-\Delta z}^{z+\Delta z} \bar{u}^{-2} dz. \quad (64)$$

A jump in N across level z requires that $\beta \to 0$ as $\Delta z \to 0$. Then (60a) or (62) reduces to the two-layer model result provided by (48) and, in this case, the phase θ is independent of height.

It is readily established that $d|R|^2/d\beta^2 < 0$ in $0 \leqslant \beta < ½$, so that $|R|^2$ is a decreasing function of β in the range $|\beta| \leqslant ½$. As β increases, $|R|^2$ becomes an oscillatory function that decreases in amplitude, with $|R|^2 \to 0$ as $\beta \to \infty$. $|R|^2$ also vanishes when

$$\gamma \ln(S^2/S_0{}^2) = 2n\pi, \quad n = 0, 1, 2, \cdots. \quad (65)$$

The case $n = 0$ corresponds to $S^2 = S_0{}^2$.

The interpretation of (65) for $n \neq 0$ is aided by assuming that:

1) $S^2 = S_0{}^2(1 + \Delta S^2/S_0{}^2)$, $\Delta S^2 \ll S_0{}^2$,
2) $\bar{u} =$ constant,
3) $\gamma = (4\beta^2 - 1)^{1/2} \approx 2\beta$.

Then (65) becomes, for $n = 1$,

$$\beta\Delta N/N_0 \approx \pi, \quad (66)$$

and $\beta = ls^{-1} \approx l_0 s^{-1}$, where $s \approx d(\Delta N/N_0)dz$. Elimination of β yields

$$\Delta l/s \approx \pi, \quad (67)$$

where $\Delta l = l - l_0$ and, as a consequence of assumption 2), $\Delta l \propto \Delta S^2$.

This expression (67) is analogous to similar expressions obtained by analyses of three-layer models, e.g., Rayleigh (1880); Eliassen and Palm (1961). In a homogeneous medium, reflection does not occur because the characteristic impedance is uniform. Reflection is always present in a two-layer model because $l \neq l_0$ at some level. However, in a three-layer model, it is possible to match the characteristic impedances between the bottom and top layers to permit unreflected vertical propagation of a wave. The essence of this matching is revealed by (67), if the middle layer depth is interpreted to be $d \sim s^{-1}$. When the basic state parameters vary continuously, then (65) reveals that $|R|^2$ can only vanish at discrete levels.

5. Example

In order to illustrate some of the properties presented in Sections 3 and 4, a basic state satisfying (56) must be determined. We choose

$$N = N_0 \,\mathrm{sech}\, a(z_0 - z), \quad 0 \le z \le z_0 \tag{68}$$

where a is a constant. Integration of (56), and choosing the integration constant $c = N_0^2(a\beta)^{-1}$ yields

$$\bar{u} = N_0(a\beta)^{-1}[1 - \tanh a(z_0 - z)] \cosh a(z_0 - z), \tag{69}$$

where $\beta > 1/2$. This choice for β restricts attention to flows for which (60b) is the appropriate solution for the reflection coefficient.

A nonreflecting upper layer, $z \ge z_0$, characterized by $S^2 = N\bar{u}$ = constant will also be employed. A typical example of the type of flows that characterize a nonreflecting layer has recently been examined by Blumen and Cox (1984), in connection with the derivation of an approximate solution of (13). Here, we choose

$$N = N_0[1 - \tanh a(z_0 - z)]^{-1}, \tag{70}$$

$$\bar{u} = N_0(a\beta)^{-1}[1 - \tanh a(z_0 - z)]. \tag{71}$$

Both N and $\bar{u}$ are continuous at $z = z_0$, $d\bar{u}/dz$ is continuous but dN/dz is discontinuous at $z = z_0$. However, this discontinuity in the gradient of N does not produce a spurious reflection because the continuity of R only depends on the continuity of $N\bar{u}$ at this level. In addition, the Richardson number, $\mathrm{Ri} = N^2(d\bar{u}/dz)^{-2}$ is continuous at $z = z_0$, where it achieves its minimum value $\mathrm{Ri} = \beta^2 > 1/4$. Consequently, Kelvin-Helmholtz instability is not a consideration when $\beta > 1/2$.

The parameter values used to determine the reflection coefficient R are $\beta = \pi$, $N_0 = 10^{-2}\ \mathrm{s}^{-1}$, $a = 1.061 \times 10^{-4}\ \mathrm{m}^{-1}$, $z_0 = 9.425 \times 10^3$ m ($az_0 = 1$). At $z = z_0$, $\bar{u}_0 = N_0(a\beta)^{-1} = 30\ \mathrm{m\ s}^{-1}$ and $\mathrm{Ri} = \pi^2$. Since $\gamma^2 = 4\beta^2 - 1 \approx 4\pi^2$, (60b) may be expressed as

$$\left.\begin{aligned} |R|^2 &\approx (8\pi^2)^{-1}[1 - \cos(2\pi \ln S^2/S_0^2)] \\ \tan\theta &\approx -\left[\frac{1 - \cos(2\pi \ln S^2/S_0^2)}{1 + \cos(2\pi \ln S^2/S_0^2)}\right]^{1/2} \end{aligned}\right\}, \tag{72}$$

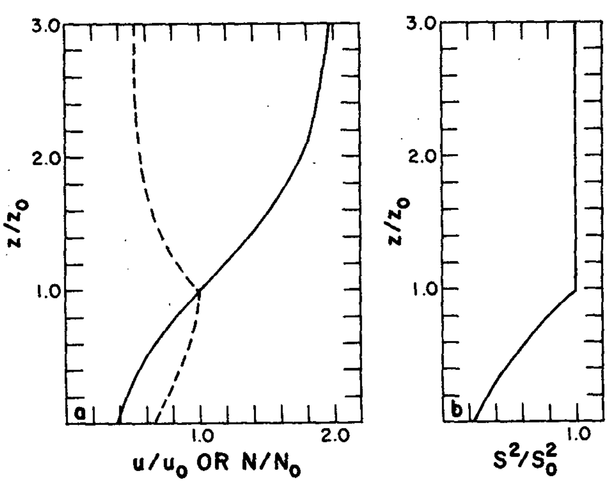


FIG. 3. (a) Distribution of the basic flow (solid) and Brunt–Väisälä frequency (dashed) corresponding to the representations given by (68)–(71). (b) Distribution of the impedance $S^2 = N\bar{u}$.

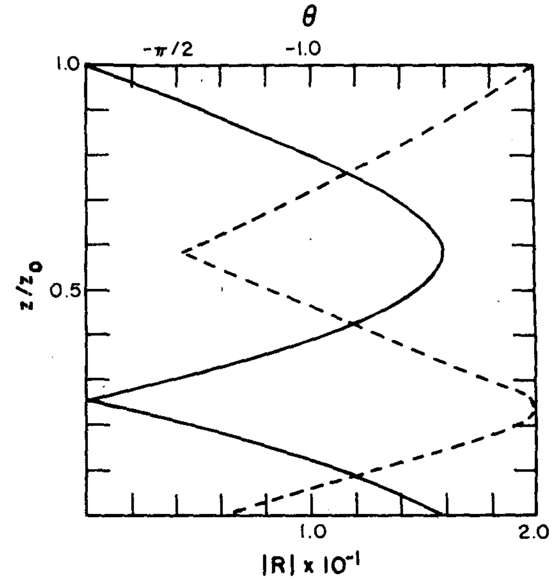


FIG. 4. The modulus of $|R|$ (solid) and phase θ (dashed) of the reflection coefficient for the basic state given in Fig. 3. $|R| = 0$ for $z/z_0 \ge 1.0$.

where

$$S^2/S_0^2 = 1 - \tanh a(z_0 - z). \tag{73}$$

The maximum value of $|R|^2$ is $|R|^2 \approx (4\pi^2)^{-1} = 2.53 \times 10^{-2}$. Consequently, (37) may be expressed as

$$\frac{\mathscr{E}/l - \mathscr{E}/l_0}{\mathscr{E}/l_0} \approx 2|R|^2(1 + |R|^2) \le 0.05,$$

where $\mathscr{E}/l_0 = \mathscr{E}/l_\infty$.

The distributions of $\bar{u}$, N and S^2 are shown in Fig. 3a and 3b, while those of $|R|$ and θ appear in Fig. 4. The zeros of $|R|$, occurring at $z/z_0 = 1$ and $z/z_0 = 0.255$, correspond respectively to $n = 0$ and $n = 1$ in (65). Although $|R|$ is relatively small in this model, the results are not atypical. For example, $|R| = 0.159$ corresponds, in the two-layer model, to a jump in N of $\Delta N = 0.378 \times 10^{-2}\ \mathrm{s}^{-1}$. This would be equivalent to a jump in lapse rates of 4.75 K km^{-1} between the troposphere and an isothermal stratosphere.

6. Remarks

The theory of continuous partial reflection of wave energy in an inhomogeneous medium was first developed to deal with electromagnetic and, then, with acoustic waves. The application to waves in a stratified shear flow has been minimal. Since the theory is appropriate for wave solutions $\phi(z)$ that satisfy the differential equation

$$d^2\phi/dz^2 + n^2(z)\phi = 0, \tag{74}$$

extension of the theory to wave reflection by a stratified shear flow is straightforward. The refractive index $n^2(z)$ depends only on the static stability, represented by the Brunt–Väisälä frequency $N(z)$ and on the basic flow velocity $\bar{u}(z)$ in the present study. This simplification leads to relatively simple interpretation of the results, and comparisons with layered-model results are not difficult to achieve. In addition, the conservation of the wave action flux leads to an equivalent conservation principle (36), involving the modulus of the reflection coefficient $|R|$ and $\mathscr{E}/l$, where $\mathscr{E}$ is the wave energy and $l = N\bar{u}^{-1}$. In effect, the Riccati equation (21) for R may be interpreted as an energy equation.

The present approach is not necessarily restricted to two-dimensional flows. For example, Blumen and Dietze (1982) have shown that a separable wave equation may be derived for basic flows that depend on both the vertical and cross-stream coordinates. The wave equation for the vertical structure may be expressed as in (13) or (74), although a separation constant must be determined. Reflection in the cross-stream direction must also be taken into account, and the wave action flux satisfies a two-dimensional continuity equation. Similar considerations apply to separable solutions for planetary waves on a beta plane, and other relevant flows. These are problems for future consideration.

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REFERENCES

Blumen, W., and S. C. Dietze, 1982: An analysis of three-dimensional mountain lee-waves in a stratified shear flow: Part II. *J. Atmos. Sci.*, **39**, 2712–2720.

——, and K. W. Cox, 1984: Analysis of a stretched variable transformation of Scorer's equation for use in mountain lee-wave studies. *J. Atmos. Sci.*, **41**, 2108–2112.

Brekhovskikh, L. M., 1980: *Waves in Layered Media*, 2nd ed., Academic Press, 503 pp.

Davis, H. T., 1962: *Introduction to Nonlinear Differential and Integral Equations*. Dover, 566 pp.

Eliassen, A., and E. Palm, 1961: On the transfer of energy in stationary mountain waves. *Geofys. Publik.*, 23 pp.

Hines, C. O., and C. A. Reddy, 1967: On the propagation of atmospheric gravity waves through regions of wind shear. *J. Geophys. Res.*, **72**, 1015–1034.

Klemp, J. B., and D. K. Lilly, 1975: The dynamics of wave-induced downslope winds. *J. Atmos. Sci.*, **32**, 320–339.

Meyer, R. E., 1975: Gradual reflection of short waves. *SIAM J. Appl. Math.*, **29**, 481–492.

Rayleigh, Lord, 1880: On reflection of vibrations at the confines of two media between which the transition is gradual. *Proc. London Math. Soc.*, **11**, 51–56.

Schelkunoff, S. A., 1951: Remarks concerning wave propagation in stratified media. *Commun. Pure Appl. Math.*, **4**, 117–128.

Tolstoy, I., 1955: Note on the propagation of normal modes in inhomogeneous media. *J. Acoust. Soc. Am.*, **27**, 274–277.

——, 1967: *Wave Propagation*, McGraw-Hill, 466 pp.

Vincent, R. A., 1969: A criterion for the use of the multilayer approximation in the study of acoustic–gravity wave propagation. *J. Geophys. Res.*, **74**, 2996–3001.

Walker, L. R., and N. Wax, 1946: Nonuniform transmission lines and reflection coefficients. *J. Appl. Phys.*, **17**, 1043–1045.